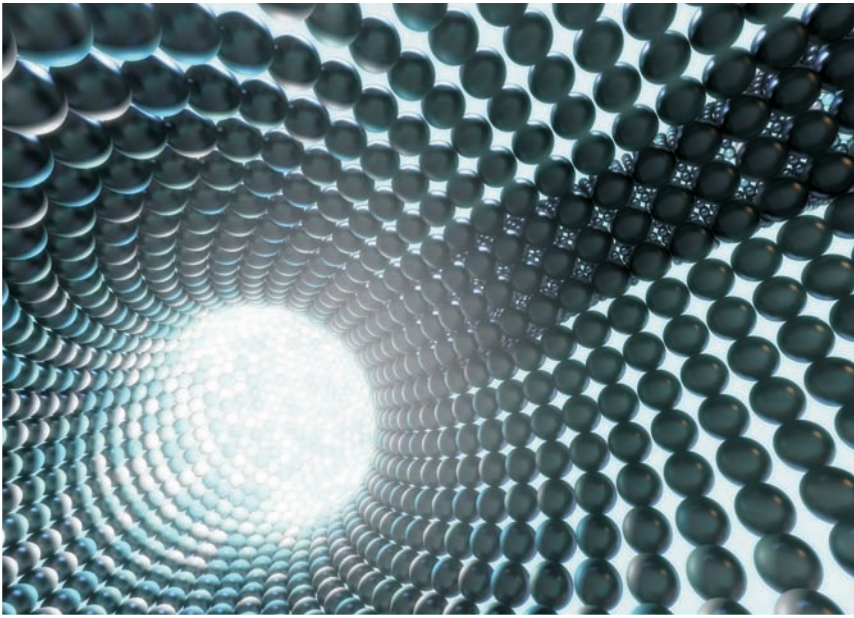
Business Mathematics



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Business Mathematics

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Contents

Preface

The objective of *Business Mathematics* is to present a wide variety of interesting applications of mathematics in a simple, flexible and accessible format. The intended readers include students who are currently studying or have previously studied mathematics at graduation or school level. It provides material on many interesting and important applications which are not usually covered in an introductory course so it can be also used as a supplement. It will also serve as a textbook for any management stream course in mathematics—a short course, a semester-long course or an independent study course.

The book contains 26 chapters, 1,000 solved examples and approximately 2,500 exercises. Each chapter of the book has been written so that it is easily accessible. The basic idea of each application is clearly explained with more sophisticated ideas given in the latter chapters. Each chapter includes analytical exercises containing routine, intermediate and challenging problems. Hint for all exercises are given at the end of the chapter. The historical background of each chapter is also described, wherever possible.

Each section of this book, though independent, is linked with some or the other section. A great deal of effort has been devoted to ensure consistency of format, writing style, accessibility, completeness and terminology. The book covers a broad range of applications in different areas of mathematics, arranged in three sections: Theory, Computing and Exercise.

If students read this book in a logical manner, they will be able to handle not only the examination questions but will also be able to apply the book’s concepts in real life.

I would like to thank the dynamic and creative team of professionals of Pearson, who have made a significant contribution in every manner in bringing out this book. I would specially like to thank Dhiraj Pandey for fully organizing and reviewing the contents and for his valuable inputs.

I would appreciate valuable suggestions and constructive comments from students, readers and academicians.

Kashyap Trivedi

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1

Surds

LEARNING OBJECTIVE

This chapter will enable you to learn the concepts and application of :

- Pure surds
- Mixed surds
- Rational and irrational surds
- Various application of surds

INTRODUCTION

If x is a positive real number which cannot be expressed as the nth power of some rational number then irrational number $\sqrt[n]{x}$ or $x^{1/n}$, which is the positive nth root of x, is said to be a surd or a radical.

Symbol

$\sqrt[n]{\ }$ = radical sign

n = order of surd

x = rational number

$\sqrt[n]{x}$ = irrational number

Types of Surds

There are two types of surds. These are described as under:

Pure Surd	Mixed Surd
A surd, which has unity as its rational factor, and the other factor being irrational is called a pure surd, e.g. $\sqrt{7}$, $3\sqrt{5}$, $7\sqrt{4}$ are pure surds.	A surd, which has a rational factor other than unity, and the other factor being irrational is called a pure surd e.g. $\sqrt{5}$, $3\sqrt[4]{9}$, $2\sqrt[3]{19}$ are mixed surds

RATIONALIZATION OF SURDS

If $\sqrt[n]{x}\sqrt[n]{y} = a$ is a rational number, then each of the two surds is known as rationalizing factor of the other, e.g. $2 \times \sqrt{3} \times \sqrt{3} = 6$ (a rational number)

Therefore, $\sqrt{3}$ is a rationalizing factor of $2\sqrt{3}$. Surds of second order are called two quadratic surds (second degree surds). Binomial expression containing surds, which differ only with (+) or (−) sign containing them, are said to be conjugate surds to each other, e.g. $\sqrt{a} + \sqrt{b}$ *and* $\sqrt{a} - \sqrt{b}$ are conjugate surds to each other.

$$\text{A surd} = \sqrt[\text{Index}]{(\text{radical})^{\text{exponent}}}$$

Rapid Information List

Rapid Information Data	Mathematical Operation
Removed radical sign	$\sqrt[y]{a^x} = a^{x/y}$
Partial transfer of index from outside to inside of the radical sign	$\sqrt[xy]{a^z} = \sqrt[x]{a^{z/y}} = \sqrt[y]{a^{z/x}} = a^{z/xy}$
Change of index	$\sqrt[x]{a^y} = \sqrt[m]{a^{my/x}}$
Transfer of exponent from inside to the outside of radical sign	$\sqrt[z]{a^{xy}} = \left(\sqrt[z]{a^x}\right)^y = \left(\sqrt[z]{a^y}\right)^x = \left(\sqrt[z]{a}\right)^{xy}$
One rational number under more than one radical sign	$\sqrt[p]{\sqrt[q]{\sqrt[r]{a^x}}} = \sqrt[pqr]{a^x} = \left(a^x\right)^{1/pqr}$
More than one rational number under more than one radical sign	$\sqrt[p]{a^x \sqrt[q]{b^y \sqrt[r]{c^z}}} = \sqrt[pqr]{a^{xqr} b^{yr} c^z}$
For same index	
Multiplication	$\sqrt[p]{a^x} \times \sqrt[p]{b^y} = \sqrt[p]{q^x b^y}$
Division	$\dfrac{\sqrt[p]{a^x}}{\sqrt[p]{b^y}} = \sqrt[p]{\dfrac{a^x}{b^y}}$

ILLUSTRATIONS

Illustration 1 If $x = 3 + 2\sqrt{2}$ then show that $x^{1/2} + x^{-1/2} = 2\sqrt{2}$

Solution

$$x = 3 + 2\sqrt{2}$$

$$\sqrt{x} = \sqrt{3 + \sqrt[2]{2}}$$

$$= \sqrt{2 + 1 + 2\sqrt{2} \times 1}$$

$$\sqrt{x} = 1 + \sqrt{2} \qquad (1)$$

$$\text{and } \frac{1}{\sqrt{x}} = \frac{1}{1+\sqrt{2}}$$

$$= \left(\frac{1}{\sqrt{2}+1}\right)\left(\frac{\sqrt{2}-1}{\sqrt{2}-1}\right)$$

$$= \frac{\sqrt{2}-1}{2-1}$$

$$= \sqrt{2}-1 \qquad (2)$$

Now the value of $x^{1/2} + x^{-1/2} = x^{1/2} + \frac{1}{x^{1/2}}$

$$= \sqrt{2}+1+\sqrt{2}-1$$

$$= 2\sqrt{2}$$

Illustration 2 Find the square root of $3+\sqrt{5}$

Solution

$$\sqrt{3+\sqrt{5}}$$

$$= \sqrt{3+\frac{2}{2}\sqrt{5}}$$

$$= \sqrt{3+2\sqrt{\frac{5}{4}}}$$

$$= \sqrt{3+2\sqrt{\frac{5}{2}\times\frac{1}{2}}}$$

$$= \sqrt{\frac{5}{2}+\frac{1}{2}+2\sqrt{\frac{5}{2}\times\frac{1}{2}}}$$

$$= \sqrt{\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{1}{2}}\right)^2}$$

$$= \sqrt{\frac{5}{2}}+\sqrt{\frac{1}{2}}$$

Illustration 3 If $x = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$ and $y = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ then find the value of $x + y$

Solution

$$\text{Now } x + y = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} + \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$$

$$= \frac{\left(\sqrt{3}+\sqrt{2}\right)^2 + \left(\sqrt{3}-\sqrt{2}\right)^2}{\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}+\sqrt{2}\right)}$$

$$= \frac{3+2+2\sqrt{6}+3+2-2\sqrt{6}}{1}$$

$$= 10$$

Illustration 4 If $a = \sqrt{5}+3$ then prove that value of $3a^3 - 16a^2 + 4 = -4$

Solution

$$a = \sqrt{5}+3$$

$$a-3 = \sqrt{5}$$

$$(a-3)^2 = \left(\sqrt{5}\right)^2$$

$$a^2 - 6a + 9 = 5$$

$$a^2 - 6a + 4 = 0$$

$$\text{Now } 3a^3 - 16a^2 + 4$$

$$= 3a\left(a^2 - 6a + 4\right) + 2\left(a^2 - 6a + 4\right) - 4$$

$$= 3a(0) + 2(0) - 4$$

$$= -4$$

Illustration 5 If $2a = \sqrt{\frac{5}{3}} - \sqrt{\frac{3}{5}}$ then prove that the value of $\frac{5\sqrt{1+a^2}}{a+\sqrt{1+a^2}} = 4$

Solution

$$2a = \frac{\sqrt{5}}{\sqrt{3}} - \frac{\sqrt{3}}{\sqrt{5}}$$

$$= \frac{5-3}{\sqrt{15}}$$

$$2a = \frac{2}{\sqrt{15}}$$

$$a = \frac{1}{\sqrt{15}}$$

$$1+a^2 = 1+\frac{1}{15}$$

$$1+a^2 = \frac{16}{15}$$

$$\therefore \frac{5\sqrt{1+a^2}}{a+\sqrt{1+a^2}}$$

$$= \frac{5\left(4/\sqrt{15}\right)}{\left(1/\sqrt{15}\right)+\left(4/\sqrt{15}\right)}$$

$$= \frac{\left(20/\sqrt{15}\right)}{\left(5/\sqrt{15}\right)}$$

$$= \frac{20}{5}$$

$$= 4$$

Illustration 6 Find the square root of $68 + 48\sqrt{2}$

Solution

$$\sqrt{68 + 48\sqrt{2}}$$

$$= \sqrt{68 + 2 \times 24\sqrt{2}}$$

$$= \sqrt{68 + 2\sqrt{24 \times 24 \times 2}}$$

$$= \sqrt{36 + 32 + 2\sqrt{36 \times 32}}$$

$$= \sqrt{\left(\sqrt{36} + \sqrt{32}\right)^2}$$

$$= \sqrt{36} + \sqrt{32}$$

$$= 6 + 4\sqrt{2}$$

Illustration 7 Find the square root of $12 - \sqrt{68 + 48\sqrt{2}}$

Solution

$$12 - \sqrt{68 + 48\sqrt{2}}$$

$$= \sqrt{12 - \sqrt{\left(6 + 4\sqrt{2}\right)^2}} \qquad \text{(Refer to Illustration 6)}$$

$$= \sqrt{12 - \left(6 + 4\sqrt{2}\right)}$$

$$= \sqrt{12 - 6 - 4\sqrt{2}}$$

$$= \sqrt{6 - 4\sqrt{2}}$$

$$= \sqrt{6 - 2\sqrt{4 \times 2}}$$

$$= \sqrt{4 + 2 - 2\sqrt{4 \times 2}}$$

$$= \sqrt{\left(\sqrt{4} - \sqrt{2}\right)^2}$$

$$= \sqrt{\left(2 - \sqrt{2}\right)^2}$$

$$= 2 - \sqrt{2}$$

Illustration 8 Find the square root of $7 + \sqrt{15} + \sqrt{18} + \sqrt{30}$

Solution

$$\sqrt{7 + \sqrt{15} + \sqrt{18} + \sqrt{30}}$$

$$= \sqrt{7 + 2\sqrt{\frac{15}{4}} + 2\sqrt{\frac{18}{4}} + 2\sqrt{\frac{30}{4}}}$$

$$= \sqrt{3 + \frac{5}{2} + \frac{3}{2} + 2\sqrt{\frac{15}{4}} + 2\sqrt{\frac{18}{4}} + 2\sqrt{\frac{30}{4}}}$$

$$= \sqrt{\left(\sqrt{3} + \sqrt{\frac{5}{2}} + \sqrt{\frac{3}{2}}\right)^2}$$

$$= \sqrt{3} + \sqrt{\frac{5}{2}} + \sqrt{\frac{3}{2}} = \frac{1}{\sqrt{2}}\left(\sqrt{6} + \sqrt{5} + \sqrt{3}\right)$$

Illustration 9 If $z = \dfrac{\sqrt{x+a} - \sqrt{x-a}}{\sqrt{x+a} + \sqrt{x-a}}$ then prove that $z + \dfrac{1}{z} = \dfrac{2x}{a}$

Solution

$$z = \frac{\sqrt{x+a} - \sqrt{x-a}}{\sqrt{x+a} + \sqrt{x-a}}$$

$$\therefore z = \frac{\sqrt{x+a} - \sqrt{x-a}}{\sqrt{x+a} + \sqrt{x-a}}\left(\frac{\sqrt{x+a} - \sqrt{x-a}}{\sqrt{x+a} - \sqrt{x-a}}\right)$$

$$\therefore z = \frac{x + a + x - a - 2\sqrt{x+a} \times \sqrt{x-a}}{x + a - x + a}$$

$$= \frac{2x - 2\sqrt{x^2 - a^2}}{2a}$$

$$z = \frac{x - \sqrt{x^2 - a^2}}{a}$$

$$az = x - \sqrt{x^2 - a^2}$$

$$az - x = -\sqrt{x^2 - a^2}$$

$$(az - x)^2 = x^2 - a^2 \qquad (\because \text{ squaring both sides})$$

$$a^2z^2 - 2axz + x^2 = x^2 - a^2$$

$$a^2z^2 = 2axz - a^2$$

$$az^2 = 2xz - a$$

$$a(1 + z^2) = 2xz$$

$$\frac{(1 + z^2)}{z} = \frac{2x}{a}$$

$$\frac{1}{z} + z = \frac{2x}{a}$$

Illustration 10 Evaluate the value of $\frac{5 - \sqrt{6}}{15\sqrt{3} - 8\sqrt{2} + 2\sqrt{50} - 8\sqrt{12}}$

Solution

$$\frac{5 - \sqrt{6}}{15\sqrt{3} - 8\sqrt{2} + 2\sqrt{50} - 8\sqrt{12}}$$

$$= \frac{5 - \sqrt{6}}{15\sqrt{3} - 8\sqrt{2} + 10\sqrt{2} - 16\sqrt{3}}$$

$$= \frac{5 - \sqrt{6}}{2\sqrt{2} - \sqrt{3}}$$

$$= \frac{5 - \sqrt{6}}{2\sqrt{2} - \sqrt{3}}\left(\frac{2\sqrt{2} + \sqrt{3}}{2\sqrt{2} + \sqrt{3}}\right) \qquad \text{(Rationalizing)}$$

$$= \frac{10\sqrt{2} + 5\sqrt{3} - 2\sqrt{12} - \sqrt{18}}{(2\sqrt{2})^2 - (\sqrt{3})^2}$$

$$= \frac{10\sqrt{2} + 5\sqrt{3} - 4\sqrt{3} - 3\sqrt{2}}{8 - 3}$$

$$= \frac{7\sqrt{2} + \sqrt{3}}{5}$$

Illustration 11 Evaluate $\frac{6\sqrt{2}}{\sqrt{3}+\sqrt{6}} - \frac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}} + \frac{2\sqrt{6}}{\sqrt{3}+\sqrt{2}}$

Solution

$$= \frac{6\sqrt{2}}{\sqrt{3}+\sqrt{6}}\left(\frac{\sqrt{6}-\sqrt{3}}{\sqrt{6}-\sqrt{3}}\right) - \frac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}\left(\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}}\right) + \frac{2\sqrt{6}}{\sqrt{3}+\sqrt{2}}\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right)$$

$$= \frac{6\sqrt{12}-6\sqrt{6}}{6-3} - \frac{4\sqrt{18}-4\sqrt{6}}{6-2} + \frac{2\sqrt{18}-2\sqrt{12}}{3-2}$$

$$= 4\sqrt{3} - 2\sqrt{6} - 3\sqrt{2} + \sqrt{6} + 6\sqrt{2} - 4\sqrt{3}$$

$$= 3\sqrt{2} - \sqrt{6}$$

Illustration 12 If $M = 3 + \sqrt{8}$ then find the value of $M^4 + \frac{1}{M^4}$

Solution

Here $M = 3 + \sqrt{8}$

$$\therefore \frac{1}{M} = \frac{1}{3+\sqrt{8}} = \frac{1}{3+\sqrt{8}}\left(\frac{3-\sqrt{8}}{3-\sqrt{8}}\right) = \frac{3-\sqrt{8}}{9-8}$$

$$\frac{1}{M} = 3 - \sqrt{8}$$

$$M + \frac{1}{M} = 3 + \sqrt{8} + 3 - \sqrt{8} = 6$$

$$M^2 + \frac{1}{M^2} = \left(M + \frac{1}{M}\right)^2 - 2M\frac{1}{M} = (6)^2 - 2 = 34$$

$$M^4 + \frac{1}{M^4} = \left(M^2 + \frac{1}{M^2}\right)^2 - 2M^2\frac{1}{M^2}$$

$$= (34)^2 - 2$$

$$= 1156 - 2$$

$$= 1154$$

Illustration 13 If $N = \frac{5-\sqrt{21}}{2}$ then verify that

$$\left(N^3 + \frac{1}{N^3}\right) - 5\left(N^2 + \frac{1}{N^2}\right) + \left(N + \frac{1}{N}\right) = 0$$

Solution

$$\frac{1}{N} = \frac{2}{5-\sqrt{21}} = \frac{2}{5-\sqrt{21}}\left(\frac{5+\sqrt{21}}{5+\sqrt{21}}\right)$$

$$= \frac{2\left(5+\sqrt{21}\right)}{25-21} = \frac{5+\sqrt{21}}{2}$$

$$\text{Now } N + \frac{1}{N} = \frac{5-\sqrt{21}}{2} + \frac{5+\sqrt{21}}{2} = \frac{10}{2} = 5$$

$$N^2 + \frac{1}{N^2} = \left(N + \frac{1}{N}\right)^2 - 2N\frac{1}{N} = (5)^2 - 2 = 23$$

$$N^3 + \frac{1}{N^3} = \left(N + \frac{1}{N}\right)\left(N^2 + \frac{1}{N^2} - 1\right) = 5(23-1) = 110$$

$$\left(N^3 + \frac{1}{N^3}\right) - 5\left(N^2 + \frac{1}{N^2}\right) + \left(N + \frac{1}{N}\right)$$

$$= 110 - 5(23) + 5$$

$$= 0$$

Illustration 14 If $N = 5 - \sqrt{24}$ then find the value of

$$\left(N^3 + \frac{1}{N^3}\right) - 10\left(N^2 + \frac{1}{N^2}\right) + 4\left(N + \frac{1}{N}\right) - 30$$

Solution

$$\frac{1}{N} = \frac{1}{5-\sqrt{24}} = \frac{1}{5-\sqrt{24}}\left(\frac{5+\sqrt{24}}{5+\sqrt{24}}\right) = \frac{5+\sqrt{24}}{25-24} = 5+\sqrt{24}$$

$$N + \frac{1}{N} = 5 - \sqrt{24} + 5 + \sqrt{24} = 10$$

$$N^2 + \frac{1}{N^2} = \left(N + \frac{1}{N}\right)^2 - 2N\frac{1}{N} = (10)^2 - 2 = 98$$

$$N^3 + \frac{1}{N^3} = \left(N + \frac{1}{N}\right)\left(N^2 + \frac{1}{N^2} - 1\right) = 10(98-1) = 970$$

$$\left(N^3 + \frac{1}{N^3}\right) - 10\left(N^2 + \frac{1}{N^2}\right) + 4\left(N + \frac{1}{N}\right) - 30$$

$$= 970 - 10(98) + 4(10) - 30$$

$= 970 - 980 + 40 - 30$
$= 970 - 970$
$= 0$

Illustration 15 If $x = \dfrac{4\sqrt{15}}{\sqrt{5}+\sqrt{3}}$ then find the value of $\dfrac{x+\sqrt{20}}{x-\sqrt{20}} + \dfrac{x+\sqrt{12}}{x-\sqrt{12}}$

Solution

$$x = \frac{4\sqrt{15}}{\sqrt{5}+\sqrt{3}} = \frac{\sqrt{16\times 15}}{\sqrt{5}+\sqrt{3}}$$

$$\therefore x = \frac{\sqrt{20}\times\sqrt{12}}{\sqrt{5}+\sqrt{3}}$$

$$\therefore \frac{x}{\sqrt{20}} = \frac{\sqrt{12}}{\sqrt{5}+\sqrt{3}} \qquad \text{or} \qquad \therefore \frac{x}{\sqrt{12}} = \frac{\sqrt{20}}{\sqrt{5}+\sqrt{3}}$$

$$\therefore \frac{x}{\sqrt{20}} = \frac{2\sqrt{3}}{\sqrt{5}+\sqrt{3}} \qquad \frac{x}{\sqrt{12}} = \frac{2\sqrt{5}}{\sqrt{5}+\sqrt{3}}$$

$$\therefore \frac{x+\sqrt{20}}{x-\sqrt{20}} = \frac{2\sqrt{3}+\sqrt{5}+\sqrt{3}}{2\sqrt{3}-\sqrt{5}-\sqrt{3}} \qquad \therefore \frac{x+\sqrt{12}}{x-\sqrt{12}} = \frac{2\sqrt{5}+\sqrt{5}+\sqrt{3}}{2\sqrt{5}-\sqrt{5}-\sqrt{3}}$$

(Componendo) (Dividendo)

$$\therefore \frac{x+\sqrt{20}}{x-\sqrt{20}} = \frac{3\sqrt{3}+\sqrt{5}}{\sqrt{3}-\sqrt{5}} \qquad \therefore \frac{x+\sqrt{12}}{x-\sqrt{12}} = \frac{3\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} \qquad (2)$$

$$\therefore \frac{x+\sqrt{20}}{x-\sqrt{20}} = \frac{-3\sqrt{3}-\sqrt{5}}{\sqrt{5}-\sqrt{3}} \qquad (1)$$

By adding Eqs. (1) and (2) we get

$$\frac{x+\sqrt{20}}{x-\sqrt{20}} + \frac{x+\sqrt{12}}{x-\sqrt{12}} = \frac{-3\sqrt{3}-\sqrt{5}}{\sqrt{5}-\sqrt{3}} + \frac{3\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$$

$$= \frac{-3\sqrt{3}-\sqrt{5}+3\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$$

$$= \frac{2\sqrt{5}-2\sqrt{3}}{\sqrt{5}-\sqrt{3}}$$

$$= \frac{2\left(\sqrt{5}-\sqrt{3}\right)}{\left(\sqrt{5}-\sqrt{3}\right)} = 2$$

Illustration 16 If $x = \dfrac{1}{\sqrt{2}-1}$ then verify that $x^2 + \dfrac{1}{x^2} - 6 = 0$

Solution

$$x = \frac{1}{\sqrt{2}-1}\left(\frac{\sqrt{2}+1}{\sqrt{2}+1}\right) = \frac{\sqrt{2}+1}{2-1} = \sqrt{2}+1$$

$$\therefore \frac{1}{x} = \frac{1}{\sqrt{2}+1} = \frac{1}{\sqrt{2}+1}\left(\frac{\sqrt{2}-1}{\sqrt{2}-1}\right) = \frac{\sqrt{2}-1}{2-1} = \sqrt{2}-1$$

and $x^2 + \dfrac{1}{x^2} = \left(x + \dfrac{1}{x}\right)^2 - 2x\dfrac{1}{x} = \left(2\sqrt{2}\right)^2 - 2$

$$x^2 + \frac{1}{x^2} - 6$$

$= 6 - 6$
$= 0$

Illustration 17 If $m = \dfrac{\sqrt{5}-2}{\sqrt{5}+2}$ then show that $m^4 + \dfrac{1}{m^4} = 103682$

Solution

$$m = \frac{\sqrt{5}-2}{\sqrt{5}+2} = \frac{\sqrt{5}-2}{\sqrt{5}+2}\left(\frac{\sqrt{5}-2}{\sqrt{5}-2}\right)$$

$$= \frac{\left(\sqrt{5}-2\right)^2}{5-4} = \left(\sqrt{5}-2\right)^2$$

$$= 5 + 4 - 4\sqrt{5}$$

$$m = 9 - 4\sqrt{5} \qquad (1)$$

and $\dfrac{1}{m} = \dfrac{\sqrt{5}+2}{\sqrt{5}-2} = \dfrac{\sqrt{5}+2}{\sqrt{5}-2}\left(\dfrac{\sqrt{5}+2}{\sqrt{5}+2}\right)$

$$= \frac{\left(\sqrt{5}+2\right)^2}{5-4} = 5 + 4 + 4\sqrt{5} = 9 + 4\sqrt{5} \qquad (2)$$

From Eqs. (1) and (2), we can say that

$$m + \frac{1}{m} = 9 - 4\sqrt{5} + 9 + 4\sqrt{5} = 18$$

Now $m^2 + \dfrac{1}{m^2} = \left(m + \dfrac{1}{m}\right)^2 - 2m\dfrac{1}{m} = \left(18\right)^2 - 2 = 322$

and $m^4 + \frac{1}{m^4} = \left(m^2 + \frac{1}{m^2}\right)^2 - 2m^2 \frac{1}{m^2}$

$$= (322)^2 - 2$$
$$= 1,03,682$$

Illustration 18 Evaluate $\frac{1}{\sqrt{11-2\sqrt{30}}} - \frac{3}{\sqrt{7-2\sqrt{10}}} - \frac{4}{\sqrt{8+4\sqrt{3}}}$

Solution

$$\frac{1}{\sqrt{11-2\sqrt{30}}} - \frac{3}{\sqrt{7-2\sqrt{10}}} - \frac{4}{\sqrt{8+2\sqrt{12}}}$$

$$= \frac{1}{\sqrt{6+5-2\sqrt{6\times 5}}} - \frac{3}{\sqrt{5+2-2\sqrt{5\times 2}}} - \frac{4}{\sqrt{6+2+2\sqrt{6\times 2}}}$$

$$= \frac{1}{\sqrt{\left(\sqrt{6}-\sqrt{5}\right)^2}} - \frac{3}{\sqrt{\left(\sqrt{5}-\sqrt{2}\right)^2}} - \frac{4}{\sqrt{\left(\sqrt{6}+\sqrt{2}\right)^2}}$$

$$= \frac{1}{\sqrt{6}-\sqrt{5}} - \frac{3}{\sqrt{5}-\sqrt{2}} - \frac{4}{\sqrt{6}+\sqrt{2}}$$

$$= \frac{1}{\sqrt{6}-\sqrt{5}}\left(\frac{\sqrt{6}+\sqrt{5}}{\sqrt{6}+\sqrt{5}}\right) - \frac{3}{\sqrt{5}-\sqrt{2}}\left(\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}\right) - \frac{4}{\sqrt{6}+\sqrt{2}}\left(\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}}\right)$$

$$= \frac{\sqrt{6}+\sqrt{5}}{6-5} - \frac{3\left(\sqrt{5}+\sqrt{2}\right)}{5-2} - \frac{4\left(\sqrt{6}-\sqrt{2}\right)}{6-2}$$

$$= \sqrt{6}+\sqrt{5}-\sqrt{5}-\sqrt{2}-\sqrt{6}+\sqrt{2}$$

$$= 0$$

Illustration 19 If $x = \frac{\sqrt{3}}{2}$ then find the value of $\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$

Solution

Here $\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$

$$= \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right)$$

$$= \frac{\left(\sqrt{1+x}-\sqrt{1-x}\right)^2}{(1+x)-(1-x)}$$

$$= \frac{1 + x + 1 - x - 2\sqrt{(1 + x)(1 - x)}}{2x}$$

$$= \frac{2 - 2\sqrt{1 - x^2}}{2x}$$

$$= \frac{1 - \sqrt{1 - x^2}}{x}$$

$$= \frac{1 - \sqrt{1 - (3/4)}}{(\sqrt{3}/2)}$$

$$= \frac{1 - (\sqrt{1/4})}{(\sqrt{3}/2)}$$

$$= \frac{1 - (1/2)}{(\sqrt{3}/2)}$$

$$= \frac{(1/2)}{(\sqrt{3}/2)}$$

$$= \frac{1}{\sqrt{3}}$$

Illustration 20 If $p = \frac{\sqrt{2} + 1}{\sqrt{2} - 1}$ and $q = \frac{\sqrt{2} - 1}{\sqrt{2} + 1}$ then find the value of

$$\frac{p^2 + pq + q^2}{p^2 - pq + q^2}$$

Solution

$$p + q = \frac{\sqrt{2} + 1}{\sqrt{2} - 1} + \frac{\sqrt{2} - 1}{\sqrt{2} + 1} = \frac{(\sqrt{2} + 1)^2 + (\sqrt{2} - 1)^2}{(\sqrt{2})^2 - (1)^2}$$

$$= \frac{2 + 1 + 2\sqrt{2} + 2 + 1 - 2\sqrt{2}}{1} = 6$$

and $pq = \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1}\right)\left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1}\right) = 1$

$$\therefore \frac{p^2 + pq + q^2}{p^2 - pq + q^2} = \frac{(p + q)^2 - pq}{(p + q)^2 - 3pq}$$

$$= \frac{(6)^2 - 1}{(6)^2 - 3(1)}$$

$$= \frac{35}{33}$$

Illustration 21 If $y = \dfrac{\sqrt{a+2b} + \sqrt{a-2b}}{\sqrt{a+2b} - \sqrt{a-2b}}$ then show that $by^2 - ay + b = 0$

Solution

Here $\dfrac{y}{1} = \dfrac{\sqrt{a+2b} + \sqrt{a-2b}}{\sqrt{a+2b} - \sqrt{a-2b}}$

$$\Rightarrow \frac{y+1}{y-1} = \frac{\sqrt{a+2b} + \sqrt{a-2b} + \sqrt{a+2b} - \sqrt{a-2b}}{\sqrt{a+2b} + \sqrt{a-2b} - \sqrt{a+2b} + \sqrt{a-2b}}$$

(Componendo–dividendo)

$$\Rightarrow \frac{y+1}{y-1} = \frac{\sqrt{a+2b}}{\sqrt{a-2b}}$$

$$\Rightarrow \left(\frac{y+1}{y-1}\right)^2 = \left(\frac{\sqrt{a+2b}}{\sqrt{a-2b}}\right)^2 \qquad (\because \text{squaring both sides})$$

$$\Rightarrow \frac{y^2 + 2y + 1}{y^2 - 2y + 1} = \frac{a+2b}{a-2b}$$

Again by Componendo–dividendo

$$\Rightarrow \frac{y^2 + 2y + 1 + (y^2 - 2y + 1)}{y^2 + 2y + 1 - (y^2 - 2y + 1)} = \frac{a + 2b + (a - 2b)}{a + 2b - (a - 2b)}$$

$$\Rightarrow \frac{2(y^2 + 1)}{4y} = \frac{2a}{4b}$$

$$\Rightarrow \frac{y^2 + 1}{y} = \frac{a}{b}$$

$$\Rightarrow by^2 + b = ay$$

$$\Rightarrow by^2 - ay + b = 0$$

Illustration 22 If $a = \frac{\sqrt{5}+1}{\sqrt{5}-1}$ and $b = \frac{\sqrt{5}-1}{\sqrt{5}+1}$ then find the value of $5a^2 - 3ab + 5b^2$

Solution

Here $a + b = \frac{\sqrt{5}+1}{\sqrt{5}-1} + \frac{\sqrt{5}-1}{\sqrt{5}+1}$

$$= \frac{(\sqrt{5}+1)^2 + (\sqrt{5}-1)^2}{(\sqrt{5})^2 - (1)^2}$$

$$= \frac{5+1+2\sqrt{5}+5+1-2\sqrt{5}}{5-1}$$

$$= \frac{12}{4} = 3$$

and $ab = \left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right)\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right) = 1$

Now $5a^2 - 3ab + 5b^2$

$= (5a^2 + 10ab + 5b^2) - 13ab$

$= 5(a+b)^2 - 13(1)$

$= 45 - 13$

$= 32$

Illustration 23 If $(x+y)^{1/3} + (y+z)^{1/3} + (z+x)^{1/3} = 0$ then prove that $(x+y+z)^3 = 9(x^3+y^3+z^3)$

Solution

$\because (x+y)^{1/3} + (y+z)^{1/3} + (z+x)^{1/3} = 0$

$\therefore (x+y)^{1/3} + (y+z)^{1/3} = -(z+x)^{1/3}$

Now cubing both sides we get

$$\left[(x+y)^{1/3} + (y+z)^{1/3}\right]^3 = \left[-(z+x)^{1/3}\right]^3$$

$$\therefore (x+y) + (y+z) + 3(x+y)^{1/3}(y+z)^{1/3}\left[(x+y)^{1/3} + (y+z)^{1/3}\right] = -(z+x)$$

$$\therefore (x+y) + (y+z) + 3(x+y)^{1/3}(y+z)^{1/3}\left[-(z+x)^{1/3}\right] = -(z+x)$$

$$\therefore 2(x+y+z) = 3(x+y)^{1/3}(y+z)^{1/3}(z+x)^{1/3}$$

Cubing both sides

$$\therefore 8(x+y+z)^3 = 9\times 3(x+y)(y+z)(z+x)$$

$$\therefore 8(x+y+z)^3 = 9\left[(x+y+z)^3 - x^3 - y^3 - z^3\right]$$

$$\Rightarrow (x+y+z)^3 = 9\left(x^3+y^3+z^3\right)$$

Illustration 24 If $x = \sqrt{\dfrac{n+1}{n-1}}$ then show that $\left(\dfrac{x}{x-1}\right)^2 + \left(\dfrac{x}{x+1}\right)^2 = n(n+1)$

Solution

$$\left(\frac{x}{x-1}\right)^2 + \left(\frac{x}{x+1}\right)^2$$

$$= x^2\left[\frac{(x+1)^2+(x-1)^2}{\left(x^2-1\right)^2}\right]$$

$$= x^2\left[\frac{2\left(x^2+1\right)}{\left(x^2-1\right)^2}\right]$$

$$= \left(\frac{n+1}{n-1}\right)\left\{\frac{2\left[\left[(n+1)/(n-1)\right]+1\right]}{\left[\left[(n+1)/(n-1)\right]-1\right]^2}\right\}$$

$$= \left(\frac{n+1}{n-1}\right)\left\{\frac{2\left[(n+1+n-1)/(n-1)\right]}{\left[(n+1-n+1)/(n-1)\right]^2}\right\}$$

$$= \left(\frac{n+1}{n-1}\right)\left\{\frac{2\left[2n/(n-1)\right]}{\left[4/(n-1)^2\right]}\right\}$$

$$= \left(\frac{n+1}{n-1}\right)\left(\frac{n}{n-1}\right)\left[(n-1)^2\right]$$

$$= n(n+1)$$

Illustration 25 $\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z} = 0$ then prove that $(x+y+z)^3 = 27xyz$

Solution

Hint: $\left(a+b+c=0 \Rightarrow a^3+b^3+c^3 = 3abc\right)$

$$\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z} = 0$$

$$\Rightarrow \left(\sqrt[3]{x}\right)^3 + \left(\sqrt[3]{y}\right)^3 + \left(\sqrt[3]{z}\right)^3 = 3\sqrt[3]{x} \times \sqrt[3]{y} \times \sqrt[3]{z}$$

$$\Rightarrow x + y + z = 3x^{1/3}\, y^{1/3}\, z^{1/3}$$

$$\Rightarrow (x + y + z)^3 = \left(3x^{1/3} \times y^{1/3} \times z^{1/3}\right)^3$$

$$\Rightarrow (x + y + z)^3 = 27xyz$$

ANALYTICAL EXERCISES

1. Simplify $\dfrac{2}{\sqrt{7} - \sqrt{5}} - \dfrac{3}{\sqrt{5} - \sqrt{2}} - \dfrac{5}{\sqrt{7} + \sqrt{2}}$

2. Simplify $\dfrac{2\sqrt{3}}{2\sqrt{3} + \sqrt{5}} + \dfrac{\sqrt{5}}{2\sqrt{3} - \sqrt{5}}$

3. Simplify $\left(\sqrt{2} - \sqrt{3}\right)\left(\sqrt{2} - 4\sqrt{3}\right)\left(14 + 5\sqrt{6}\right)$

4. Simplify $\dfrac{1}{2 + \sqrt{3}} + \dfrac{2}{2 + \sqrt{6}} - \dfrac{3}{\sqrt{3} + \sqrt{6}}$

5. Find the square root of $24 + 2\sqrt{119}$

6. Find the square root of $7 + 4\sqrt{3}$

7. Find the square root of $14 - \sqrt{180}$

8. Find the square root of $5 + \sqrt{21}$

9. Find the square root of $3 - \dfrac{1}{3}\sqrt{56}$

10. Find the square root of $\dfrac{13}{4} + \sqrt{3}$

11. If $a = \dfrac{1}{2 - \sqrt{3}}$ then find the value of $\left(a^2 - 4a + 6\right)^2$

12. If $m = 3 + \sqrt{5}$ then find the value of $3m^3 - 16m^2 + 4$

13. If $m = \dfrac{1}{5 - 2\sqrt{6}}$ then find the value of $\left(m^2 - 10m + 11\right)^3$

14. If $a = \dfrac{1 - \sqrt{5}}{4}$ then find the value of $8a^4 - 16a^3 - 12a^2 + 11a + 7$

15. If $M = \dfrac{1}{2-\sqrt{3}}$ and $N = \dfrac{1}{2+\sqrt{3}}$ then find the value of $M^3 - N^3$

16. If $\dfrac{1}{\left(\sqrt{12}-\sqrt{3}+1\right)^2} = x + y\sqrt{3}$ then find x and y

17. Simplify $\dfrac{2}{\sqrt{5}-\sqrt{3}} + \dfrac{4}{\sqrt{10+\sqrt{84}}} - \dfrac{2}{\sqrt{7}-\sqrt{5}}$

18. Simplify $\dfrac{3+\sqrt{7}}{\sqrt{2}-\sqrt{4-\sqrt{7}}} - \dfrac{3-\sqrt{7}}{\sqrt{2}+\sqrt{4+\sqrt{7}}} - \dfrac{2}{\sqrt{7}-\sqrt{5}}$

19. Simplify $\dfrac{3}{\sqrt{5}-\sqrt{2}} - \dfrac{2}{\sqrt{8+\sqrt{60}}} - \dfrac{1}{\sqrt{3}-\sqrt{2}}$

20. Simplify $\dfrac{2+\sqrt{18}-\sqrt{50}}{\sqrt{72}-\sqrt{50}-1}$

21. Simplify $\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}} + \dfrac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}$

22. Simplify $\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}} + \dfrac{\sqrt{7}}{\sqrt{5}-\sqrt{7}} + \dfrac{5\sqrt{5}}{4\sqrt{3}-2\sqrt{7}}$

23. Find the square root of $\sqrt{60}+\sqrt{40}+\sqrt{24}+10$

24. Find the square root of $2x+2y+\sqrt{3x^2+12xy}$

25. Find the square root of $17-\sqrt{72}-\sqrt{192}+\sqrt{96}$

26. Find the square root of $13-\sqrt{40}+\sqrt{48}-\sqrt{120}$

27. Find the square root of $x+\sqrt{x^2-y^2}$

28. Find the square root of $2x+2\sqrt{x^2-y^2}$

29. Find the fourth root of $\dfrac{7+3\sqrt{5}}{2}$

30. Find the fourth root of $28-16\sqrt{3}$

31. Find the fourth root of $14+6\sqrt{5}$

32. Find the fourth root of $28+4\sqrt{48}$

33. Find the fourth root of $\dfrac{7}{4}-\sqrt{3}$

34. Find the fourth root of $7-4\sqrt{3}$

35. Find the fourth root of $241-44\sqrt{30}$

36. Find the fourth root of $153 - 36\sqrt{18}$

37. Find the fourth root of $89 - 28\sqrt{10}$

38. Find the fourth root of $184 + 40\sqrt{21}$

39. Find the fourth root of $56 + 12\sqrt{20}$

40. Find the square root of $124 + 32\sqrt{15}$

41. If $a^2 > 25$ then prove that $\dfrac{a}{a - \sqrt{a^2 - 25}} = \dfrac{a^2}{25} + \dfrac{a\sqrt{a^2 - 25}}{25}$

42. If $2m = \sqrt{\dfrac{5}{3}} - \sqrt{\dfrac{3}{5}}$ then prove that $\dfrac{5\sqrt{1 + m^2}}{m + \sqrt{1 + m^2}} = 4$

43. If $M = a + \sqrt{a^2 + 1}$ then prove that $M^3 + \dfrac{1}{M^3} = 8a^3 - 6a$

44. If $M = \sqrt[3]{4 + \sqrt{15}} + \dfrac{1}{\sqrt[3]{4 + \sqrt{15}}}$ then prove that $M^3 = 3M + 8$

45. If $\dfrac{\sqrt{x} - 1}{\sqrt{x} + 1} = \dfrac{\sqrt{3}}{2}$ then prove that $x^{3/4} + x^{-3/4} = 52$

46. If $y = 3 - 2\sqrt{2}$ then prove that $\left(y^2 - 6y + 4\right)^3 = 27$

47. Show that $\left(5 - 2\sqrt{6}\right)^{-1/2} - \left(5 + 2\sqrt{6}\right)^{-1/2} = 2\sqrt{2}$

48. Show that $\left(6 - 2\sqrt{5}\right)^{-1/2} - \left(6 + 2\sqrt{5}\right)^{-1/2} = \dfrac{1}{2}$

49. Prove that $\dfrac{1}{1 + \sqrt{2}} + \dfrac{1}{\sqrt{2} + \sqrt{3}} + \dfrac{1}{\sqrt{3} + \sqrt{4}} + \dfrac{1}{\sqrt{n} + \sqrt{n + 1}} = \sqrt{n + 1} - 1$

50. Simplify $\left(\dfrac{9 + 4\sqrt{5}}{9 - 4\sqrt{5}}\right)^{1/2} + \left(\dfrac{9 - 4\sqrt{5}}{9 + 4\sqrt{5}}\right)^{1/2}$

51. If $\dfrac{2}{\sqrt{2}} + \dfrac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}} + \dfrac{\sqrt{3} - \sqrt{2}}{\sqrt{3} + \sqrt{2}} = x + \sqrt{y}$ then find the value of x and y

52. If $a = \sqrt{2} - \sqrt{3}$ then find the value of $\dfrac{a + 1}{a - 1} + \dfrac{a - 1}{a + 1}$

53. If $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$ then prove that $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$
where $\sqrt{xy}$ is surd.

54. If $\sqrt[4]{\dfrac{7}{4} - \sqrt{3}} = x + y\sqrt{3}$ then prove that $x + y = 0 \ (x, y \in Q)$

55. Find the value of $\dfrac{3}{1-\sqrt{2}+\sqrt{3}}+\dfrac{1}{1-\sqrt{2}-\sqrt{3}}-\dfrac{2}{1+\sqrt{2}-\sqrt{3}}+\dfrac{3}{\sqrt{2}}$

56. If $M = 2+2^{2/3}+2^{1/3}$ then find the value of M^3-6M^2+6M

57. Prove that the square root of $\dfrac{\sqrt{26-15\sqrt{3}}}{5\sqrt{2}-\sqrt{38+5\sqrt{3}}}$ is a rational number

58. Find the square root of $1+a^2+\sqrt{1+a^2+a^4}$

59. If $a=\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$ and $b=\dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$ then find the value of $3a^2+4ab-3b^2$

60. If $x=\dfrac{2ab}{1+b^2}$ and $b^2-1\geq 0$ then show that $\dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}=\dfrac{1}{b}$

61. Simplify $\dfrac{1+x}{1+\sqrt{1+x}}+\dfrac{1-x}{1-\sqrt{1-x}}$ when $x=\dfrac{\sqrt{3}}{2}$

62. Evaluate $\dfrac{(3+\sqrt{3})(3+\sqrt{5})(\sqrt{5}-2)}{(5-\sqrt{5})(1+\sqrt{3})}$

63. If $x=\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$ and $y=\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$ then find the value of $\dfrac{x^2+xy+y^2}{x^2-xy+y^2}$

64. If $2x=\sqrt{a}+\dfrac{1}{\sqrt{a}}$ then prove that $\dfrac{\sqrt{x^2-1}}{x-\sqrt{x^2-1}}=\dfrac{1}{2}(a-1)$

65. If $x=9+4\sqrt{5}$ then find the value of $\sqrt{x}-\dfrac{1}{\sqrt{x}}$

66. If $x=\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$ and $y=\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ then show that $\dfrac{x^2+xy+y^2}{x^2-xy+y^2}=\dfrac{15}{13}$

67. If $2x=a+\dfrac{1}{a}$ and $2y=b+\dfrac{1}{b}$ then prove that

$$xy+\sqrt{(x^2-1)(y^2-1)}=\frac{1}{2}\left(ab+\frac{1}{ab}\right)$$

68. If $\sqrt{x}+\sqrt{a-x}=\sqrt{y}+\sqrt{a-y}=\sqrt{z}+\sqrt{a-z}$

then prove that $(x-y)(y-z)(z-x)=0$

69. If $x=2+\sqrt[3]{2}+\sqrt[3]{4}$ then prove that $x^3-6x^2+6x+7=0$

70. If $\left(x+\sqrt{x^2-bc}\right)\left(y+\sqrt{y^2-ac}\right)\left(z+\sqrt{z^2-ab}\right)$ then $=\left(x-\sqrt{x^2-bc}\right)\left(y-\sqrt{y^2-ac}\right)\left(z-\sqrt{z^2-ab}\right)$

show that each side $=\pm abc$

71. If $\sqrt[3]{a+\sqrt{b}}=x+\sqrt{y}$ then prove that $\sqrt[3]{a-\sqrt{b}}=x-\sqrt{y}$

72. If $\sqrt{\left(x-\sqrt{a^2-b^2}\right)^2+y^2}+\sqrt{\left(x+\sqrt{a^2-b^2}\right)^2+y^2}=2a$

then prove that $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

73. If $x=\dfrac{q-\sqrt{p^2-4q}}{q+\sqrt{p^2-4q}}$ then show that

$\left(q^2-p^2+4q\right)\left(x^2+1\right)-2\left(p^2+q^2-4q\right)x=0$

74. If x is a positive integer then find x from the following equation

$$\frac{\sqrt{5+2\sqrt{6}}-\sqrt{5-2\sqrt{6}}}{\sqrt{5+2\sqrt{6}}+\sqrt{5-2\sqrt{6}}}=\sqrt{\frac{x}{16}}$$

75. Solve $\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}=2$

76. Solve $4x^2+6x+\sqrt{2x^2+3x+4}=13$

ANSWERS

(1) 0

(2) $\dfrac{17}{7}$

(3) 46

(4) 0

(5) $\sqrt{7}+\sqrt{17}$

(6) $2+\sqrt{3}$

(7) $3-\sqrt{5}$

(8) $\sqrt{\dfrac{7}{2}}+\sqrt{\dfrac{3}{2}}$

(9) $\sqrt{\dfrac{7}{3}}-\sqrt{\dfrac{2}{3}}$

(10) $\sqrt{3}+\dfrac{1}{2}$

(11) 25

(12) -4

(13) 1,000

(14) 3

(15) $30\sqrt{3}$

(16) $x=1\,;\,y=-\dfrac{1}{2}$

(17) 0

(18) $6\sqrt{14}$

(19) 0

(20) $\sqrt{2}-1$

(21) $\sqrt{2}$

(22) $\frac{1}{2}$

(23) $\sqrt{2}+\sqrt{3}+\sqrt{5}$

(24) $\sqrt{\frac{3x}{2}}+\sqrt{\frac{x+4y}{2}}$

(25) $\sqrt{3}-\sqrt{6}+\sqrt{8}$

(26) $\sqrt{2}-\sqrt{5}+\sqrt{6}$

(27) $\sqrt{\frac{x+y}{2}}+\sqrt{\frac{x-y}{2}}$

(28) $\left(\sqrt{x+y}\right)+\left(\sqrt{x-y}\right)$

(29) $\frac{1}{2}\left(\sqrt{5}+1\right)$

(30) $\sqrt{3}-1$

(31) $\frac{1}{2}\left(\sqrt{5}+1\right)$

(32) $\sqrt{3}+1$

(33) $\frac{1}{2}\left(\sqrt{3}-1\right)$

(34) $\frac{1}{\sqrt{2}}\left(\sqrt{3}-1\right)$

(35) $\sqrt{5}+\sqrt{6}$

(36) $\sqrt{6}-\sqrt{3}$

(37) $\sqrt{5}-\sqrt{2}$

(38) $\sqrt{7}+\sqrt{3}$

(39) $\sqrt{5}+\sqrt{1}$

(40) $\sqrt{3}+\sqrt{5}$

(50) 18

(51) $x = 10$ and $y = 2$

(52) $-\sqrt{6}$

(55) 1

(56) 2

(58) $\pm\frac{1}{\sqrt{2}}\left(\sqrt{1+a+a^2}+\sqrt{1-a+a^2}\right)$

(59) $\frac{1}{3}\left(12+56\sqrt{10}\right)$

(61) 1

(62) $\frac{1}{5}\sqrt{15}$

(63) $\frac{63}{61}$

(65) 4

(74) $\frac{32}{3}$

(75) $x=\pm\frac{2}{\sqrt{5}}$

(76) $x=1,\ -\frac{5}{2}$

2

Indices and Logarithm

LEARNING OBJECTIVES

This chapter will enable you to learn the concepts and application of:

- Indices
- Logarighm
- Inter-relationship between indices and logarithm
- Application of indices and logarithm

INTRODUCTION

We know that when two or more quantities are multiplied together the result is called the continuous product. Each quantity is called factor of the product. But when a product consists of the same factor repeated several times, it is called a **power of the factor.**

For example

1. In the product $x \times y \times z$, x, y, and z are called the factors of the product
2. The product $m \times m \times m \times m = m^4$ is called the fourth power of m.

EXPONENT (INDEX) OF THE POWER

If the number is multiplied by itself n times where n being a positive integer, i.e. in continuous product $a \times a \times \ldots n$ times $= a^n$ is called nth power of a where it is said to be based and n is said to be exponent or power of a.

For example

$$2 \times 2 \times 2 \times 2 \times 2 = 2^5$$
$$(-3) \times (-3) \times (-3) \times (-3) = (-3)^4$$

Laws of Indices

The eight laws of indices are as follows:

1. $a^m \times a^n = a^{m+n}$

2. $\left(a^m\right)^n = a^{mn}$
3. $\left(ab\right)^m = a^m b^m$
4. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
5. $$\begin{tikzcd} & a^{m-n} \quad m>n \\ \frac{a^m}{a^n} = \arrow[ur] \arrow[r] \arrow[dr] & 1 \quad m=n \\ & \frac{1}{a^{n-m}} \quad m<n \end{tikzcd}$$
6. $a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$
7. $a^0 = 1$ (anything raised to the power 0 is equal to unity); where $a \neq 0$
8. $0^a = 0$ (0 raised to any power is equal to zero)

DEFINITION OF LOGARITHM

The logarithm of a number to a given base is the index or the power of which the base must be raised to obtain that number. In other words to we can say that If $a^m = b$ where $b > 0$ (where $a \neq 1$, $a > 0$) then the exponent m is said to be logarithm of b to the base a. We represent it as follows:

$$a^m = b \qquad \text{(Exponent form)} \qquad (1)$$

$$m = \log_a^b \qquad \text{(Logarithm form)} \qquad (2)$$

$$\therefore \quad a^m = b \Leftrightarrow \quad m = \log_a^b$$

Also the number b in (1) and (2) is called the antilogarithm of m to the base a it can be expressed as follows:

$$b = \text{antilog}(m) \ \text{(Antilog form)} \qquad (3)$$

Following illustrations give relationship between indices and logarithms.

ILLUSTRATIONS

1. $2^4 = 16$ (exponential form) then $4 = \log_2^{16}$ (logarithm form)
2. $3^{-2} = \frac{1}{9}$ (exponential form) then $-2 = \log_3^{1/9}$ (logarithm form)

Laws of Logarithms

1. $\log_a^{mn} = \log_a^m + \log_a^n$ (Product formula)
2. $\log_a^{m/n} = \log_a^m - \log_a^n$ (Division formula)
3. $\log_a^{m^n} = n\log_a^m$ (Power formula)
4. $\log_a^m = \dfrac{\log_a^m}{\log_a^n}$ (Change base)
5. $\log_n^m \times \log_m^n = 1 \Rightarrow \left(\begin{array}{l} \log_n^m = \dfrac{1}{\log_m^n} \\ \log_m^n = \dfrac{1}{\log_n^m} \end{array} \right)$ (Reciprocal relation of logarithm)
6. $e^{\log_e^x} = x$ where $x > 0$
7. If $m > 1$ and $a > 1$ then $\log_a^m > 0$
8. $\log_a^1 = 0$ (Logarithm of 1 to any base is equal to zero)
9. $\log_m^m = 1$ (Logarithm of any number to the same base is 1.)
10. If $0 < a < 1$ and $0 < m < 1$ then $\log_a^m > 0$
11. If $0 < a < 1$ and $m > 1$ then $\log_a^m < 0$
12. If $a > 1$ and $0 < m < 1$ then $\log_a^m < 0$
13. $\log_{a^n}^m = \dfrac{1}{n}\log_a^m$
14. $\log_{a^y}^{m^x} = \dfrac{x}{y}\log_a^m$
15. Systems of logarithm

There are two systems of logarithm:

1. **Common logarithm:** The logarithm to the base 10 is called the common logarithms. **For example**

$$\log_{10}^{10} = 1; \quad \log_{10}^{0.0001} = -4$$

2. **Natural Logarithm:** The logarithm to the base e ($e = 2.718281...$) is called the natural logarithm. It is used in theoretical everywhere in logarithm.
   For Example

   $\log_e^e = 1$

   i.e. $e = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + ... = 2.718281...\ (\approx 2.72)$

   Here e is called Napier's constant.

CHARACTERISTIC AND MANTISSA

In common logarithms, the base is 10. As most of the numbers are not exact power of 10, so the numbers or logarithm of such numbers consist of two parts:

1. Integral part
2. Decimal or fractional part

Integral part of a logarithm of a number is called the characteristic and the positive decimal part is called the mantissa.

Standard Form

$x = y \times 10^z$

Taking logarithm to the base 10 on both sides

$\log x = \log^y + \log_{10}^{10^z}$

$\therefore \log x = \log y + z \log_{10}^{10}$

$\therefore \log x = z + \log^y$

Here x is a characteristic and $\log^y$ is the mantissa of $\log x$.

Method to Find log *x*:

1. Put the given number x in the standard form, i.e. $x = y \times 10^z$ where z is an integer – positive or negative or zero – and $0 \leq y < 10$.
2. Read the characteristic z of $\log x$ from the expression (exponent of 10) of (1).
3. Find the value of $\log^y$ from the log tables.
4. Write $\log x = z + \log y$.

Rules to Find Characteristic

There are two rules to find the characteristic of the logarithm of a number by inspection:

Rule 1

The characteristic of the logarithm of a number greater than unity is positive and is less than the number of digits in the integral part of the number.

Rule 2

The characteristic of a number less than 1 is negative and numerically one more than the number of zeros immediately after the decimal point.

e.g. $10^3 = 1000$ $\therefore \log_{10}^{1000} = 3$

$10^4 = 10{,}000$ $\therefore \log_{10}^{10{,}000} = 4$

$$0.001 = 10^{-3} \qquad \therefore \log_{10}^{0.001} = -3$$

$$0.0001 = 10^{-4} \qquad \therefore \log_{10}^{0.0001} = -4$$

Rule to Find Mantissa

The mantissa of logarithm of all numbers consisting of the same number of digits in the same order is the same. For example,

(1)
$$\begin{aligned} \log 732.4 &= \log(100 \times 7.324) \\ &= \log^{100} + \log 7.324 \\ &= \log^{10^2} + \log 7.324 \\ &= 2 + \log 7.324 \end{aligned}$$

(2)
$$\begin{aligned} 0.01234 &= \log \frac{1.324}{100} \\ &= \log 1.324 - \log^{100} \\ &= \log 1.324 - \log^{10^2} \\ &= \log 1.324 - 2 \end{aligned}$$

ILLUSTRATIONS

Illustration 1 Solve $2x^{1/3} + 8x^{-1/3} - 8 = 0$

Solution

$$2x^{1/3} + \frac{8}{x^{1/3}} - 8 = 0$$

Let $x^{1/3} = m$

$$\therefore 2m^2 - 8m + 8 = 0$$

$$\therefore m^2 - 4m + 4 = 0$$

$$\therefore (m-2)^2 = 0$$

$$\therefore m - 2 = 0$$

$$\therefore m = 2$$

$$\therefore x^{1/3} = 2$$

$$\therefore \left(x^{1/3}\right)^3 = 2^3 \qquad \text{(Cubing both sides)}$$

$$\therefore x = 8$$

Illustration 2 Prove that $\dfrac{\left(x+y^{-1}\right)^m \left(x^{-1}+y\right)^n}{\left(1+xy\right)^n \left(1+x^{-1}y^{-1}\right)^m} = x^{m-n}$

Solution

$$\frac{\left(x+y^{-1}\right)^m \left(x^{-1}+y\right)^n}{\left(1+xy\right)^n \left(1+x^{-1}y^{-1}\right)^m}$$

$$= \frac{\left[\left(xy+1\right)/y\right]^m \left[\left(1+xy\right)/x\right]^n}{\left(1+xy\right)^n \left[\left(xy+1\right)/xy\right]^m}$$

$$= \frac{\left(xy+1\right)^m}{y^m} \; \frac{\left(xy+1\right)^n}{x^n} \; \frac{1}{\left(1+xy\right)^n} \; \frac{x^m y^m}{\left(xy+1\right)^m} = \frac{x^m}{x^n}$$

$$= x^{m-n}$$

Illustration 3 If $a^{1/3} + b^{1/3} + c^{1/3} = 0$ then prove that $\left(a+b+c\right)^3 = 27abc$

Solution

$\left(\text{Hint: } a + b + c = 0 \text{ then } a^3 + b^3 + c^3 = 3abc\right)$

$a^{1/3} + b^{1/3} + c^{1/3} = 0$

$\Rightarrow \left(a^{1/3}\right)^3 + \left(b^{1/3}\right)^3 + \left(c^{1/3}\right)^3 = 3a^{1/3} b^{1/3} c^{1/3}$

$\Rightarrow a + b + c = 3a^{1/3} b^{1/3} c^{1/3}$

$\Rightarrow \left(a+b+c\right)^3 = \left(3a^{1/3} b^{1/3} c^{1/3}\right)^3$ (Cubing both sides)

$\Rightarrow \left(a+b+c\right)^3 = 27abc$

Illustration 4 If $x > 0$ then prove that

$$\frac{1}{1+x^{a-b}+x^{a-c}} + \frac{1}{1+x^{b-c}+x^{b-a}} + \frac{1}{1+x^{c-a}+x^{c-b}} = 1$$

Solution

$$\frac{1}{1+x^{a-b}+x^{a-c}} + \frac{1}{1+x^{b-c}+x^{b-a}} + \frac{1}{1+x^{c-a}+x^{c-b}}$$

$$= \frac{1}{1+\left(x^{-b}/x^{-a}\right)+\left(x^{-c}/x^{-a}\right)} + \frac{1}{1+\left(x^{-c}/x^{-b}\right)+\left(x^{-a}/x^{-b}\right)}$$

$$+ \frac{1}{1+\left(x^{-a}/x^{-c}\right)+\left(x^{-b}/x^{-c}\right)}$$

$$= \frac{x^{-a}}{x^{-a} + x^{-b} + x^{-c}} + \frac{x^{-b}}{x^{-b} + x^{-a} + x^{-c}} + \frac{x^{-c}}{x^{-c} + x^{-b} + x^{-c}}$$

$$= \frac{x^{-a} + x^{-b} + x^{-c}}{x^{-a} + x^{-b} + x^{-c}}$$

$$= 1$$

Illustration 5 If $e^x = y + \sqrt{1 + y^2}$ then prove that $y = \frac{1}{2}\left(e^x - e^{-x}\right)$

Solution

L.H.S. = $e^x = y + \sqrt{1 + y^2}$

$$\therefore e^x - y = \sqrt{1 + y^2}$$

$$\therefore \left(e^x - y\right)^2 = \left(\sqrt{1 + y^2}\right)^2 \qquad \text{(Squaring both sides)}$$

$$\therefore e^{2x} - 2ye^x + y^2 = 1 + y^2$$

$$\therefore e^{2x} - 2ye^x = 1$$

$$\therefore 2ye^x = e^{2x} - 1$$

$$\therefore y = \frac{1}{2}\left(\frac{e^{2x-1}}{e^x}\right)$$

$$\therefore y = \frac{1}{2}\left(e^x - e^{-x}\right) = \text{R.H.S}$$

Illustration 6 If $x = 5^{1/3} + 5^{-1/3}$ then prove that $5x^3 - 15x = 26$

Solution

Here $x = 5^{1/3} + 5^{-1/3}$

$$\therefore x^3 = \left(5^{1/3} + 5^{-1/3}\right)^3$$

$$\therefore x^3 = 5 + 5^{-1} + 3 \times 5^{1/3} \times 5^{-1/3}\left(5^{1/3} + 5^{-1/3}\right)$$

$$\therefore x^3 = 5 + \frac{1}{5} + 3x$$

$$\therefore x^3 - 3x = 5 + \frac{1}{5}$$

$$\therefore x^3 - 3x = \frac{25 + 1}{5}$$

$$\therefore 5x^3 - 15x = 26$$

Illustration 7 Simplify $\left(\dfrac{m^x}{m^y}\right)^{x+y} \times \left(\dfrac{m^y}{m^z}\right)^{y+z} \div 3\left(m^x\, m^z\right)^{x-z}$

Solution

$$\left(\frac{m^x}{m^y}\right)^{x+y} \times \left(\frac{m^y}{m^z}\right)^{y+z} \div 3\left(m^x\, m^z\right)^{x-z}$$

$$= \frac{\left(m^{x-y}\right)^{x+y} \times \left(m^{y-z}\right)^{y+z}}{3\left(m^{x+z}\right)^{x-z}}$$

$$= \frac{\left(m^{x^2-y^2}\right)\left(m^{y^2-z^2}\right)}{3m^{x^2-z^2}}$$

$$= \frac{\left(m^{x^2-y^2}\right)\left(m^{y^2-z^2}\right)\left(m^{z^2-x^2}\right)}{3}$$

$$= \frac{m^{x^2-y^2+y^2-z^2+z^2-x^2}}{3}$$

$$= \frac{m^0}{3} = \frac{1}{3}$$

Illustration 8 If $a = x\,y^{m-1}$; $b = x\,y^{n-1}$; $c = x\,y^{p-1}$ then prove that $a^{np} \times b^{pm} \times c^{mn} = 1$

Solution

$$an - p \times bp - m \times cm - n$$

$$= \left(x\,y^{m-1}\right)^{n-p}\left(x\,y^{n-1}\right)^{p-m}\left(x\,y^{p-1}\right)^{m-n}$$

$$\left(x^{n-p}\,x^{p-m}\,x^{m-n}\right)\left[\left(y^{m-1}\right)^{n-p}\left(y^{n-1}\right)^{p-m}\left(y^{p-1}\right)^{m-n}\right]$$

$$= \left(x^{n-p+p-m+m-n}\right)\left(y^{mn-mp-n+p} \times y^{np-mn-p+m} \times y^{mp-np-m+n}\right)$$

$$= \left(x^0\right)\left(y^{mn-mp-n+p+np-mn-p+m+mp-np-m+n}\right)$$

$$= (1)\left(y^0\right)$$

$$= 1 \times 1 = 1$$

Illustration 9 Simplify $\left(\dfrac{x^{ab}}{x^{a^2+b^2}}\right)^{a+b} \left(\dfrac{x^{b^2+c^2}}{x^{bc}}\right)^{b+c} \left(\dfrac{x^{ac}}{x^{c^2+a^2}}\right)^{c+a}$

Solution

$$= \left(\frac{x^{ab}}{x^{a^2+b^2}}\right)^{a+b} \left(\frac{x^{b^2+c^2}}{x^{bc}}\right)^{b+c} \left(\frac{x^{ac}}{x^{c^2+a^2}}\right)^{c+a}$$

$$= \left(\frac{1}{x^{a^2+b^2-ab}}\right)^{a+b} \left(x^{b^2+c^2-bc}\right) \left(\frac{1}{x^{c^2+a^2-ac}}\right)^{c+a}$$

$$= \frac{1}{x^{a^3 + b^3}} \times x^{b^3 + c^3} \times \frac{1}{x^{c^3 + a^3}}$$

$$= x^{-a^3 - b^3} \times x^{b^3 + c^3} \times x^{c^3 - a^3}$$

$$= x^{-a^3 - b^3 + b^3 + c^3 - c^3 - a^3}$$

$$= x^{-2a^3}$$

Illustration 10 Simplify $\frac{2^{x+3} \times 3^{2x-y} \times 5^{x+y+3} \times 6^{y+1}}{6^{x+1} \times 10^{y+3} \times 15^{x}}$

Solution

$$= \frac{2^{x+3} \times 3^{2x-y} \times 5^{x+y+3} \times 6^{y+1}}{6^{x+1} \times 10^{y+3} \times 15^{x}}$$

$$= \frac{2^{x+3} \times 3^{2x-y} \times 5^{x+y+3} \times 2^{y+1} \times 3^{y+1}}{2^{x+1} \times 3^{x+1} \times 2^{y+3} \times 5^{y+3} \times 3^{x} \times 5^{x}}$$

$$= \frac{2^{x+3+y+1} \times 3^{2x-y+y+1} \times 5^{x+y+3}}{2^{x+1+y+3} \times 3^{x+1+x} \times 5^{x+y+3}}$$

$$= 2^{x+y+4-x-y-4} \times 3^{2x+1-2x-1} \times 5^{x+y+3-x-y-3}$$

$$= 2^{0} \times 3^{0} \times 5^{0}$$

$$= 1$$

Illustration 11 Simplify $\log_{11}^{121\sqrt{14641}/\sqrt[3]{1331}}$

Solution

$$\log_{11}^{121\sqrt{14641}/\sqrt[3]{1331}} = \log_{11}^{11^2\sqrt{11^4}/\sqrt[3]{11^3}}$$

$$= \log_{11}^{\left[\left(11^2 11^2\right)/11\right]} = \log_{11}^{11^3} = 3\log_{11}^{11} = 3$$

Illustration 12 Simplify $\log_{2}^{16\sqrt{8}} + \log_{5}^{\sqrt[4]{25}/625}$

Solution

$$\log_{2}^{16\sqrt{8}} + \log_{5}^{\sqrt[4]{25}/625}$$

$$= \log_{2}^{\left(2^4 \times 2^{3/2}\right)} + \log_{5}^{\left(5^2\right)^{1/4}/5^4}$$

$$= \log_{2}^{2^{11/2}} + \log_{5}^{5^{-7/2}}$$

$$= \frac{11}{2} - \frac{7}{2} = 2$$

Illustration 13 Simplify $\log_3^{\sqrt[4]{729\sqrt[3]{\frac{1}{9}\,(27)^{-4/3}}}}$

Solution

$$\log_3^{\sqrt[4]{729\sqrt[3]{\frac{1}{9}\,(27)^{-4/3}}}}$$

$$= \log_3^{\sqrt[4]{3^6\sqrt[3]{(3)^{-2}\left(3^3\right)^{-4/3}}}}$$

$$= \log_3^{\sqrt[4]{3^6\sqrt[3]{3^{-2}\times 3^{-4}}}}$$

$$= \log_3^{\sqrt[4]{3^6\sqrt[3]{3^{-6}}}}$$

$$= \log_3^{\sqrt[4]{3^6\times 3^{-2}}}$$

$$= \log_3^{\sqrt[4]{3^4}}$$

$$= \log_3^{3}$$

$$= 1$$

Illustration 14 If $\log\sqrt{11+4\sqrt{7}} = \log\left(2+x\right)$ then prove that $x=\sqrt{7}$

Solution

$$\log\sqrt{11+4\sqrt{7}} = \log(2+x)$$

$$\therefore \log\sqrt{7+4+4\sqrt{7}} = \log(2+x)$$

$$\therefore \log\sqrt{7+4+2\times 2\sqrt{7}} = \log(2+x)$$

$$\therefore \log\sqrt{(\sqrt{7})^2+(2)^2+2\times 2\sqrt{7}} = \log(2+x)$$

$$\therefore \log\sqrt{\left(\sqrt{7}+2\right)^2} = \log(2+x)$$

$$\therefore \log\left(\sqrt{7}+2\right) = \log(2+x)$$

$$\therefore \sqrt{7}+2 = x+2$$

$$\therefore x = \sqrt{7}$$

Illustration 15 If $\log_{27}\left(\log_4^{\sqrt{n^9}}\right) = \frac{2}{3}$ then prove that value of $n = 16$

Solution

$$\log_{27}\left(\log_4^{\sqrt{2^9}}\right) = \frac{2}{3}$$

$$\therefore \log_4^{\sqrt{n^9}} = (27)^{2/3}$$

$$\therefore \log_4^{\sqrt{n^9}} = \left(3^3\right)^{2/3}$$

$$\therefore \log_4^{\sqrt{n^9}} = 9$$

$$\therefore \sqrt{n^9} = 4^9$$

$$\therefore n^9 = \left(4^9\right)^2 \qquad \text{(Squaring both sides)}$$

$$\therefore n^9 = \left(4^2\right)^9$$

$$\therefore n^9 = (16)^9$$

$$n = 16$$

Illustration 16 If $\log_4\left(\log_3^x\right) = \frac{1}{2}$ then prove that $x = 4$

Solution

$$\log_4\left(\log_3^x\right) = \frac{1}{2}$$

$$\therefore \log_3^x = (4)^{1/2}$$

$$\therefore \log_3^x = 2$$

$$x = 3^2 = 9$$

Illustration 17 Prove that $\log_6^{216\sqrt{6}} = \frac{7}{2}$

Solution

$$\begin{aligned}
&\log_6^{216\sqrt{6}} \\
&= \log_6^{6^3 \times 6^{1/2}} \\
&= \log_6^{6^{3+1/2}} \\
&= \log_6^{6^{7/2}} \\
&= \frac{7}{2}\log_6^6 \\
&= \frac{7}{2}
\end{aligned}$$

Illustration 18 Prove that "the sum of n terms
$\log_a^b + \log_{a^2}^{b^2} + \log_{a^3}^{b^3} + \ldots + \log_{a^n}^{b^n} = n\log_a^b$"

Solution

$$\begin{aligned}
&\log_a^b + \log_{a^2}^{b^2} + \log_{a^3}^{b^3} + \ldots + \log_{a^n}^{b^n} \\
&= \frac{\log^b}{\log^a} + \frac{\log^{b^2}}{\log^{a^2}} + \frac{\log^{b^3}}{\log^{a^3}} + \ldots + \frac{\log^{b^n}}{\log^{a^n}} \\
&= \frac{\log^b}{\log^a} + \frac{2\log^b}{2\log^a} + \frac{3\log^b}{3\log^a} + \ldots + \frac{n\log^b}{n\log^a} \\
&= \frac{\log^b}{\log^a} + \frac{\log^b}{\log^a} + \ldots + \frac{n\log^b}{n\log^a} \\
&= \log_a^b + \log_a^b + \ldots \log_a^b \\
&= n\log_a^b \\
&= \log_a^{b^n}
\end{aligned}$$

Illustration 19 If $\log_{10}^{t} + \log_{10}^{(t-3)} = 1$ then show that $t = 5$

Solution

$$\log_{10}^{t} + \log_{10}^{(t-3)} = 1$$
$$\therefore \log_{10}^{t(t-3)} = \log_{10}^{10}$$
$$\therefore t(t-3) = 10$$
$$\therefore t^2 - 3t - 10 = 0$$
$$\therefore (t-5)(t+2) = 0$$
$$\therefore \; t - 5 = 0 \text{ or } t + 2 = 0$$
$$t = 5 \text{ or } t = -2$$

but $t = -2$ is not applicable in logarithm

$\therefore$ $t = 5$ is the required solution

Illustration 20 Prove that $\log_5^5 \log_4^9 \log_3^2 = 1$

Solution

$$\log_5^5 \log_4^9 \log_3^2$$
$$= 1 \frac{\log^9}{\log^4} \frac{\log^2}{\log^3}$$
$$= \frac{\log^{3^2}}{\log^{2^2}} \frac{\log^2}{\log^3}$$
$$= \frac{2\log^3}{2\log^2} \frac{\log^2}{\log^3}$$
$$= 1$$

Illustration 21 Find the value of $\log_2\left[\log_2\left\{\log_3\left(\log_3^{27^3}\right)\right\}\right]$

Solution

$$\log_2\left[\log_2\left\{\log_3\left(\log_3^{27^3}\right)\right\}\right]$$
$$= \log_2\left[\log_2\left\{\log_3\left(\log_3^{3^9}\right)\right\}\right]$$
$$= \log_2\left[\log_2\left\{\log_3\left(9\log_3^3\right)\right\}\right]$$
$$= \log_2\left[\log_2\left(\log_3^9\right)\right]$$
$$= \log_2\left[\log_2\left(\log_3^{3^2}\right)\right]$$
$$= \log_2\left[\log_2\left(2\log_3^3\right)\right]$$
$$= \log_2\left(\log_2^2\right)$$
$$= \log_2^1 = 0$$

Illustration 22 If $x^2 + y^2 = 6xy$ then prove that $2\log(x+y) = \log x + \log y + 3\log 2$

Solution

$$x^2 + y^2 = 6xy$$
$$\Rightarrow x^2 + y^2 + 2xy = 8xy$$
$$\Rightarrow (x+y)^2 = 8xy$$

(Taking log both sides)

$$\Rightarrow \log(x+y)^2 = \log 8xy$$
$$\Rightarrow 2\log(x+y) = \log 8 + \log x + \log y$$
$$\Rightarrow 2\log(x+y) = \log x + \log y + \log 2^3$$
$$\Rightarrow 2\log(x+y) = \log x + \log y + 3\log 2$$

Illustration 23 If a series of numbers be in geometrical progression (G.P) show that their logarithms are in arithmetical progression (A.P).

Solution

Suppose the numbers in G.P. are $a, ar, ar^2, ar^3, \ldots, ar^{n-1}$

$$\therefore \log a = \log a$$
$$\therefore \log ar = \log a + \log r$$
$$\therefore \log ar^2 = \log a + 2\log r$$
$$\therefore \log\left(ar^{n-1}\right) = \log a + (n-1)\log r$$

$\therefore$ From above equations we can say that $\log a$, $\log(ar)$, $\log(ar^2) \ldots \log(ar^{n-1})$ are in A.P. because difference between consecutive terms is $\log r$.

Illustration 24 If $2\log_{10}^{(x^2y)} = 3\log x - \log y$ where x and y are both positive then express the value of y in terms of x

Solution

$$2\log_{10}^{(x^2y)} = 3\log x - \log y$$
$$\therefore 2\log_{10}^{x^2} + 2\log_{10}^{y} = 3\log_{10}^{x} - \log_{10}^{y}$$
$$\therefore 4\log_{10}^{x} - \log_{10}^{x} + 3\log_{10}^{y} = 3$$
$$\therefore 3\left(\log_{10}^{x} + \log_{10}^{y}\right) = 3$$
$$\therefore \log_{10}^{xy} = 1$$
$$\therefore xy = 10^1 = 10$$
$$\therefore y = \frac{10}{x} = 10x^{-1}$$

Illustration 25 By usual notation prove that

$$\log_2^{75/16} - 2\log_2\left[\frac{\left[\sqrt[4]{(25/81)^3}\right]\left[\sqrt[3]{(25/81)}\right]}{\sqrt[12]{(25/81)^7}}\right] + \frac{1}{3}\log_2^{\left(2^{15}\times 3^{-15}\right)} = 1$$

Solution

L.H.S.

$$= \log_2^{(75/16)} - 2\log_2^{(25/81)^{3/4\ +\ 1/3\ -\ 7/12}} + \frac{1}{3}\log_2^{\left(2^{15}\times 3^{-15}\right)}$$

$$= \log_2^{\left[\left(5^2\times 3\right)/2^4\right]} - 2\log_2^{(25/81)^{1/2}} + \frac{1}{3}\log_2^{\left(2^{15}\times 3^{-15}\right)}$$

$$= 2\log_2^5 + \log_2^3 - 4\log_2^2 - \log_2^{25} + \log_2^{81} + 5\log_2^2 - 5\log_2^3$$

$$= 2\log_2^5 + \log_2^3 - 4 - 2\log_2^5 + 4\log_2^3 + 5 - 5\log_2^3$$

$= 1$ (R.H.S.)

Illustration 26 Show that $\log_3^{\sqrt{3\sqrt{3\sqrt{3\ldots\infty}}}} = 1$

Solution

Suppose $x = \sqrt{3\sqrt{3\sqrt{3...\infty}}}$

Now squaring both sides

$x^2 = 3\sqrt{3\sqrt{3\sqrt{3...\infty}}}$

$\therefore x^2 = 3x$

$\therefore x^2 - 3x = 0$

$\therefore x(x-3) = 0$

$\therefore x = 0$ or $x = 3$ $\quad (\because x \neq 0)$

$\therefore \log_3^{\sqrt{3\sqrt{3\sqrt{3\ldots\infty}}}} = 1$

Illustration 27 The first and last term of a G.P. are a and k_1 respectively. If again the number of terms be n then prove that

$$n = 1 + \left(\frac{\log k_1 - \log a}{\log r}\right)$$

Solution

Here first term of G.P. is a and last term is $k_1 = ar^{n-1}$

Taking log both sides

$\therefore \log k_1 = \log a + \log r^{n-1}$

$$\therefore \log k_1 = \log a + (n-1)\log r$$
$$\therefore (n-1)\log r = \log k_1 - \log a$$
$$\therefore (n-1) = \left(\frac{\log k_1 - \log a}{\log r}\right)$$
$$\therefore n = 1 + \left(\frac{\log k_1 - \log a}{\log r}\right)$$

Illustration 28 Solve $3^{5x} \times 9^{4x-2} = \dfrac{(27)^{3x-8}}{(81)^{-3x}}$

Solution

$$3^{5x} \times 9^{4x-2} = \frac{(27)^{3x-8}}{(81)^{-3x}}$$
$$\Rightarrow 3^{5x} \times \left(3^2\right)^{4x-2} = \frac{\left(3^3\right)^{3x-8}}{\left(3^4\right)^{-3x}}$$
$$\Rightarrow 3^{5x} \times (3)^{8x-4} = \frac{(3)^{9x-24}}{(3)^{-12x}}$$
$$\Rightarrow 3^{5x+8x-4} = 3^{9x-24+12x}$$
$$\Rightarrow 3^{13x-4} = 3^{21x-24}$$
$$\Rightarrow 21x - 24 = 13x - 4$$
$$\Rightarrow 21x - 13x = 24 - 4$$
$$\Rightarrow 8x = 20$$
$$\Rightarrow x = \frac{20}{8}$$
$$\Rightarrow x = \frac{5}{2}$$

Illustration 29 Solve $3^{2x+1} = \left(\sqrt{3}\right)^{x+3}$

Solution

$$3^{2x+1} = \left(\sqrt{3}\right)^{x+3}$$
$$\Rightarrow 3^{2x+1} = \left(3^{1/2}\right)^{x+3}$$
$$\Rightarrow 3^{2x+1} = (3)^{(x+3)/2}$$
$$\Rightarrow 2x + 1 = \frac{x+3}{2}$$
$$\Rightarrow 4x + 2 = x + 3$$
$$\Rightarrow 4x - x = 3 - 2$$
$$\Rightarrow 3x = 1$$
$$\Rightarrow x = \frac{1}{3}$$

ANALYTICAL EXERCISES

1. Prove that $\dfrac{5\left(300\right)^{1/2} - 3^{1/2}}{40 \times 3^{1/2} + 3^{5/2}} = 1$

2. Prove that $\dfrac{3^2 \times 10^4 \times 5^2}{2^3 \times \left(3^{-1}\right)^{-1} \times 5^5} = 30$

3. Prove that $\left(\dfrac{1}{1 + x^{-1}}\right)^{-1} - \left(\dfrac{x - 1}{1 - x^{-1}}\right)^{-1} = 1$

4. If $x, y, z > 0$ and $\sqrt[3]{\dfrac{x}{y}} + \sqrt[3]{\dfrac{y}{z}} + \sqrt[3]{\dfrac{z}{x}} = 0$ then prove that $xy^2 + yz^2 + zx^2 = 3$

5. Prove that $\dfrac{a^2 - bc}{x^{(a + b)(a + c)}} \dfrac{b^2 - ac}{x^{(b + c)(b + a)}} \dfrac{c^2 - ab}{x^{(b + c)(c + a)}} = 1$

6. If $5^{x+1} + 5^{x-1} = 26$ then find the value of x

7. If $\left(\begin{array}{l} 9^x - 5^{2y} = 56 \\ 3^x - 5^y \;\, = 4 \end{array}\right)$ then find the value of x and y

8. $x^{1/l} = y^{1/m} = z^{1/n}$ and $xyz = 1$ then find the value of $l + m + n$

9. If $2^x + 2^{x+1} + 2^{x+2} = 56$ then find the value of x

10. If $5^{x+1} + 5^{2-x} = 126$ then find the value of x

11. Prove that $\dfrac{\left(x^{2/3} \div x^{1/3}\right)^3 \left(x^{5/4} \times x^{-1/2}\right)^{4/3}}{\left(x^{1/3} \times x^{-1/9}\right)^5} = x^{8/9}$

12. Prove that The value of $\left\{\dfrac{1}{5}\left[\dfrac{1}{5}\left(\dfrac{1}{5}\right)^{-1/2}\right]^{-1/2}\right\}^{-1/2} \times \left(\dfrac{1}{5}\right)^{-1/8} = \sqrt{5}$

13. If $\left(x\sqrt{x}\right)^x = \left(x\right)^{x\sqrt{x}}$ then find the value of x

14. If $\sqrt{3^{m^2} \div \left(3^m\right)^2} = 81$ then find the value of m

15. Prove that $\left(1 + x^{a-b}\right)^{-1} + \left(1 + x^{b-a}\right)^{-1} = 1$

16. Prove that $\left(\dfrac{x^a}{x^b}\right)^{a^2 + ab + b^2} \left(\dfrac{x^b}{x^c}\right)^{b^2 + bc + c^2} \left(\dfrac{x^c}{x^a}\right)^{c^2 + ca + a^2} = 1$

17. If $a = \sqrt[3]{4} + \dfrac{1}{\sqrt[3]{4}}$ and $b = \sqrt[4]{3} + \dfrac{1}{\sqrt[4]{3}}$ then prove that $\left(a^2 + b^2\right)^2 = \dfrac{64}{3}$

18. If $f(x+y)=f(xy); \forall\ x, y \in R$ and $f(2010)=2010$ then the value of $f(-2010)$

19. If $\begin{pmatrix} 3^{x+1}+4^{y}=4 \\ 3^{x}+4^{y+1}=5 \end{pmatrix}$ then find the value of x and y

20. If $a^2 \neq 1$ then prove that $\dfrac{a^2}{a^{2/3}-1}+\dfrac{1}{a^{2/3}+1}-\dfrac{a^{4/3}}{a^{2/3}+1}-\dfrac{1}{a^{2/3}-1}=a^{4/3}+2$

21. If $\sqrt[a]{x^{b-c}}\cdot\sqrt[b]{x^{c-a}}\cdot\sqrt[c]{x^{a-b}}=1$ then prove that among a, b and c any two are equal.

22. Prove that $\dfrac{9^{x}+1+9^{-x}}{3^{x}+3^{-x}}+\dfrac{9^{x}-1+9^{-x}}{3^{x}-3^{-x}}=\dfrac{2\left(3^{5x}\right)}{3^{4x}-1}$

23. Prove that $\dfrac{\left(x^{2}-y^{-2}\right)^{x}\left(x-y^{-1}\right)^{y-x}}{\left(y^{2}-x^{-2}\right)^{y}\left(y-x^{-1}\right)^{x-y}}=\left(\dfrac{x}{y}\right)^{x+y}$

24. If $\dfrac{9^{n-1}\times 3^{2}\times\left(3^{-n}\right)^{-1}-3(27)^{n-1}}{3^{4}\times 2^{3}}=1$ then find the value of n

25. Prove that $\dfrac{2^{n/3}-2^{n+1}}{2\times 2^{(n+3)}}+2^{-3}=\dfrac{1}{2}$

26. If $2^{a}=4^{b}=8^{c}$ then find the value of $\dfrac{1}{2a}+\dfrac{1}{4b}+\dfrac{1}{6c}$

27. If $2^{a}=3^{b}=6^{-c}$ then find the value of $\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)$

28. If $2^{x+4}-2^{x+2}=3$ then find the value of x

29. If $x=\sqrt[3]{\sqrt{2}+1}+\sqrt[3]{\sqrt{2}-1}$ then find the value of $x^{3}+3x-2$

30. If $(64)^{-1/2}-(-32)^{-4/5}=y$ then find the value of y

31. If $\left(\dfrac{x}{y}\right)^{m-1}=\left(\dfrac{y}{x}\right)^{m-3}$ then find the value of m

32. If $ax^{2/3}+bx^{1/3}+c=0$ then find the value of $a^{3}x^{2}+b^{3}x+c^{3}$

33. If $a=5^{1/3}+5^{-1/3}$ then find the value of $5a^{3}-15a$

34. Prove that $\left[\dfrac{x^{a/(a-b)}}{x^{a/(a+b)}}\div\dfrac{x^{b/(b-a)}}{x^{b/(b+a)}}\right]^{a+b}=1$

35. If $2^{n}-2^{n-1}=4$ then find the value of n^{n}

36. If $x=\sqrt{6+\sqrt{6+\sqrt{6+\ldots\infty}}}$ then find the value of x

37. If $4^x = 8^y$ then find the value of $\frac{x}{y} - 1$

38. Prove that $\dfrac{\left[p^2 - (1/q)^2\right]^q \left[p - (1/q)\right]^{p-q}}{\left[q^2 - (1/p)^2\right]^q \left[q - (1/p)\right]^{p-q}} = \left(\dfrac{p}{q}\right)^{p+q}$

39. If $z = \left(\dfrac{x^{-1}y^2}{x^2y^{-4}}\right)^7 \div \left(\dfrac{x^3y^{-5}}{x^{-2}y^3}\right)^{-5} \left(\dfrac{x^{7/2}y^{-1/3}}{x^{5/2}y^{10/3}}\right)^{2/3}$ then prove that $z + 1 = 2$

40. If $\left(\dfrac{a}{b}\right)^{y-1} = \left(\dfrac{b}{a}\right)^{y-3}$ then prove that $y - 2 = 0$

41. If $y = 2 + \sqrt{3}$ then prove that $y^3 - 2y^2 - 7y + 4 = 0$

42. If $x = \dfrac{1}{1 + a^{(n-m)}} + \dfrac{1}{1 + a^{(m-n)}}$ then prove that $x + 1 = 2$

43. If $p^x = q^y = r^z$ and $pqr = 1$ then prove that $xy + yz + zx = 0$

44. If $m = \dfrac{\sqrt{2}+1}{\sqrt{2}-1}$; $n = \dfrac{\sqrt{2}-1}{\sqrt{2}+1}$ and $z = m^2 + mn\ n^2$ then find the value of $z - 5$

45. Prove that $\sqrt{x^{-1}y} \times \sqrt{y^{-1}z} \times \sqrt{z^{-1}x} = 1$

46. If $a^x = b^y = c^z$ *and* $b^2 = ac$ then prove that $y = \dfrac{2xz}{(x+z)}$

47. Prove that $\left[(81)^{-3/4} \times \dfrac{16^{1/4}}{6^{-1/2}} \times \left(\dfrac{1}{27}\right)^{4/3}\right]^{1/3} = \sqrt{6}$

48. Prove that $\dfrac{(bc)^{b-c}(ca)^{c-a}(ab)^{a-b}}{\left(a^{b-c}b^{c-a}c^{a-b}\right)^{-1}} = 1$

49. Prove that $\left(\dfrac{4^{m+1/4}\sqrt{2 \times 2^m}}{2\sqrt{2^{-m}}}\right)^{1/m} = 8$

50. Prove that $\dfrac{a^{3/2} + ab}{ab - b^3} - \dfrac{\sqrt{a}}{\sqrt{a} - b} = \dfrac{\sqrt{a}}{b}$

51. Prove that $\left[x^{(b+c)/(c-a)}\right]^{1/(a-b)} \times \left[x^{(c+a)/(a-b)}\right]^{1/(b-c)} \times \left[x^{(a+b)/(b-c)}\right]^{1/(c-a)} = 1$

52. Prove that $\sqrt[ln]{\dfrac{x^l}{x^n}} \times \sqrt{\dfrac{x^n}{x^m}} \times \sqrt[ml]{\dfrac{x^m}{x^l}} = 1$

53. Prove that $\sqrt[bc]{\frac{x^{b/c}}{x^{c/b}}} \times \sqrt[ca]{\frac{x^{c/a}}{x^{a/c}}} \times \sqrt[ab]{\frac{x^{a/b}}{x^{b/a}}} = 1$

54. Prove that $\frac{1}{1+a^{-m}b^{n}+a^{-m}c^{p}} + \frac{1}{1+b^{-n}c^{p}+b^{-n}a^{m}} + \frac{1}{1+c^{-p}a^{m}+c^{-p}b^{n}} = 1$

55. Prove that $\frac{a^{2}}{(1-a)^{n}} + \frac{2}{(1-a)^{n-1}} - \frac{1}{(1-a)^{n-2}} = \frac{1}{(1-a)^{n}}$

56. Prove that $\frac{2^{a}\left(2^{a-1}\right)^{a}}{2^{a+1}\times 2^{a-1}}\left(\frac{8^{a/3}}{a}\right)^{-a} = 1$

57. Prove that $\frac{2^{n}\times 6^{m+1}\times 10^{m-n}\times 15^{m+n-2}}{4^{m}\times 3^{2m+n}\times 25^{m+1}} = \frac{2}{3}$

58. If $a = x^{q+r}\times y^{p}; b = x^{r+p}\times y^{q}; c = x^{p+q}\times y^{r}$ then prove that $a^{q-r}b^{r-p}c^{p-q} = 1$

59. If $p^{x} = q^{y} = r^{z}$ and $q^{2} = pr$ then prove that $\frac{1}{x}+\frac{1}{z}=\frac{2}{y}$

60. Prove that $\sqrt{a^{2}b^{-2/3}c^{-7/6}} \div \sqrt[3]{a^{4}b^{-1}c^{5/4}} = \frac{a^{1/6}}{c}$

61. Prove that $\sqrt[4]{a^{2}b^{3}c^{4}} \div \sqrt[3]{a^{4}b^{6}} \times \left(a^{1/2}b^{1/3}c^{1/4}\right)^{-4} = a^{-17/6}\times b^{-31/32}$

62. If $x = 3-\sqrt{3}$ then prove that $x^{4}-24x^{2}+36=0$

63. Solve $3^{2x+1} = \left(\sqrt{3}\right)^{x+3}$

64. Prove that $\left(8x^{3} \div 27a^{-3}\right)^{2/3} \times \left(64x^{3} \div 27a^{-3}\right)^{-2/3} = \frac{1}{4}$

65. Simplify $\frac{x^{4/3}+a^{2/3}x^{2/3}+a^{4/3}}{x^{2/3}+x^{1/3}a^{1/3}+a^{2/3}}$

66. Simplify $\left[\left(a^{-1/2}\right)^{2/3}\right]^{-3} \times \left[\left(b^{-1}\right)^{1/2}\right]^{-2/3} \sqrt{a^{2}+b^{2}}$

67. If $a^{x} = b^{x}$ then prove that $a = b$ or $x = 0$

68. If $\left(\frac{y}{z}\right)^{a}\left(\frac{z}{x}\right)^{b}\left(\frac{x}{y}\right)^{c} = 1$ then prove that $\left(\frac{y}{z}\right)^{1/(b-c)} = \left(\frac{z}{x}\right)^{1/(c-a)} = \left(\frac{x}{y}\right)^{1/(a-b)}$

69. Simplify $\left[\frac{1}{a^{1/(x-y)}}\right]^{1/(x-z)}\left[\frac{1}{a^{1/(y-z)}}\right]^{1/(y-x)}\left[\frac{1}{a^{1/(z-x)}}\right]^{1/(z-y)}$

70. Solve $\begin{pmatrix} 8\times 2^{xy} = 4^{y} \\ 9^{x}\times 3^{xy} = \frac{1}{27} \end{pmatrix}$

71. Prove that $\log\left(\frac{11}{15}\right) + \log\left(\frac{490}{297}\right) - 2\log\left(\frac{7}{9}\right) = \log 2\frac{1}{2}$

72. Prove that $7\log\left(\frac{8}{5}\right) - 6\log\left(\frac{4}{15}\right) + 3\log\left(\frac{5}{72}\right) = 2\log 5$

73. If $\log 2 + \log(x + 2) - \log(3x - 5) = \log 3$ then show that $x = \frac{19}{7}$

74. Prove that $(\log x)^2 - (\log y)^2 - \log xy \log\left(\frac{x}{y}\right) = 0$

75. Solve $\log_5^{(x+12)} - \log_5^x = 1$

76. If $\log_2\left[\log_3(\log_4 x)\right] = 2$ then prove that $x = 4^{81}$

77. Prove that $\log\left(\sqrt{x^2+1} + x\right) + \log\left(\sqrt{x^2+1} - x\right) = 0$

78. If $a = \log_{2x}^x;\ b = \log_{3x}^{2x};\ c = \log_{4x}^{3x}$ then prove that $abc + 1 = 2bc$

79. If $a = \log_z^{xy};\ b = \log_x^{yz}$ and $c = \log_y^{zx}$ then prove that $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 1$

80. Show that $\log_6^{216\sqrt{6}} = \frac{7}{2}$

81. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$ then prove that $f\left(\frac{x_1 + x_2}{1 + x_1x_2}\right) = f(x_1) + f(x_2)$

82. If $\frac{\log a}{y - z} = \frac{\log}{z - x} = \frac{\log c}{x - y}$ then show that $abc = a^x b^y c^z = a^{y+z} b^{z+x} c^{x+y}$

83. If $\frac{\log x}{b + c - 2a} = \frac{\log y}{c + a - 2b} = \frac{\log z}{a + b - 2c}$ then show that $xyz = 1$

84. Show that $\frac{\log_3^8}{\log_9^{16}\ \log_4^{10}} = 3\log 2$

85. If $x = \frac{e^y - e^{-y}}{e^y + e^{-y}}$ then prove that $y = \log_e^{\sqrt{\left(\frac{1+x}{1-x}\right)}}$

86. Prove that $(a + b)\log\left(\frac{x^a}{x^b}\right) + (b + c)\log\left(\frac{x^b}{x^c}\right) + (c + a)\log\left(\frac{x^c}{x^a}\right) = 0$

87. Prove that $\log_{10}^{\sqrt{10\sqrt{10\sqrt{10...\infty}}}} = 1$

88. If $\log\left(\frac{a + b}{2}\right) = \frac{1}{2}(\log a + \log b)$ then prove that $a = b$

89. If a, b, c are in geometric progression then prove that $\log_b^a + \log_b^c = 2$

90. Prove that $\log\left\{1 - \left[1 - \left(1 - x^2\right)^{-1}\right]^{-1}\right\}^{-1/2} = \log x$

91. If $\log_{1/2}^{\left[\log_t\left(\log_4^{32}\right)\right]} = 2$ then prove that $t = \dfrac{625}{16}$

92. If $x^3 + y^3 = 0$ then prove that $\log(x + y) - \dfrac{1}{2}(\log x + \log y + \log 3) = 0$

93. If $\dfrac{\log x}{a + b - 2c} = \dfrac{\log y}{b + c - 2a} = \dfrac{\log z}{c + a - 2b}$ then prove that $x^2y^2z^2 = 1$

94. If $\log(a + b) = \log\left(\dfrac{3a - 2b}{2}\right)$ then prove that $\log a - \log b = \log 5$

95. If $\log_{10}^{\left(98 + \sqrt{m^2 - 12m + 36}\right)} = 2$ then find the value of m

96. If $y = a^{1/\left(1 - \log_a^x\right)}$; $z = a^{1/\left(1 - \log_a^y\right)}$ and $x = a^k$ then show that $k = \dfrac{1}{1 - \log_a^z}$

97. If $4^m + 2^{2m - 1} = 3^{m + 1/2} + 3^{m - 1/2}$ then show that $m = \dfrac{3}{2}$

98. If $\log(2x - 3y) = \log x - \log y$ then show that $x = \dfrac{3y^2}{2y - 1}$

99. If $\log\dfrac{x}{y} + \log\dfrac{y}{x} = \log(x + y)$ then prove that $x + y = 1$

100. If a, b, c are three consecutive integers then show that $\log(ac + 1) = 2\log b$

101. If $\log\left(\dfrac{x + y}{5}\right) = \dfrac{1}{2}(\log x + \log y)$ then show that $\dfrac{x}{y} + \dfrac{y}{x} = 23$

102. If $\log_2^x + \log_4^x + \log_{16}^x = \dfrac{21}{4}$ then find x

103. Prove that $7\log\dfrac{16}{15} + 5\log\dfrac{25}{24} + 3\log\dfrac{81}{80} = \log 2$

104. Prove that $\dfrac{3 + \log_{10}^{343}}{2 + 1/2\left[\log(49/4)\right] + 1/3\left[\log(1/125)\right]} = 3$

105. If $\left(\dfrac{21}{10}\right)^x = 2$ then prove that $x = \dfrac{\log 2}{\log 3 + \log 7 - 1}$

106. If $\log\left(\dfrac{a + b}{5}\right) = \dfrac{1}{2}(\log a + \log b)$ then prove that $a^2 + b^2 = 23ab$

107. Prove that $\log\left(\dfrac{a^2}{bc}\right) + \log\left(\dfrac{b^2}{ac}\right) + \log\left(\dfrac{c^2}{ab}\right) = 0$

108. Prove that $\left(\dfrac{1}{\log_{p/q}^x} + \dfrac{1}{\log_{q/r}^x} + \dfrac{1}{\log_{r/p}^x}\right) = 0$

109. If $\log_2^{(x^2+x)} - \log_5^{(x+1)} = 2$ then find x

110. Find the value of $\log_2\left[\log_{\sqrt{2}}\left(\log_3^{81}\right)\right]$

111. If $x = \log_{24}^{12}$, $y = \log_{36}^{24}$ *and* $z = \log_{48}^{36}$ then find the value of $(2yz - xyz)$

112. If $\log_2^{10} = 0.3010$ then number of digits in 5^{20} is what?

113. Prove that $\log_5^5 \log_4^9 \log_3^2 = 1$

114. If $\log_e^x + \log_e^{(x+1)} = 0$ then prove that $x^2 + x = 0$

115. Solve $2^{x-3} = 5^{x-3}$ $\left(\text{Hint: } \log_{10}^2 = 0.3010\right)$

116. Solve $\dfrac{1}{\log_x^{10}} + 2 = \dfrac{2}{\log_5^{10}}$

117. Simplify $\sqrt[10]{a^8\sqrt{a^6\sqrt{a^{-4}}}}$

118. Simplify $\left(4a^{2/3} - 10a^{1/3}b^{1/3} + 25b^{2/3}\right)\left(2a^{1/3} + 5b^{1/3}\right)$

119. Simplify $\left\{1 - 1\left[1 - \left(1 - x^3\right)^{-1}\right]^{-1}\right\}^{-1/3}$

120. If $\sqrt[x]{a} = \sqrt[y]{b} = \sqrt[z]{c}$ and $abc = 1$ then prove that $x + y + z = 0$

121. If $x = 2 + 2^{2/3} + 2^{1/3}$ then prove that $x^3 - 6x^2 + 6x - 2 = 0$

122. If $a^x = b^y = c^z = d^w$ and $ab = cd$ then show that $(x + y)zw = xy(z + w)$

123. Simplify $\dfrac{\left(2^{2n} - 3\times 2^{2n-2}\right)\left(3^n - 2\times 3^{n-2}\right)}{3^{n-4}\left(4^{n+3} - 2^{2n}\right)}$

124. Solve $2^{x+3} + 2^{x+1} = 320$

ANSWERS

(6) $x = 1$

(7) $x = 2; y = 1$

(8) 0

(9) $x = 3$

(10) $x = 2$ or -1

(13) $x = \dfrac{9}{4}$

(14) $m = 4$ or -2

(18) 2010

(19) $x = 0; y = 0$

(24) $n = 2$

(26) $\dfrac{24}{7}$

(27) 0

(28) $x = -2$

(29) 0

(30) $y = \dfrac{1}{16}$

(31) $m = 2$

(32) $-3abcx$

(33) 26

(35) 27

(36) $x = 3$

(37) $\frac{1}{2}$

(44) 30

(63) $x = 1/3$

(65) $x^{2/3} - a^{1/3}x^{1/3} + a^{2/3}$

(66) $ab^{1/3}$

(69) 1

(70) $(x, y) = (-1, 1)(x, y) = (3, -3)$

(75) $x = 3$

(95) $m = 4$ or 8

(102) $x = 8$

(109) $x = 25$

(110) 3

(111) 1

(112) 14

(115) $x = 3$

(116) $x = \frac{1}{4}$

(117) a

(118) $8a + 125b$

(123) $\frac{1}{4}$

(124) $x = 5$

3

Quadratic Equation

LEARNING OBJECTIVES

This chapter will enable you to learn the concepts and application of :

- Quadratic equitation
- Advance application

INTRODUCTION

$p(x) = ax^2 + bx + c = 0;\ a \neq 0,\ a,\ b,\ c \in R$ is said to be a quadratic equation or second degree equation.

SOLUTION SET OF QUADRATIC EQUATION

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here $b^2 - 4ac = \Delta$ (Discrimination function)

$$\therefore x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$$\therefore x = \left(\frac{-b + \sqrt{\Delta}}{2a};\ \frac{-b - \sqrt{\Delta}}{2a}\right)$$

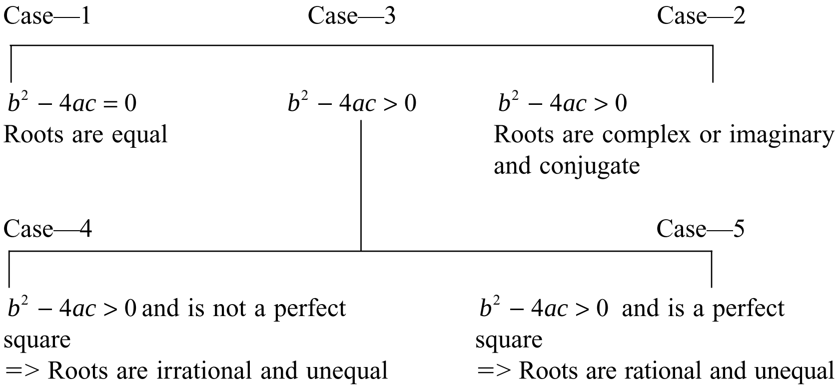
Here $\frac{-b + \sqrt{\Delta}}{2a} = \alpha$ and $\frac{-b - \sqrt{\Delta}}{2a} = \beta$

Here α and β are the roots of the equation.

NATURE OF THE ROOTS OF QUADRATIC EQUATION

1. $\Delta > 0$ then roots are rational and unequal
2. $\Delta = 0$ then roots are rational and equal
3. $\Delta < 0$ then roots are not real (roots are complex or imaginary)

$$\Delta = b^2 - 4ac$$



SUM AND PRODUCT OF THE ROOTS

(1) $\alpha + \beta = \dfrac{-b + \sqrt{\Delta}}{2a} + \dfrac{-b - \sqrt{\Delta}}{2a}$

$$\therefore \alpha + \beta = \alpha + \beta = -\frac{b}{a} = -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$$

(2) $\alpha\beta = \left(\dfrac{-b + \sqrt{\Delta}}{2a}\right)\left(\dfrac{-b - \sqrt{\Delta}}{2a}\right)$

$$= \frac{b^2 - \Delta}{4a^2}$$

$$= \frac{b^2 - \left(b^2 - 4ac\right)}{4a^2}$$

$$= \frac{4ac}{4a^2}$$

$$\therefore \alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

USING ROOTS OF THE EQUATION TO FORM THE QUADRATIC EQUATION

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

Following factors are useful to get sum and product of root i.e. $\alpha + \beta$ and $\alpha\beta$

(1) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

(2) $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$

(3) $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}$

(4) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}$

(5) $\alpha^3+\beta^3=(\alpha+\beta)\left(\alpha^2+\beta^2-\alpha\beta\right)$

(6) $\alpha-\beta=\sqrt{(\alpha+\beta)^2-4\alpha\beta}$

(7) $\alpha^2-\beta^2=(\alpha-\beta)(\alpha+\beta)$

(8) $\alpha^3-\beta^3=(\alpha-\beta)\left(\alpha^2+\beta^2+\alpha\beta\right)$

(9) $\frac{1}{\beta}-\frac{1}{\alpha}=\frac{\alpha-\beta}{\alpha\beta}$

(10) $\frac{1}{\beta^2}-\frac{1}{\alpha^2}=\frac{\alpha^2-\beta^2}{\alpha^2\beta^2}$

(11) $\left(\alpha^3+\beta^3\right)=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$

Nature of the Roots from the Values *a, b* and *c:*

(1) If $b = 0$ then both the roots are additive inverse of each other

(2) If $c = 0$ then one root is zero and the other root is $-\frac{b}{a}$

(3) If $b = 0$ and $c = 0$ then both the roots are 0

(4) If $a = c$ then both the roots are reciprocal of each other

(5) If $a + b + c = 0$ then one root is 1 and the other root is $\frac{c}{a}$

Nature of the Roots from the Signs of *a, b* and *c:*

(1) If *a* and *c* have the like sign and *a* and *b* have unlike sign then both the roots are positive.

(2) If *a, b* and *c* all have the like signs then both the roots are negative.

(3) If *a* and *c* have unlike signs then one root of the equation is positive and the other is negative

Quadratic Inequalities

(A) For $a > 0$ solution of $ax^2 + bx + c > 0$ and $ax^2 + bx + c < 0$

	$ax^2 + bx + c > 0$	$ax^2 + bx + c < 0$
$\Delta<0$	R	$\varnothing$
$\Delta=0$	$R-\left(\frac{-b}{2a}\right)$	$\varnothing$
$\Delta>0$	$R-(\alpha,\beta)$	(α,β)

(B) For $a < 0$ solution of $ax^2 + bx + c > 0$ and $ax^2 + bx + c < 0$

	$ax^2 + bx + c > 0$	$ax^2 + bx + c < 0$
$\Delta<0$	$\varnothing$	R
$\Delta=0$	$\varnothing$	$R - \left\{\frac{-b}{2a}\right\}$
$\Delta>0$	(α,β)	$R - \{\alpha,\beta\}$

where α and β are the roots of the corresponding equation $ax^2 + bx + c = 0$ and $\alpha<\beta$

ILLUSTRATIONS

Illustration 1 If α and β are the roots of the equation $px^2 + qx + r = 0$ then find the value of $(\alpha^3\beta + \alpha\beta^3)$

Solution

$$px^2 + qx + r = 0$$

$$\therefore \alpha + \beta = \frac{-b}{a} = \frac{-q}{p}$$

$$and\ \alpha\beta = \frac{c}{a} = \frac{r}{p}$$

$$\therefore \alpha^3\beta + \beta^3\alpha = \alpha\beta(\alpha^2 + \beta^2)$$

$$= \alpha\beta\left[(\alpha + \beta)^2 - 2\alpha\beta\right]$$

$$= \frac{r}{p}\left(\frac{q^2}{p^2} - \frac{2r}{p}\right)$$

$$= \frac{r}{p}\left(\frac{q^2 - 2rp}{p^2}\right)$$

$$= \frac{r(q^2 - 2rp)}{p^3}$$

Illustration 2 If α, β are roots of the equation $3x^2 - 2x + 5 = 0$ which is the equation whose roots are $(2\alpha + 3\beta)$ and $(3\alpha + 2\beta)$

Solution

$$3x^2 - 2x + 5 = 0$$

$$\alpha + \beta = \frac{-b}{a}$$

$$= -\frac{-2}{-3}$$

$$= \frac{2}{3}$$

and $\alpha\beta = \frac{c}{a}$

$$= \frac{5}{3}$$

Now $(2\alpha + 3\beta) + (3\alpha + 2\beta)$

$= 5\alpha + 5\beta$

$= 5(\alpha + \beta)$

$= 5\left(\frac{2}{3}\right)$

$= \frac{10}{3}$

And $(2\alpha + 3\beta)(3\alpha + 2\beta)$

$= 6\alpha^2 + 4\alpha\beta + 9\alpha\beta + 8\beta^2$

$= 6(\alpha^2 + \beta^2) + 13\alpha\beta$

$= 6\left[(\alpha + \beta)^2 - 2\alpha\beta\right] + 13\alpha\beta$

$= 6(\alpha + \beta)^2 - 12\alpha\beta + 13\alpha\beta$

$= 6(\alpha + \beta)^2 + \alpha\beta$

$= 6\left(\frac{4}{9}\right) + \frac{5}{3}$

$= \frac{24}{9} + \frac{5}{3}$

$= \frac{24 + 15}{9}$

$= \frac{39}{9}$

$\therefore$ required equation is

$x^2 - (2\alpha + 3\beta + 3\alpha + 2\beta) + (2\alpha + 3\beta)(3\alpha + 2\beta) = 0$

$\therefore x^2 - \frac{10}{2}x + \frac{39}{9} = 0$

$\therefore x^2 - 5x + \frac{39}{9} = 0$

$\therefore 9x^2 - 45x + 39 = 0$

Illustration 3 If α and β are the roots of $x^2+2x+1=0$ then which is the equation whose roots are $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$?

Solution

$$x^2+2x+1=0$$

$$\therefore \alpha+\beta=\frac{-b}{a}=\frac{-2}{1} \text{ and } \alpha\beta=\frac{c}{a}=1$$

Now

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}$$

$$=\frac{\alpha^2+\beta^2}{(\alpha\beta)^2}$$

$$=\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}$$

$$=\frac{4-2}{2}$$

$$=2$$

$$\text{and } \left(\frac{1}{\alpha^2}\right)\left(\frac{1}{\beta^2}\right)=\frac{1}{(\alpha\beta)^2}=1$$

$\therefore$ required equation is

$$x^2-\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)x+\frac{1}{\alpha^2}\frac{1}{\beta^2}=0$$

$$\therefore\ x^2+2x+1=0$$

Illustration 4 How many the number of real solutions of $x^2-3|x|+2=0$?

Solution

$$x^2-3|x|+2=0$$

$$|x|=\pm x$$

If $_i |x|=x$

$$\therefore x^2-3x+2=0$$

$$(x-2)(x-1)=0$$

$x=2$ or $x=1$

If $_i |x|=-x$

$$\therefore x^2 + 3x + 2 = 0$$
$$\therefore (x+2)(x+1) = 0$$
$$\therefore x = -2 \text{ or } x = -1$$

Illustration 5 If the roots of the equation $x^2 - px + q = 0$ differ by unity then prove that $p^2 = 4q + 1$

Solution

$$x^2 - px + q = 0$$
$$\therefore \alpha + \beta = \frac{-b}{a} = -\frac{-p}{1} = p \text{ and}$$
$$\alpha\beta = \frac{c}{a} = \frac{q}{1} = q$$

Here

$$\therefore \alpha - \beta = 1 \qquad (\because given)$$
$$\therefore (\alpha - \beta)^2 = 1$$
$$\therefore (\alpha + \beta)^2 - 4\alpha\beta = 1$$
$$\therefore p^2 - 4q = 1$$
$$\therefore p^2 = 4q + 1$$

Illustration 6 For what value of p the difference between the roots of the equation $x^2 - px + 8 = 0$ is 2?

Solution

$$x^2 - px + 8 = 0$$
$$\therefore \alpha + \beta = p, \alpha\beta = 8$$

Here $\alpha - \beta = 2$ (*given*)

$$\therefore (\alpha - \beta)^2 = 4$$
$$\therefore (\alpha + \beta)^2 - 4\alpha\beta = 4$$
$$p^2 - 4(8) = 4$$
$$p^2 = 36$$
$$p = \pm 6$$

Illustration 7 If the sum of the squares of the roots of the equation $x^2 + x - p = 0$ is 10 then find the value of p

Solution

$$x^2 + x - p = 0$$
$$\therefore \alpha + \beta = -1, \alpha\beta = -p$$

Here $\alpha^2+\beta^2=10$

$$\therefore (\alpha+\beta)^2-2\alpha\beta=10$$

$$1+2p=10$$

$$2p=9$$

$$p=\frac{9}{2}$$

Illustration 8 Solve $6x^{1/3}+x^{-1/3}=5$

Solution

$6x^{1/3}+x^{-1/3}=5$ let $x^{1/3}=3$

$$\therefore 6m+m^{-1}=5$$

$$\therefore 6m+\frac{1}{m}=5$$

$$\therefore \frac{6m^2+1}{m}=5$$

$$\therefore 6m^2+1=5m$$

$$\therefore 6m^2-5m+1=0$$

$$\therefore 6m^2-3m-2m+1=0$$

$$\therefore 3m(2m-1)-1(2m-1)=0$$

$$\therefore (2m-1)(3m-1)=0$$

$$\therefore 2m-1=0 \quad \text{or} \quad 3m-1=0$$

$$\therefore 2m-1=0 \qquad 3m-1=0$$

$$\therefore m=\frac{1}{2} \qquad m=\frac{1}{3}$$

$$\therefore m=x^{1/3}=\frac{1}{2} \qquad m=x^{1/3}=\frac{1}{3}$$

$$\therefore x=\left(\frac{1}{2}\right)^3 \qquad x=\left(\frac{1}{3}\right)^3$$

$$\therefore x=\frac{1}{8} \qquad x=\frac{1}{27}$$

Illustration 9 Solve $\dfrac{3}{3x-2}+\dfrac{2}{2x+3}=\dfrac{1}{x+2}$

Solution

$$\frac{3}{3x-2}+\frac{2}{2x+3}=\frac{1}{x+2}$$

$$\therefore 3(2x+3)(x+2)+2(3x-2)(x+2)=(3x-2)(2x+3)$$
$$\therefore 3(2x^2+7x+6)+2(3x^2+4x-4)=(6x^2+5x-6)$$
$$\therefore 6x^2+21x+18+6x^2+8x-8=6x^2+5x-6$$
$$\therefore 6x^2+24x+16=0$$
$$\therefore 3x^2+12x+8=0$$

Here $a = 3$, $b = 12$, $c = 8$

$\Delta = b^2 - 4ac = 144 - 48 = 98 > 0$

$\therefore$ roots are real and distinct

$$\therefore (\alpha,\beta)=\frac{-b\pm\sqrt{\Delta}}{2a}$$
$$=\frac{-12\pm\sqrt{98}}{6}$$
$$=\frac{-12\pm 4\sqrt{6}}{6}=\frac{-6\pm 2\sqrt{6}}{3}$$
$$(\alpha,\beta)=\frac{2\sqrt{6}}{3}$$
$$\therefore \text{solution set} = \left(-2+\frac{2\sqrt{6}}{3}; -2-\frac{2\sqrt{6}}{3}\right)$$

Illustration 10 Solve $x(x+5)(x+7)(x+12)+150=0$

Solution

$$x(x+5)(x+7)(x+12)+150=0$$
$$(x^2+12x)(x+5)(x+7)+150=0$$
$$(x^2+12)(x^2+12x+35)+150=0$$

let $x^2 + 12x = m$

$$\therefore m(m+35)+150=0$$
$$\therefore m^2+35m+150=0$$

Here

$$\Delta = b^2 - 4ac$$
$$= (35)^2-4(1)(150)$$
$$= 1225 - 600$$
$$= 625$$
$$\therefore m=\frac{-b\pm\sqrt{\Delta}}{2a}$$
$$=\frac{-35\pm\sqrt{625}}{2}$$
$$=\frac{-35\pm 25}{2}$$

$\therefore m = \frac{-35+25}{2}$ or $\therefore m = \frac{-35-25}{2}$

$m = -5$ or $m = -30$

but $m = x^2 + 12x$

or

$x^2 + 12x = -5$

$x^2 + 12x + 5 = 0$

$\Delta = b^2 - 4ac$

$= (12)^2 - 4(1)(5)$

$= 124 > 0$

$\therefore$ roots are real and distinct

$\therefore x = \frac{-b \pm \sqrt{\Delta}}{2a}$

$= \frac{-12 \pm \sqrt{124}}{2}$

$= \frac{-12 \pm 2\sqrt{31}}{2}$

$x^2 + 12x = -30$

$x^2 + 12x + 30 = 0$

$\Delta = b^2 - 4ac$

$= (12)^2 - 4(1)(30)$

$= 24 > 0$

$\therefore$ roots are real and distinct

$\therefore x = \frac{-b \pm \sqrt{\Delta}}{2a}$

$= \frac{-12 \pm \sqrt{24}}{2}$

$= \frac{-12 \pm 2\sqrt{6}}{2}$

$= -6 \pm \sqrt{6}$

$x = -6+$ or $-6-$ $x = -6+$ or $-6-$

Thus solution set $= \{-6+, -6-, -6+, -6-\}$

Illustration 11 If $2 + \sqrt{3}$ is one root of the equation $x^2 - 4x + k = 0$ then find k and the second root

Solution

Here $2 + \sqrt{3}$ is one root of the equation $x^2 - 4x + k = 0$

$\therefore \left(2+\sqrt{3}\right)^2 - 4\left(2+\sqrt{3}\right) + k = 0$

$\therefore 4 + 3 + 4\sqrt{3} - 8 - 4\sqrt{3} + k = 0$

$\therefore k - 1 = 0$

$\therefore k = 1$

Now the equation will be $x^2 - 4x + 1 = 0$

$\therefore a = 1 , b = 4 , c = 1$

$\therefore \Delta = b^2 - 4ac$

$= (-4)^2 - 4(1)(1)$

$= 12$

$\therefore (\alpha, \beta) = \frac{-b \pm \sqrt{\Delta}}{2a}$

$$= \frac{4 \pm 2\sqrt{3}}{2}$$

$$\therefore (\alpha, \beta) = \left(2 + \sqrt{3}\,, 2 - \sqrt{3}\right)$$

$\therefore$ the second root is $2 - \sqrt{3}$

Illustration 12 If the equation $ax^2 - 6x + c + 9 = 0$ has one root $3 + 4i$ (where $a\,, c \in R\,; a \neq 0$) then find a and c and the second root.

Solution

If one root of $ax^2 - 6x + c + 9 = 0$ has one root $3 + 4i$ then its second root is $3 - 4i$

$$\therefore \alpha = 3 + 4i \text{ and } \therefore \beta = 3 - 4i$$

$$\therefore \alpha + \beta = 3 + 4i + 3 - 4i$$

$$\therefore \frac{6}{a} = 6$$

$$\therefore a = 1$$

Now $\alpha\beta = (3 + 4i)(3 - 4i)$

$$\therefore \frac{c}{a} = 9 - 16i^2$$

$$\therefore \frac{c + 9}{1} = 25$$

$$\therefore c + 9 = 25$$

$$\therefore c = 16$$

Illustration 13 Construct the quadratic equation having the roots $\frac{3 + \sqrt{5}}{7}$ and $\frac{3 - \sqrt{5}}{7}$

Solution

Here $\alpha = \frac{3 + \sqrt{5}}{7}$ and $\beta = \frac{3 - \sqrt{5}}{7}$

$$\therefore \alpha + \beta = \frac{3 + \sqrt{5}}{7} + \frac{3 - \sqrt{5}}{7} = \frac{6}{7}$$

$$\alpha\beta = \left(\frac{3 + \sqrt{5}}{7}\right)\left(\frac{3 - \sqrt{5}}{7}\right) = \frac{9 - 5}{49} = \frac{4}{49}$$

$\therefore$ The required quadratic equation form

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

$$\therefore x^2 - \left(\frac{6}{7}\right)x + \frac{4}{49} = 0$$

$$\therefore 49x^2 - 42x + 4 = 0$$

Illustration 14 Construct the quadratic equation having the following roots:

$$\frac{1+i}{1-i} \text{ and } \frac{1-i}{1+i}$$

Solution

Here $\alpha = \frac{1+i}{1-i}$ and $\beta = \frac{1-i}{1+i}$

$$\therefore \alpha+\beta = \frac{1+i}{1-i} + \frac{1-i}{1+i} = \frac{(1+i)^2 + (1-i)^2}{1-i^2}$$

$$\therefore \alpha+\beta = \frac{1+2i+i^2+1-2i+i^2}{2} \qquad (\because i^2 = -1)$$

$$= \frac{2+2i^2}{2} = \frac{2-2}{2} = \frac{0}{2} = 0$$

and $\therefore \alpha\beta = \left(\frac{1+i}{1-i}\right)\left(\frac{1-i}{1+i}\right) = 1$

$\therefore$ The required quadratic equation form

$x^2 - (\alpha+\beta)x + \alpha\beta = 0$

$x^2 - (0)x + 1 = 0$

$x^2 + 1 = 0$

Illustration 15 If α and β are the roots of the equation $ax^2 + bx + c = 0$ then construct a quadratic equation whose roots are $\frac{1}{\alpha+\beta}$ and $\frac{1}{\alpha} + \frac{1}{\beta}$

Solution

Here $a(x) = ax^2 + bx + c = 0$

$$\therefore \alpha+\beta = -\frac{b}{a}; \alpha\beta = \frac{c}{a}$$

$$\therefore \frac{1}{\alpha+\beta} + \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = \frac{1}{\alpha+\beta} + \left(\frac{\alpha+\beta}{\alpha\beta}\right)$$

$$= \frac{1}{-b/a} + \left(\frac{-b/a}{c/a}\right)$$

$$= \frac{-a}{b} - \frac{b}{c} = -\left(\frac{ac+b^2}{bc}\right)$$

and $\left(\frac{1}{\alpha+\beta}\right)\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = \left(\frac{1}{\alpha+\beta}\right)\left(\frac{\alpha+\beta}{\alpha\beta}\right)$

$$= \frac{1}{\alpha\beta} = \frac{1}{c/a} = \frac{a}{c}$$

$\therefore$ The required quadratic equation form

$$x^2 - \left(\frac{1}{\alpha+\beta} + \frac{1}{\alpha} + \frac{1}{\beta}\right)x + \left(\frac{1}{\alpha+\beta}\right)\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = 0$$

$$\therefore x^2 + \left(\frac{b^2 + ac}{bc}\right)x + \frac{a}{c} = 0$$

$$\therefore \frac{bcx^2 + \left(b^2 + ac\right)x + ab}{bc} = 0$$

$$\therefore bcx^2 + \left(b^2 + ac\right)x + ab = 0$$

Illustration 16 If α and β are the roots of the equation $ax^2 + bx + c = 0$ then find the value of the following

(1) $\alpha^2 + \beta^2$ (2) $\alpha^4 + \beta^4$ (3) $\alpha - \beta$

(4) $\frac{1}{\alpha} + \frac{1}{\beta}$ (5) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

Solution

Here $ax^2 + bx + c = 0$

$$\therefore \alpha + \beta = \frac{-b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

(1) $\alpha^2 + \beta^2$

$$= (\alpha + \beta)^2 - 2\alpha\beta$$

$$= \left(\frac{-b}{a}\right)^2 - \frac{2c}{a}$$

$$= \frac{b^2}{a^2} - \frac{2c}{a}$$

$$= \frac{b^2 - 2ac}{a^2}$$

(2) $\alpha^4 + \beta^4$

$$= \left(\alpha^2 + \beta^2\right)^2 - 2\alpha^2\beta^2$$

$$= \left(\frac{b^2 - 2ac}{a^2}\right)^2 - \frac{2c^2}{a^2}$$

(3) $\alpha - \beta = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$

$$= \sqrt{\left(\frac{-b}{a}\right)^2 - \frac{4c}{a}}$$

$$= \sqrt{\frac{b^2 - 4ac}{a^2}}$$

(4) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$

$$= \frac{-b/a}{c/a} = \frac{-b}{c}$$

(5) $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2 + \beta^2}{(\alpha\beta)^2}$

$$= \frac{\left(b^2 - 2ac\right)/a^2}{\left(c^2/a^2\right)}$$

$$= \frac{b^2 - 2ac}{c^2}$$

Illustration 17 If the roots of the equation $x^2 + 3kx + 2 = 0$ are in the ratio of 1: 2 then find the value of k.

Solution

If α and β are the roots of the equation $x^2 + 3kx + 2 = 0$ then

$$\alpha + \beta = \frac{-b}{a} = \frac{-3k}{1} = -3k$$

and $\alpha\beta = \frac{c}{a} = \frac{2}{1} = 2$

but $\frac{\alpha}{\beta} = \frac{1}{2}$

$\therefore \beta = 2\alpha$

Now $\alpha + \beta = -3k$ and $\alpha\beta = 2$

$\therefore \alpha + 2\alpha = -3k$ and $\alpha(2\alpha) = 2$

$\therefore 3\alpha = -3k$ and $2\alpha^2 = 2$

$\therefore k = -\alpha$ and $\alpha = \pm 1$

$\therefore \alpha = +1 \Rightarrow k = -1$

$\therefore \alpha = +1 \Rightarrow k = -1$

$\therefore k = \pm 1$

Illustration 18 If the difference of the roots of the equation $x^2 + 2kx + 3k = 0$ is 4 then find the value of k

Solution

$x^2 + 2kx + 3k = 0$

$$\therefore \alpha + \beta = \frac{-b}{a} = \frac{2k}{1} = 2k \quad (1)$$

and $\alpha\beta = \frac{c}{a} = \frac{3k}{1} = 3k$

Also $\alpha - \beta = 4 \quad (\because \alpha > \beta)$ (2)

By adding and subtracting Eqs. (1) and (2) we get

$2\alpha = 2k + 4$ and $2\beta = 2k - 4$

$\therefore \alpha = k + 2$ and $\beta = k - 2$

and $\alpha\beta = k^2 - 4$

$\therefore 3k = k^2 - 4$

$\therefore k^2 - 3k - 4 = 0$

$\therefore (k - 4)(k + 1) = 0$

$\therefore k = 4$ *or* $k = -1$

Illustration 19 Solve $a(b-c)x^2+b(c-a)x+c(a-b)=0$

Solution

Here we take P = $a(b-c)$, Q = $b(c-a)$ and R = $c(a-b)$

$\therefore$ P + Q + R

$$= a(b-c)+b(c-a)+c(a-b)=0$$

$$= ab-ac+bc-ab+ca-cb=0$$

$$\therefore \alpha = 1 \text{ and } \beta = \frac{c(a-b)}{a(b-c)}$$

(*Q* refer to theory concept)

Illustration 20 Solve $(ax+b)(bx+a)=(a+b)^2$

Solution

$$(ax+b)(bx+a) = a^2 + 2ab + b^2 = 0$$

$$\therefore abx^2 + a^2x + b^2x + ab - a^2 - 2ab - b^2 = 0$$

$$\therefore abx^2 + (a^2+b^2)x + (-a^2-ab-b^2) = 0$$

Here let P = ab Q = $a^2 + b^2$ and R = $-a^2 - ab - b^2$

P + Q + R = $ab + a^2 + b^2 - a^2 - ab - b^2 = 0$

$$\therefore \alpha = 1 \text{ and } \beta = \frac{-(a^2+ab+b^2)}{ab}$$

(*Q* refer to theory concept)

Illustration 21 For which value of k the roots of the equation $2x^2 - 3kx + 18 = 0$ are (i) real and unequal (ii) real and equal (iii) complex numbers?

Solution

$2x^2 - 3kx + 18 = 0$

Here $a = 2$, $b = -3k$, $c = 18$

$$\therefore \Delta = b^2 - 4ac$$

$$= (-3k)^2 - 4(2)(18)$$

$$= 9k^2 - 9(16)$$

$$= 9(k^2 - 16)$$

(i) If the roots are real and unequal then $\Delta > 0$

$\therefore 9(k^2 - 16) > 0$

$\therefore k^2 - 16 > 0$

$\therefore k < -4$ or $k > 4$

$\therefore k \in (-\infty, -4) \cup (4, \infty)$

(ii) If the roots are real and equal then $\Delta = 0$

$\therefore 9(k^2 - 16) > 0$

$\therefore k^2 - 16 = 0$
$\therefore k^2 = 16$
$\therefore k^2 = \pm 4$

(iii) If the roots are complex then $\Delta < 0$

$\therefore 9\left(k^2 - 16\right) < 0$
$\therefore k^2 - 16 < 0$
$\therefore k^2 < 16$
$\therefore k > -4$ or $k < 4$
$\therefore k \in \left(-4 , 4\right)$

Illustration 22 If a, b, c, d are real numbers and $ac = 2\left(b + d\right)$ then show that at least one equation out of $x^2 + ax + b = 0$ and $x^2 + cx + d = 0$ has real roots

Solution

Here for equation $x^2 + ax + b = 0$
$\Delta_1 = a^2 - 4b$
and for the equation $x^2 + cx + d = 0$
$\Delta_2 = c^2 - 4d$
Now if both the equations do not have real roots then
$\Delta_1 < 0$ and $\Delta_2 < 0$
$\therefore \Delta_1 + \Delta_2 < 0$
$\therefore a^2 - 4d + c^2 - 4d < 0$
$\therefore a^2 + c^2 - 4\left(b + d\right) < 0$
$\therefore a^2 + c^2 - 2ac < 0 \quad \left[\because ac = 2\left(b + d\right)\right]$
$\therefore \left(a - c\right)^2 < 0$
which is not possible because a, b, c, d are real numbers
$\therefore$ Our supposition is wrong
Thus at least one equation should have real roots

Illustration 23 If one of the roots of the distinct equations $x^2 + cx + ab = 0$ and $x^2 + bx + ac = 0$ is common then prove that their other roots satisfy the equation $x^2 + ax + bc = 0$

Solution

Suppose α and β be the roots of the equation $x2 + bx + ac = 0$. Let α and γ be the roots of the equation $x^2 + cx + ab = 0$
Thus $\alpha^2 + b\alpha + ac = 0$ and $\alpha^2 + c\alpha + ab = 0$
$\therefore \alpha^2 + b\alpha + ac = \alpha^2 + c\alpha + ab$
$\therefore \left(b - c\right)\alpha + a\left(c - b\right) = 0$

$\Rightarrow \alpha = a$

(Q If $b = c$ then clearly $x^2 + bx + ac = 0$ and $x^2 + cx + ab = 0$ are not distinct equations.)

Now $\alpha\beta = ca$ and $\alpha\gamma = ab$

$\therefore \beta = c$ and $\gamma = b \qquad (\because \alpha = a)$ (1)

Also $\alpha = a$ is a root of the equation $x^2 + bx + ac = 0$

$\therefore a^2 + ba + ca = 0$

$\therefore a + b + c = 0$

$\therefore b + c = -a$ (2)

Now from (1) and (2) we can say that $\beta + \gamma = b + c = -a$

and $\beta\gamma = bc$

Thus the quadratic equation with the roots β and γ will be

$x^2 - (\beta + \gamma)x + \beta\gamma = 0$

$\therefore x^2 + ax + bc = 0$

Illustration 24 If one of the roots of the equations $ax^2 + bx + c = 0$ and $px^2 + qx + r = 0$ is common then prove that $(qr - pc)^2 = (br - q)(aq - pb)$

Solution

Suppose α be the common root of the two quadratic equations

$\therefore a\alpha^2 + b\alpha + c = 0$ and $p\alpha^2 + q\alpha + r = 0$

Now according to Crammer rule (cross multiplication method)

$$\alpha^2 = \frac{\begin{vmatrix} b & c \\ q & r \end{vmatrix}}{\begin{vmatrix} a & b \\ p & q \end{vmatrix}} = \frac{br - qc}{aq - pb} \text{ and } \alpha = \frac{\begin{vmatrix} a & c \\ p & r \end{vmatrix}}{\begin{vmatrix} a & b \\ p & q \end{vmatrix}} = \frac{(ar - pc)}{aq - bp}$$

Now $\alpha^2 = (\alpha)^2$

$$\frac{br - qc}{aq - pb} = \frac{(ar - pc)^2}{(aq - bp)^2}$$

$\therefore (ar - pc)^2 = (br - qc)(aq - pb)$

Illustration 25 Solve $5x - 3 - 2x^2 > 0$

Solution

$5x - 3 - 2x^2 > 0$

Here $a = -2$, $b = 5$, $c = -3$

and $\Delta = b^2 - 4ac$

$= (5)^2 - 4(-2)(-3) = 1 > 0$

Thus $x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-5 \pm 1}{-4}$

$x = \frac{-5 + \sqrt{1}}{-4}$ or $= \frac{-5 - 1}{-4}$

$\therefore x = 1$ or $\therefore x = \frac{3}{2}$

Here $\alpha < 0$ and $\Delta < 0$ then the solution set will be $\left(1, \frac{3}{2}\right)$ that $\alpha > \beta$

Illustration 26 Solve $x^2 + 7x + 6 > 0$

Solution

$x^2 + 7x + 6 > 0$

Here $a = 1, b = 7, c = 6$

$$\Delta^2 = b^2 - 4ac$$
$$= (7)^2 - 4(1)(6)$$
$$= 25$$

$\therefore$ The roots of the equation $x^2 + 7x + 6 = 0$ are

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}$$
$$= \frac{-7 \pm \sqrt{25}}{2}$$

$x = \frac{-7 + 5}{2}$ or $x = \frac{-7 - 5}{2}$

$x = -1$ or $x = -6$

Here $a > 0$ and $\Delta > 0$ the solution set of $x^2 + 7x + 6 > 0$ is

$(-\infty, -6) \cup (-1, \infty)$ or

$R - (-6, -1)$

Illustration 27 Solve $x^2 + x(p - q) - 2pq = 2(x - q)^2$

Solution

$x^2 + x(p - q) - 2pq = 2(x - q)^2$

$\therefore x^2 + px - qx - 2pq = 2(x^2 - 2qx + q^2)$

$\therefore x^2 + px - qx - 2pq = 2x^2 - 4qx + 2q^2$

$\therefore x^2 - (p + 3q)x + (2q^2 + 2pq) = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{(p+3q) \pm \sqrt{(p+3q)^2 - 4(2q^2 + 2pq)}}{2}$$

$$= \frac{(p+3q) \pm \sqrt{p^2 + 6pq + 9q^2 - 8q^2 - 8pq}}{2}$$

$$= \frac{(p+3q) \pm \sqrt{p^2 - 2pq + q^2}}{2}$$

$$= \frac{(p+3q) \pm \sqrt{(p-q)^2}}{2}$$

$$= \frac{(p+3q) \pm (p-q)}{2}$$

$$\therefore \alpha = \frac{p+3q+p-q}{2}\,;\ \beta = \frac{p+3q-p+q}{2}$$

$$\alpha = \frac{2(p+q)}{2}\,;\ \beta = \frac{4q}{2}$$

$$\alpha = p+q\,;\ \beta = 2q$$

Illustration 28 If the roots of $(a-b)x^2 + (b-c)x + (c-a) = 0$ are equal then prove that $2a = b + c$

Solution

If the roots of $(a-b)x^2 + (b-c)x + (c-a) = 0$ are equal then

$\Delta = 0$

$\therefore b^2 - 4ac = 0$

$\therefore (b-c)^2 - 4(a-b)(c-a) = 0$

$\therefore (b^2 - 2bc + c^2) - 4(ac - a^2 - bc + ab) = 0$

$\therefore b^2 - 2bc + c^2 - 4ac + 4a^2 + 4bc - 4ab = 0$

$\therefore 4a^2 + b^2 + c^2 - 4ab + 2bc - 4ac = 0$

$\therefore (2a - b - c)^2 = 0$

$\therefore 2a - b - c = 0$

$\therefore 2a = b + c$

Illustration 29 For what value of k the roots of $(k+1)^2 + 2(k+3)x + (2k+3) = 0$ are equal?

Solution

$(k+1)^2 + 2(k+3)x + (2k+3) = 0$

but $a = k+1$, $b = 2(k+3)$, $c = (2k+3)$

$\therefore \Delta = b^2 - 4ac = 0$

$\therefore 4(k+3)^2 - 4(k+1)(2k+3) = 0$

$\therefore 4(k^2 + 6k + 9) - 4(2k^2 + 5k + 3) = 0$

$\therefore 4k^2 + 24k + 36 - 8k^2 - 20k - 12 = 0$

$\therefore -4k^2 + 4k + 24 = 0$

$\therefore -4(k^2 - k - 6) = 0$

$\therefore k^2 - k - 6 = 0$

$\therefore (k-3)(k+2) = 0$

$\therefore k = 3$ or $k = -2$

Illustration 30 If the roots of $(x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a) = 0$ are equal then prove that $a = b = c$

Solution

$(x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a) = 0$

$\therefore x^2 - bx - ax + ab + x^2 - bx - cx + bc + x^2 - ax - cx + ac = 0$

$\therefore 3x^2 - 2(a+b+c)x + (ab+bc+ca) = 0$

As roots are equal

$\therefore \Delta = 0$

$\therefore b^2 - 4ac = 0$

$\therefore 4(a+b+c)^2 - 4(3)(ab+bc+ca) = 0$

$\therefore 4a^2 + 4b^2 + 4c^2 + 8ab + 8bc + 8ca - 12ab - 12bc - 12ca = 0$

$\therefore 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = 0$

$\therefore (a-b)^2 + (b-c)^2 + (c-a)^2 = 0$

$\therefore$ we can say that

$\therefore (a-b)^2 = 0 ; (b-c)^2 = 0 ; (c-a)^2 = 0$

$\therefore a - b = 0 ; b - c = 0 ; c - a = 0$

$\therefore a = b ; b = c ; c = a$

$\therefore a = b = c$

Illustration 31 If the ratio of the roots of the equation $x^2 - 5x + k = 0$ is 2 : 3 then find the value of k

Solution

Suppose the roots of $x^2 - 5x + k = 0$ are 2α and 3α

$\therefore 2\alpha + 3\alpha = \dfrac{-b}{a} = 5$

$\therefore 5\alpha = 5$

$\therefore \alpha = 1$

$$\text{and } \therefore (2\alpha)(3\alpha) = \frac{c}{a} = k$$

$$\therefore 6\alpha^2 = k$$

$$\therefore k = 6 \qquad (\because \alpha = 1)$$

Illustration 32 If one root of the equation $4x^2 - 8x + k = 0$ exceeds the other root by 1, then find the value of k

Solution

Suppose the roots are α and $\alpha + 1$

$$\therefore \text{Sum of the roots } \alpha + (\alpha + 1) = \frac{-b}{a}$$

$$\therefore 2\alpha + 1 = -\frac{(-8)}{4}$$

$$\therefore 2\alpha + 1 = 2$$

$$\therefore 2\alpha = 1$$

$$\therefore \alpha = \frac{1}{2}$$

$$\text{and product of the roots } = \frac{c}{a}$$

$$\therefore \alpha(\alpha + 1) = \frac{k}{4}$$

$$\therefore \frac{1}{2}\left(\frac{1}{2} + 1\right) = \frac{k}{4}$$

$$\frac{1}{2}\left(\frac{3}{2}\right) = \frac{k}{4}$$

$$\therefore \frac{k}{4} = \frac{3}{4}$$

$$\therefore k = 3$$

Illustration 33 If the ratio of the roots of $ax^2 + bx + c = 0$ is $m : n$ then prove that $mnb^2 = ac(m + n)^2$

Solution

Suppose the roots of $ax^2 + bx + c = 0$ are $m\alpha$ and $n\alpha$

$$\therefore \text{sum of the roots } = \frac{-b}{a}$$

$$\therefore m\alpha + n\alpha = \frac{-b}{a}$$

$$\therefore \alpha(m+n) = \frac{-b}{a}$$

$$\therefore \alpha = \frac{-b}{a(m+n)}$$

and product of the roots $= \dfrac{c}{a}$

$$\therefore (m\alpha)(n\alpha) = \frac{c}{a}$$

$$\therefore mn\alpha^2 = \frac{c}{a}$$

$$\therefore mn\left[\frac{-b}{a(m+n)}\right]^2 = \frac{c}{a}$$

$$\therefore \frac{mnb^2}{a^2(m+n)^2} = \frac{c}{a}$$

$$\therefore mnb^2 = \frac{ca^2(m+n)^2}{a}$$

$$\therefore mnb^2 = ac(m+n)^2$$

Illustration 34 If one root of $x^2 + 4x + k = 0$ is $-2 + i$ then find the value of k

Solution

In a quadratic equation if one root is complex the other root is its conjugate complex.

$\therefore$ other root is $-2 - i$

Now product of the roots $= \dfrac{c}{a}$

$$\therefore \alpha\beta = \frac{c}{a}$$

$$\therefore (-2+i)(-2-i) = \frac{k}{1}$$

$$\therefore (-2)^2 - (i)^2 = k$$

$$4 - (-1) = k \qquad (\because i^2 = 1)$$

$$\therefore k = 5$$

ANALYTICAL EXERCISES

1. Solve $\dfrac{4}{x+2} + \dfrac{2}{x+3} = \dfrac{3}{x+4}$
2. For what value of k the roots of the equation $4x^2 - 12x + k = 0$ are equal?
3. Prove that the roots of $(x - p)(x - q) = r^2$ are real

4. If α and β are the roots of the equation $x^2 - 2kx + 8 = 0$ and if $\alpha = 2\beta$ find the value of k

5. For what value of k the ratio of the roots of $x^2 + 3x - 6 = k(x-1)^2$ is 2 : 1 $(k \neq 1)$?

6. If the roots of $4x^2 + 4x - 3 = 0$ are α and β find the value of the following

(1) $\alpha^2 + \beta^2$ (2) $\alpha^3 + \beta^3$ (3) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

7. If the roots of $2x^2 - 4x + 1 = 0$ are α and β find the equation whose roots are $\alpha^2 + \beta$ and $\beta^2 + \alpha$

8. If the roots of $x^2 + x + 2 = 0$ are α, β construct equations with the following roots

(i) $\frac{\alpha}{\beta}; \frac{\beta}{\alpha}$ (ii) $\alpha + 2; \beta + 2$

(iii) $\alpha + \frac{1}{\beta}; \beta + \frac{1}{\alpha}$ (iv) $\frac{\alpha - 1}{\alpha + 1}; \frac{\beta - 1}{\beta + 1}$

9. If α, β are the roots of $x^2 - px + q = 0$ obtain an equation with root $\alpha\beta + \alpha + \beta$ and $\alpha\beta - \alpha - \beta$

10. If α, β are the roots of $x^2 - px + q = 0$ construct an equation with roots $\frac{q}{p - \alpha}$ and $\frac{q}{p - \beta}$

11. Solve $x^{2/5} - 5x^{1/5} + 6 = 0$

12. Solve $(x - 1)(x - 2) = 110$

13. Solve $x(x - p) = a(a + p)$

14. Solve $x + \frac{1}{x} = \frac{10}{3}$

15. Solve $x^2 - 2ax + 2ab - b^2 = 0$

16. Solve $\frac{3}{2x - 1} + \frac{2}{3x + 1} = \frac{1}{x + 3}$

17. Solve $x^2 - 22x + 201 = 0$

18. Solve $\sqrt{\frac{x}{1 - x}} + \sqrt{\frac{1 - x}{x}} = \frac{13}{6}$

19. Solve $x = \sqrt{6 + x}$

20. Solve $x^{2/3} + x^{1/3} - 2 = 0$

21. Solve $x^{-4} + 4 = \frac{5}{x^2}$

22. Solve $\frac{1}{x + 7} + \frac{1}{x + 2} = \frac{1}{x + 3}$

23. Solve $x^2 = px + 1$

24. For what value of k the root of the equation $4x^2 + 12x + k - 1 = 0$ are equal?

25. Prove that the roots of the equation $x^2 - 2mx + m^2 + n^2 = 0$ are complex

26. If the root of $x^2 + a^2 = 2(a + 1)x$ are equal find the value of a

27. Prove that the roots of the equation $(b + c)x^2 - (a + b + c)x + a = 0$ are rational

28. If the roots of $(p^2 + q^2)x^2 - 2q(p + r)x + (q^2 + r^2) = 0$ are equal then prove that $q^2 = pr$

29. Construct an equation with the roots $\frac{p+qi}{r}$ and $\frac{p-qi}{r}$

30. If α and β are the roots of $ax^2 + bx + c = 0$ construct an equation with the roots $\frac{\alpha}{\beta};\frac{\beta}{\alpha}$

31. If α, β are the roots of $x^2 - 2x + 3 = 0$ construct an equation with the roots $\frac{\alpha-1}{\alpha+1};\frac{\beta-1}{\beta+1}$

32. If α, β are the roots of $x^2 - px + q = 0$ construct an equation with the roots $2\alpha + 3\beta$ and $3\alpha + 2\beta$

33. If α,β are the roots of $ax^2 + bx + c = 0$ construct an equation with the roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$

34. If α,β are the roots of $ax^2 + bx + c = 0$ obtain an equation with the roots $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$

35. If the roots of $x^2 + 7x + k^2 = 0$ are equal then find the value of k

36. If α β, are the roots of $ax^2 + 2bx - a = 0$ construct an equation with the roots $2\alpha + \frac{1}{\beta}$ and $2\beta + \frac{1}{\alpha}$

37. If r is the ratio of the roots of the equation $ax^2 + bx + c = 0$ prove that $b^2r = ac(r + 1)^2$

38. If the roots of the equation $\frac{x(x-6)}{k(2k-1)} = (1 - x)$ are equal in magnitude and opposite in sign then find k

39. Obtain condition that the ratio of the roots of the equation $ax^2 + bx + c = 0$ is 2 : 3

40. If the sum of the roots of the equation $\frac{1}{x-a} + \frac{1}{x-b} = \frac{2}{c}$ is zero then prove that the product of the roots is $\frac{-1}{2}(a^2 + b^2)$

41. If roots of the equation $(2 + m)x^2 + 2(m - 2)x = 25$ are equal in magnitude and opposite in sign then find m

42. If the roots of $7x^2 - 5x + k = 0$ are reciprocal of each other then find k

43. If the ratio of the roots of $ax^2 + bx + c = 0$ is 3 : 4 then prove that $12b^2 = 49ac$

44. If one root of the equation $2x^2 - 6x + k - 1 = 0$ is $\frac{1}{2}(a + 5i)$ then find the value of a and k

45. If the difference of the roots of the equation $x^2 - 3x = 5 - k$ is 4 then find the value of k

46. Construct the equations having following roots

(i) $3 + \sqrt{5}, 3 - \sqrt{5}$ (ii) $2 + \sqrt{2}i, 2 - \sqrt{2}i$

47. Solve $6\sqrt{\frac{x-3}{3x-7}} + 5\sqrt{\frac{3x-7}{x-3}} = 13$

48. Solve $\left(\frac{x+2}{x-2}\right) + \left(\frac{x-2}{x+2}\right) = \frac{5}{2}$ $(\text{where } x \neq -2, 2)$

49. Solve $\left(\frac{7x-1}{x}\right)^2 + 3\left(\frac{7x-1}{x}\right) = 18$

50. Solve $\left(\frac{x}{x+1}\right)^2 - 5\left(\frac{x}{x+1}\right) + 6 = 0$

51. Solve $2\left(x^2 + \frac{1}{x^2}\right) + \left(x - \frac{1}{x}\right) - 10 = 0$

52. Solve $2\left(x^2 + \frac{1}{x^2}\right) - 3\left(x + \frac{1}{x}\right) - 1 = 0$

53. $y(y + 1)(y + 3)(y + 4) + 2 = 0$

54. $(x^2 - 5x + 7)^2 - (x - 2)(x - 3) = 1$

55. Solve $\frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}} = \frac{a}{x}$ $(\because x \neq 0)$

56. If $x^2 - 6x + a = 0$ has two roots α, β and $3\alpha + 2\beta = 20$ then find a

57. If $x^2 - 3x + 2$ is a factor of $x^4 - px^2 + q = 0$ then find the value of p and q

58. In the equation $k(x - 1)^2 = 5x - 7$ if one root is twice the other root then find k

59. If one root of the equation $kx^2 + 2x + 6 = 0$ is 1 then find k and the second root

60. If one root of the equation $2x^2 + 3kx + 18 = 0$ is -6 then find k and second root

61. If one root of the equation $x^2 - 6x + k = 0$ is $3 - 2i\,(k \in R)$ then find k and second root

62. If one root of the equation $ax^2 - 6x + c + 9 = 0$ is $3 + 5i$ and $a, c \in R$ then find a, c and second root

63. Construct the quadratic equations having the following roots

(i) $\frac{3+\sqrt{2}}{3-\sqrt{2}}; \frac{3-\sqrt{2}}{3+\sqrt{2}}$ (ii) $\frac{2+\sqrt{3}i}{2-\sqrt{3}i}; \frac{2-\sqrt{3}i}{2+\sqrt{3}i}$

64. Construct a quadratic equation whose roots are inverse of the roots of the equation $px^2 + qx - r = 0$; $(r \neq 0)$

65. If α , β are the roots of the equation $x^2 - 5x + 2 = 0$ then prove that the quadratic equation with roots $\left(\frac{\alpha+1}{\alpha-1}\right)$ and $\left(\frac{\beta+1}{\beta-1}\right)$ is $x^2 + x - 4 = 0$

66. If α and β are the roots of the equation $x^2 + 4x + 15 = 0$ then prove that the roots of the equation $x^2 - 11 = 0$ are $\alpha + 2$ and $\beta + 2$

67. If one root of the equation $x^2 - 9x + k = 0$ is twice the other root then find k

68. If the roots of the equation $x^2 - (k + 10)x + 9(k + 1) = 0$ are equal then find the value of k

69. If the roots of the equation $x^2 - 4kx + 6k = 0$ is 3 : 1 then find the value of k

70. If product of the roots of the equation $kx^2 + (k + 1)x + 2k + 1 = 0$ is 3 then find the value of k

71. Solve $(-a - b + 2c)x^2 + (-b - c + 2a)x + (-c - a + 2b) = 0$

72. Solve $(x - 2a)(x - 2b) = c^2 - (a - b)^2$

73. For which value of k the roots of the equation $5x^2 - 2kx + 5 = 0$ are (i) Real and unequal, (ii) Real and equal, (iii) Complex numbers?

74. If a is a real number and the roots of the equation $9x^2 + 49x + 4 = 0$ are complex numbers then show that a lies between -3 and 3

75. For which value of a the roots of the equation $x^2 + a^2 + 8x + 6a$ are
(1) Real and unequal, (2) Real and equal, (3) Complex numbers?

76. If a is a real number and the roots of the equation $x^2 - 4x - 20 = a(x - 7)$ are also real then show that $a \le 8$ or $a \ge 12$

77. For which value of a the roots of the equation $3x^2 + 2(a^2 + 1)x + (a^2 - 3a + 2) = 0$ have unlike signs?

78. If α and β are the roots of the equation $x^2 + px + q = 0$ then prove that the quadratic equation having roots $(\alpha - \beta)^2$ and $(\alpha + \beta)^2$ is $x^2 - (p^2 - 2q)x + p^2(p^2 - 4q) = 0$

79. If α and β are the roots of the equation $ax^2 + bx + c = 0$ and $\alpha + k$ and $\beta + k$ are the roots of the equation $px^2 + qx + r = 0$ then prove that

$$\frac{b^2 - 4ac}{q^2 - 4pr} = \left(\frac{q}{p}\right)^2$$

80. If one root of the equation $x^2 + acx + b = 0$ is equal then prove that $a^2(b + c) + 1 = 0$

81. M and N are the roots of the equation $x^2 + px + q = 0$. Also P and Q are the roots of the equation $x^2 + mx + n = 0$. If $\frac{M}{N} = \frac{P}{Q}$ then prove that $m^2q = np^2$

82. If the sum of the roots and the sum of the inverse of the square of the roots of the equation $ax^2 + bx + c = 0$ are equal then prove that $\frac{c}{a}, \frac{a}{b}, \frac{b}{c}$ are in arithmetic progression

83. Solve $x^2 - 5x + 7 < 0$

84. Solve $x^2 + 6x + 9 = 0$

85. Solve $2 - 4x - x2 > 0$

86. Solve $3x^2 + x + 1 > 0$

87. In equation $x^2 + mx + n = 0$, if the co-efficient of x is taken as $m + 4$ instead of m then roots of the equation are obtained as -2 and -15. Find the roots of the original equation

88. If a, b, c are real numbers and $a + b + c = 0$ then prove that the roots of $4ax^2 + 3bx + 2c = 0$ are real numbers

89. If α and β are the roots of the equation $x^2 + px - 9 = 0$ and γ and δ are the roots of the equation $x^2 + px + r = 0$ then prove that $(\alpha-\gamma)(\alpha-\delta) = (\beta-\gamma)(\beta-\delta) = q + r$

90. If one root of the equation $ax^2 + bx + c = 0$ is nth power of the other root then prove that $(ac^n)\frac{1}{n+1} + (a^nc)\frac{1}{n+1} + b = 0$

91. If α and β are the roots of the equation $x^2 + px + 1 = 0$ and γ and δ are the roots of the equation $x^2 + qx + 1 = 0$ then prove that $q2 - p^2 = (\alpha-\gamma)(\beta-\gamma)(\alpha-\delta)(\beta-\delta)$

ANSWERS

(1) $\alpha = \frac{-25 + \sqrt{73}}{6}$; $\beta = \frac{-25 - \sqrt{73}}{6}$

(2) $k = 9$

(4) $k = \pm 3$

(5) $k = -24$ or 3

(6) (1) $\frac{5}{2}$; (2) $\frac{-12}{4}$; (3) $\frac{-10}{3}$

(7) $4x^2 - 20x + 23 = 0$

(8) (i) $2x^2 + 3x + 2 = 0$;
(ii) $x^2 - 3x + 4 = 0$;
(iii) $2x^2 + 3x + 9 = 0$;
(iv) $x^2 - x + 2 = 0$

(9) $x^2 + 2qx + (q^2 - p^2) = 0$

(10) $x^2 - px + q = 0$

(11) $x = 243$ or $x = 32$

(12) 12, −9

(13) $a + p, -a$

(14) $3, \frac{1}{3}$

(15) b ; $2a - b$

(16) $\frac{-41 \pm \sqrt{1569}}{14}$

(17) $11 \pm 4\sqrt{5}\,i$

(18) $\frac{4}{13}$; $\frac{9}{13}$

(19) $3, -2$

(20) $-8, 1$

(21) $\pm 1, \pm\frac{1}{2}$

(22) $-3 \pm 2i$

(23) $\frac{p \pm \sqrt{p^2 + 4}}{2}$

(24) $k = 10$

(26) $a = -\frac{1}{2}$

(29) $r^2x^2 - 2prx + p^2 + q^2 = 0$

(30) $acx^2 - (b^2 - 2ac)x + ac = 0$

(31) $3x^2 - 2x + 1 = 0$

(32) $x^2 - 5px + 6p^2 + q = 0$

(33) $cx^2 + bx + a = 0$

(34) $c^2x^2 - \left(b^2 - 2ac\right)x + a^2 = 0$

(35) $k = \pm\frac{7}{2}$

(36) $ax^2 + 2bx - a = 0$

(38) $k = 2, \frac{-3}{2}$

(39) $6b^2 = 25ac$

(41) $m = 2$

(42) $k = \pm\frac{3}{4}$

(44) $a = 3, k = 18$

(45) $k = 18$

(46) (i) $x^2 - 6x + 4 = 0$;

(ii) $x^2 - 4x + 6 = 0$

(47) 5, $\frac{74}{32}$

(48) 6 , -6

(49) $\frac{1}{4}, \frac{1}{13}$

(50) $\frac{-3}{2}, -2$

(51) $\frac{-1}{2}, 2, -1 \pm \sqrt{2}$

(52) $\frac{1}{2}, 2, \frac{-1 \pm \sqrt{3}i}{2}$

(53) $-2 \pm \sqrt{2}, -2 \pm \sqrt{3}$

(54) $2, 3, \frac{5 \pm \sqrt{3}i}{2}$

(55) $\pm a$

(56) $a = -16$

(57) $p = 5, q = 4$

(58) $k = -25$ or 2

(59) $k = -8$, second root $= \frac{-3}{4}$

(60) $k = 5$, second root $= \frac{-3}{2}$

(61) $k = 13$, second root $= 3 + 2i$

(62) $a = 1, c = 25$, second root $= 3 - 5i$

(63) (i) $7x^2 - 22x + 7 = 0$;

(ii) $7x^2 - 2x + 7 = 0$

(67) $k = 18$

(68) $k = 8$

(69) $k = 2$

(70) $k = 1$

(71) 1, $\frac{-c - a + 2b}{-a - b + 2c}$

(72) $\left(a + b\right) \pm c$

(73) (i) $k < -5$ or $k < 5$;

(ii) $k \pm 5$

(iii) $-5 < k < 5$

(75) (1) $-2 < a < 8$;

(2) $a = -2$ or $a = 8$;

(3) $a < -2$ or $a < 8$

(77) $1 < a < 2$

(83) ϕ

(84) $R - \left(-3\right)$

(85) $\left(x \in R / -2 - \sqrt{6} < x < -2 + \sqrt{6}\right)$

(86) R

(87) -10

4

Complex Numbers

LEARNING OBJECTIVES

This chapter will enable you to learn the concepts and application of:

- What is Complex number.
- Advance application of complex number

INTRODUCTION

Remember

N is closed for addition and multiplication. The solution of $x + 5 = 3$ is not contained in *N* but in *Z*.

Solution of $5x = 3$ is not contained in *Z* but in *Q*.
Q does not cover the number line completely, *R* does.
Solution of $x^2 = 2$ is not contained in *Q* but in *R*.
Solution of $x^2 + 1 = 0$ is not contained in *R* but in *C*.

($\because$ *C* is a field)

SET OF COMPLEX NUMBERS IN *C*

If for the numbers (a, b) and (c, d) of $R \times R$, equality, addition and multiplication are defined as under then $R \times R$ is known as the set of complex numbers *C*.

Equality: $(a,b) = (c,d) \Leftrightarrow a = c \, ; b = d$

Addition: $(a,b) + (c,d) \Leftrightarrow (a + c \, ; b + d)$

Multiplication: $(a,b)/(c,d) \Leftrightarrow (ac - bd \, ; ad + bc)$

$C = [(a,b) / a \in R \, ; b \in R]$

C is a field because it satisfies the following properties of a field

(**where** $\alpha + \beta \, , \gamma \in c$)

Laws of Addition

(1) $\alpha + \beta\,, \gamma \in c$ (Closure)
(2) $\alpha + \beta = \beta + \alpha$ (Commutativity)
(3) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ (Associativity)
(4) $(0\,, 0) \in c$ (Additive identity)
(5) For $(a\,, b) \in c$ we get additive/inverse $(-a\,, -b) \in c$ (Additive inverse)

Laws of Equality

(1) $\alpha\beta \in c$ (Closure)
(2) $\alpha\,\beta = \beta\alpha$ (Commutativity)
(3) $(\alpha\,\beta)\gamma = \gamma(\beta\alpha)$ (Associativity)
(4) $(1,0) \in c$ (Multiplicative identity)
(5) Multiplicative inverse of $(a,b) \in c - (0,0)$ is

$$\left(\frac{a}{a^2 + b^2}\,;\,\frac{-b}{a^2 + b^2}\right) \in c \text{ (Multiplicative inverse)}$$

MULTIPLICATION IS DISTRIBUTIVE OVER ADDITION

$\alpha(\beta + \gamma) = \alpha\,\beta + \alpha\,\gamma$

$R_1 = [(a,0) / a \in R]$ is $R(a,0) = a$ thus $R \subset C$

COMPLEX NUMBER

$$(0,\ 1) = i$$
$$\therefore i^2 = (0,1)(0,1) = (-1,0) = -1$$
$$\therefore i^2 = -1$$

COMPLEX NUMBER

$$\begin{aligned} z &= (a,\ b) \\ &= (a,\ 0) + (0,\ b) \\ &= (a,\ 0) + (0,\ 1)(0,\ b) \\ &= a + ib \end{aligned}$$
$$\therefore z = a + ib$$

Here a is known as the real part and b is known as the imaginary part; symbolically $a = Re(z)$, and $b = Im(z)$

$$\therefore z = Re(z) + I_m(z)$$

A complex number with its real part zero and imaginary part non-zero is called 0, purely imaginary part. For example, $ai = 0 + ai$; $i - 7i$ are purely imaginary numbers

Geometric representation of a complex number is called the Argand diagram. Complex number

$$\begin{aligned} z &= x + iy \\ &= (x, y) \end{aligned}$$

is associated with a unique point (x, y) in the coordinate plane.

Thus there is one-to-one correspondence between the set c of complex numbers and the point of the coordinate plane. This plane is called a complex plane (Argand plane).

ILLUSTRATIONS

Illustration 1 Find the complex number β that satisfies the following equations

(i) $(5, -2) + \beta = (-3, 5)$ (ii) $(3, 2)\beta = (1, 0)$

Solution

(i) Suppose the required complex number $\beta = (x, y)$

$\therefore (5, -2) + \beta = (-3, 5)$

$\therefore (5, -2) + (x, y) = (-3, 5)$

$\therefore (5 + x, -2 + y) = (-3, 5)$ (Addition)

$\therefore 5 + x = -3$ and $\therefore -2 + y = 5$ (Equality)

$\therefore x = -8$ and $\therefore y = 7$

Thus $\beta = (x, y) = (-8, 7)$

(ii) Suppose the required complex number $\beta = (x, y)$

$\therefore (3, 2)\beta = (1, 0)$

$\therefore (3, 2)(x, y) = (1, 0)$

$\therefore (3x - 2y, 3y + 2x) = (1, 0)$ (Multiplication)

$\therefore 3x - 2y = 1$ and $\therefore 3x + 2x = 0$ (Equality)

$\therefore$ Solving these equations $x = \frac{3}{13}$ and $y = \frac{-2}{13}$

Thus, $\beta(x, y) = \left(\frac{3}{13}; \frac{-2}{13}\right)$

Illustration 2 Find the multiplicative inverse of the following complex numbers

(i) $(1/2, 1/2)$ (ii) $(0, 1)$

Solution

(i) Multiplicative inverse of $\left(1/2, 1/2\right)$

$$= \left[\frac{1/2}{\left(1/2\right)^2 + \left(1/2\right)^2} ; \frac{-1/2}{\left(1/2\right)^2 + \left(1/2\right)^2}\right]$$

$$= \left(\frac{1/2}{1/2} ; \frac{-1/2}{1/2}\right) = \left(1 , -1\right)$$

(ii) Multiplicative inverse of $\left(0, 1\right)$

$$= \left(\frac{0}{0+1} ; \frac{-1}{0+1}\right)$$

$$= \left(0 , -1\right)$$

Illustration 3 Express the following complex numbers in form of $x + iy$

(i) $\frac{1}{i} + \frac{2}{i^2} + \frac{3}{i^3}$ (ii) $\left(\frac{3+i}{1-i}\right)^2$

Solution

(i) $\frac{1}{i} + \frac{2}{i^2} + \frac{3}{i^3}$

$$= \frac{i}{i^{-2}} + \frac{/2}{i^{-2}} + \frac{3i}{i^{-4}}$$

$$= \frac{i}{-1} + \frac{2}{-1} + \frac{3i}{/1} \qquad \left(\because i^2 = -1\right)$$

$$= -i - 2 + 3i$$

$$= -2 + 2i$$

(ii) $$\left(\frac{3+i}{1-i}\right)^2 = \frac{9 + 6i + i^2}{1 - 2i + i^2}$$

$$= \frac{9 + 6i + -1}{1 - 2i - 1}$$

$$= \frac{8 + 6i}{-2i}$$

$$= \frac{8 + 6i}{-2i}\left(\frac{2i}{2i}\right)$$

$$= \frac{16i + 12i^2}{-4i^2}$$

$$= \frac{16i - 12}{4} \quad (\because i^2 = -1)$$

$$= -3 + 4i$$

Illustration 4 If $(1 + i)^{10} + (1 - i)^{10} = a + ib$; $(a, b \in R)$ then find a and b

Solution

$$(1+i)^{10} + (1-i)^{10} = a + ib$$

$$\Rightarrow \left[(1+i)^2\right]^5 + \left[(1-i)^2\right]^5 = a + ib$$

$$\Rightarrow (1 + 2i + i^2)^5 + (1 - 2i + i^2)^5 = a + ib$$

$$\Rightarrow (1 + 2i - 1)^5 + (1 - 2i - 1)^5 = a + ib$$

$$\Rightarrow 32i^5 - 32i^5 = a + ib$$

$$\Rightarrow 0 = a + ib$$

$$\Rightarrow a = 0 \text{ and } b = 0$$

Illustration 5 If $Z = 1 + \sqrt{3}i$ then find real and imaginary part of the following numbers

(i) $\frac{1}{Z}$ (ii) Z^2

Solution

(i) $$\frac{1}{Z} = \frac{1}{1 + \sqrt{3}i}$$

$$= \frac{1}{1 + \sqrt{3}i}\left(\frac{1 - \sqrt{3}i}{1 - \sqrt{3}i}\right)$$

$$= \frac{1 - \sqrt{3}i}{1 - \sqrt{/3}i^2}$$

$$= \frac{1 - \sqrt{3}i}{4}$$

$$= \frac{1}{4} + \left(\frac{-\sqrt{3}}{4}\right)i$$

$$\therefore \text{Re}\left(\frac{1}{z}\right) = \frac{1}{4} \text{ and } I_m\left(\frac{1}{z}\right) = \frac{-\sqrt{3}}{4}$$

(ii) $Z^2 = \left(1+\sqrt{3}i\right)^2$

$$= 1 + 2\sqrt{3}i + 3i^2$$

$$= 1 + 2\sqrt{3}i - 3$$

$$= -2 + 2\sqrt{3}i$$

$\therefore$ Re $\left(Z^2\right) = -2$ and $I_m\left(Z^2\right) = 2\sqrt{3}$

Illustration 6 Find the magnitude of the complex number $\dfrac{(1+i)^4}{\left(\sqrt{7}-i\right)^2}$

Solution

First we will express $\dfrac{(1+i)^4}{\left(\sqrt{7}-i\right)^4}$ in the form of $x + iy$

$$\frac{(1+i)^4}{\left(\sqrt{7}-i\right)^2} = \frac{\left(1+2i+i^2\right)^2}{7-2\sqrt{7}i+i^2}$$

$$= \frac{(2i)^2}{6-2\sqrt{7}i}$$

$$= \frac{4i^2}{6-2\sqrt{7}i}$$

$$= \frac{-2}{3-\sqrt{7}i}$$

$$= \frac{-2}{3-\sqrt{7}i}\left(\frac{3+\sqrt{7}i}{3+\sqrt{7}i}\right)$$

$$= \frac{-6-2\sqrt{7}i}{9-7i^2}$$

$$= \frac{-6-2\sqrt{7}i}{16}$$

$$= \frac{-3}{8} - \frac{\sqrt{7}}{8}i$$

$\therefore x = \dfrac{-3}{8}$; $y = \dfrac{-\sqrt{7}}{8}$

$\therefore$ Magnitude $= r = |z| = \sqrt{x^2 + y^2}$

$$= \sqrt{\frac{9}{64} + \frac{7}{64}}$$

$$= \sqrt{\frac{16}{64}}$$

$$= \frac{1}{2}$$

Illustration 7 Find the conjugate complex numbers of

(i) $\frac{1}{2+i}$ (ii) $4i - 3$

Solution

(i) $$\frac{1}{2+i} = \frac{1}{2+i}\left(\frac{2-i}{2-i}\right)$$

$$= \frac{2-i}{4-i^2}$$

$$= \frac{2-i}{5}$$

$$= \frac{2}{5} - \frac{i}{5}$$

So conjugate complex of $\frac{2}{5} - \frac{i}{5}$ is $\frac{2}{5} + \frac{i}{5}$

(ii) Conjugate complex numbers of $- 3 + 4i$ is $- 3 + 4i$

Illustration 8 Show that in the argand diagram, the complex numbers $-2 + 3i$, $-2 - i$ and $4 - i$ form the vertices of a right angled triangle

Solution

Suppose P, Q and R be respectively $Z_1 = - 2 + 3i$, $Z_2 = - 2 - i$ and $Z_3 = 4 - i$

Thus

$$PQ = |Z_1 - Z_2|$$

$$= |-2 + 3i + 2 + i|$$

$$= |4i| = 4$$

$$QR = |-Z_2 - Z_3|$$

$$= |-2 - i - 4 + i|$$

$$= |-6| = 6$$

and $$PR = |Z_1 - Z_3|$$

$$= |-2 + 3i - 4 + i|$$

$$= |-6 + 4i|$$
$$= \sqrt{6^2 + 4^2} = \sqrt{52}$$

$\therefore PQ^2 + QR^2 = 4^2 + 6^2 = 52 = PR^2$

Thus ΔPQR is a right angled triangle with $\overline{PR}$ as the hypotenuse that is $m < Q = 90^{o}$

Thus we can say that complex numbers $-2 + 3i$, $-2 - i$ and $4 - i$ represent the vertices of a right angled triangle.

Illustration 9 Obtain the solution in $C = x^2 + x + 1 = 0$

Solution

Let $x^2 + x + 1 = 0$ be a quadratic equation

$\therefore a = b = c = 1$

$\therefore \Delta = b^2 - 4ac = 1 - 4 = -3$

$\therefore \sqrt{\Delta} = \sqrt{-3} = \sqrt{3}i$

$$\therefore x = \frac{-b \pm \sqrt{\Delta}}{2a}$$
$$= \frac{-1 \pm \sqrt{3}i}{2}$$
$$\therefore \alpha = \frac{-1 + \sqrt{3}i}{2} \text{ or } \beta = \frac{-1 - \sqrt{3}i}{2}$$

Thus the solution set is $x = \dfrac{-1 + \sqrt{3}i}{2} ; \dfrac{-1 - \sqrt{3}i}{2}$

Illustration 10 Prove that $\dfrac{\sqrt{3} + \sqrt{2}i}{\sqrt{3} - \sqrt{2}i} + \dfrac{\sqrt{3} - \sqrt{2}i}{\sqrt{3} + \sqrt{2}i} \in Q$

Solution

$$\frac{\sqrt{3} + \sqrt{2}i}{\sqrt{3} - \sqrt{2}i} + \frac{\sqrt{3} - \sqrt{2}i}{\sqrt{3} + \sqrt{2}i}$$
$$= \frac{\left(\sqrt{3} + \sqrt{2}i\right)^2 + \left(\sqrt{3} - \sqrt{2}i\right)^2}{\left(\sqrt{3}\right)^2 - \left(\sqrt{2}i\right)^2}$$
$$= \frac{3 + 2\sqrt{6}i + 2i^2 + 3 - 2\sqrt{6}i + 2i^2}{3 - 2i^2}$$
$$= \frac{6 - 2 - 2}{3 + 2}$$
$$= \frac{2}{5} \in Q$$

Illustration 11 If $Z = x + iy$ and $|4z| = |3z - 1|$ then prove that $7(x^2 + y^2) + 6x = 1$

Solution

Here $|4z| = |3z - 1|$

$\Rightarrow |4(x + iy)| = |3(x + iy) - 1|$

$\Rightarrow |4x + 4yi| = |(3x - 1) + 3yi|$

$\Rightarrow \sqrt{(4x)^2 + (4y)^2} = \sqrt{(3x - 1)^2 + (3y)^2}$

$\Rightarrow \sqrt{16x^2 + 16y^2} = \sqrt{9x^2 - 6x + 1 + 9y^2}$

$\Rightarrow 16x^2 + 16y^2 = 9x^2 - 6x + 1 + 9y^2 \quad (\because \text{squaring})$

$\Rightarrow 7x^2 + 7y^2 + 6x = 1$

$\Rightarrow 7(x^2 + y^2) + 6x = 1$

Illustration 12 For which complex numbers $\bar{z} = |z|$?

Solution

Suppose complex number = $z = x + iy$
$\therefore \bar{z} = x - iy$
$\therefore |z| = \sqrt{x^2 + y^2}$
Comparing the imaginary part $b = 0$ and $a \in R^+$
$z = a + ib = a + i(0) = a \in R^+$

For positive real number z, that is, numbers of the type $(a, 0)$ $\bar{z} = |z|$ where $a \in R^+ \cup \{0\}$

Illustration 13 If $|z| \le 3$ then find the maximum and minimum value of $|z - 5|$

Solution

$|z - 5| \le |z| + |-5| \qquad \because |z_1 + z_2| \le |z_1| + |z_2|$

$\le 3 + 5$

≤ 8

$\therefore$ The value of $|z - 5|$ cannot be greater then 8

Now $|2 - 5| \ge ||z| - |5||$

$\ge |3 - 5| \qquad (\because |z_1 - z_2| = ||z_1| - |z_2||)$

$\therefore$ The value of $|z - 5|$ cannot be less than 2

$\therefore$ The minimum value of $|z - 5|$ is 2

Illustration 14 If $\left|z - z\right| \le 5$ then find the maximum and minimum value of $\left|z - 5 + 4i\right|$

Solution

$$\left|z - 5 + 4i\right|$$

$$= \left|\left(z - 2\right) + \left(-3 + 4i\right)\right|$$

$$\le \left|z - 2\right| + \left|-3 + 4i\right|$$

$$\le 5 + \sqrt{9 + 16}$$

$$\le 10$$

$\therefore$ The value of $\left|z - 5 + 4i\right|$ cannot be greater than 10

$\therefore$ The maximum value of $\left|z - 5 + 4i\right|$ is 10

Now $\left|z - 5 + 4i\right|$

$$= \left|z - 2 - 3 + 4i\right|$$

$$= \left|\left|z - 2\right| - \left|3 - 4i\right|\right|$$

$$\ge \left|\left|z - 2\right| - \left|3 - 4i\right|\right|$$

$$\ge 5 - \sqrt{9 + 16}$$

$$\ge 0$$

$\therefore$ The value of $\left|z - 5 + 4i\right|$ cannot be less than 0

$\therefore$ The maximum value of $\ge \left|z - 5 + 4i\right|$ is 0

Illustration 15 For any $z_1, z_2 \in c$ prove that

$$\left|z_1 + z_2\right|^2 + \left|z_1 - z_2\right|^2 = 2\left(\left|z_1\right|^2 + \left|z_2\right|^2\right)$$

Solution

$$\left|z_1 + z_2\right|^2 + \left|z_1 - z_2\right|^2$$

$$= \left(z_1 + z_2\right)\left(\overline{z_1 + z_2}\right) + \left(z_1 - z_2\right)\left(\overline{z_1 - z_2}\right)$$

$$= \left(z_1 + z_2\right)\left(\overline{z_1} + \overline{z_2}\right) + \left(z_1 - z_2\right)\left(\overline{z_1} - \overline{z_2}\right)$$

$$\left(z_1\overline{z_1} \quad \overline{z_1}\,\overline{z_2} + \overline{z_2}\,\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_1} - z_1\overline{z_2} - z_2\overline{z_1} + z_2\overline{z_2}\right)$$

$$= 2z_1\overline{z_1} + 2z_2\overline{z_2}$$

$$= 2\left(\left|\left|z_1\right|^2 + \left|z_2\right|^2\right|\right)$$

Illustration 16 For any $z_1 , z_2 \in c$ prove that $z_1\overline{z_2} + \overline{z_1}z_2 \le 2|z_1||z_2|$

Solution

Here $z_1\overline{z_2} + \overline{z_1}z_2$

$= z_1\overline{z_2} + \overline{z_1 z_2}$

$= 2\text{Re}\left(z_1\overline{z_2}\right)$ $\quad\left[\because z + \overline{z} = 2\text{Re}(z)\right]$

But $\text{Re}(z) \le |z|$

$\text{Re}\left(z_1 \cdot \overline{z_2}\right) \le 2\left|z_1\overline{z_2}\right|$

$z_1\overline{z_2} + \overline{z_1}z_2 \le 2\left|z_1\overline{z_2}\right|$

$\le 2\|z_1\|\overline{z_2}\|$

$\le 2|z_1||\overline{z_2}|$ $\quad\left(\because |z_2| = |\overline{z_2}|\right)$

$\therefore z_1\overline{z_2} + \overline{z_1}z_2 \le 2|z_1||z_2|$

$\therefore z_1\overline{z_2} + \overline{z_1}z_2 \le 2\|z_1\|z_2\|$

Illustration 17 If $x + iy = \sqrt{\dfrac{a+ib}{c+id}}$ then prove that $\left(x^2 + y^2\right)^2 = \dfrac{a^2 + b^2}{c^2 + d^2}$; where $a, b, c, d \in R$

Solution

Here $x + iy = \sqrt{\dfrac{a+ib}{c+id}}$

$$\Rightarrow (x + iy)^2 = \frac{a+ib}{c+id}$$

$$\Rightarrow x^2 + 2xyi + i^2y^2 = \frac{a+ib}{c+id}\left(\frac{c-id}{c-id}\right)$$

$$= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

$$\Rightarrow \left(x^2 - y^2\right) + 2xyi = \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{bc-ad}{c^2+d^2}\right)i$$

$$\Rightarrow x^2 - y^2 = \frac{ac+bd}{c^2+d^2}\,;\ 2xy = \frac{bc-ad}{c^2+d^2}$$

Now $\left(x^2 + y^2\right)^2 = \left(x^2 - y^2\right)^2 + 4x^2y^2$

$$= \left(\frac{ac+bd}{c^2+d^2}\right)^2 + \left(\frac{bc-ad}{c^2+d^2}\right)^2$$

$$= \frac{a^2c^2 + b^2d^2 + 2abcd + b^2c^2 + a^2d^2 - 2abcd}{\left(c^2 + d^2\right)^2}$$

$$= \frac{c^2\left(a^2 + b^2\right) + d^2\left(a^2 + b^2\right)}{\left(c^2 + d^2\right)^2}$$

$$= \frac{\left(a^2 + b^2\right)\left(c^2 + d^2\right)}{\left(c^2 + d^2\right)^2}$$

$$\Rightarrow \left(x^2 + y^2\right)^2 = \frac{a^2 + b^2}{c^2 + d^2}$$

Illustration 18 If $a - ib = \frac{1 - ix}{1 + ix}$ then show that $a^2 + b^2 = 1$ (where $a, b, x \in R$)

Solution

Here $\left|\frac{1 - ix}{1 + ix}\right| = |a - ib|$

$$\Rightarrow \frac{\sqrt{1 + x^2}}{\sqrt{1 + x^2}} = \sqrt{a^2 + b^2}$$

$$\Rightarrow \frac{1 + x^2}{1 + x^2} = a^2 + b^2$$

$$\Rightarrow a^2 + b^2 = 1$$

Illustration 19 If $\left(\frac{1 - i}{1 + i}\right)^{100} = a + ib$ then find the value of a and b.

Solution

$$\left(\frac{1 - i}{1 + i}\right)^{100} = a + ib$$

$$\Rightarrow \left[\left(\frac{1 - i}{1 + i}\right)^2\right]^{50} = a + ib$$

$$\Rightarrow \left(\frac{1 - 2i + i^2}{1 + 2i + i^2}\right)^{50} = a + ib$$

$$\Rightarrow (-1)^{50} = a + ib$$

$$\Rightarrow 1 = a + ib$$

$$\Rightarrow 1 + 0i = a + ib$$

$$\Rightarrow a = 1\,;\, b = 0$$

Illustration 20 If $x = 3 + 2i$ then show that $x^2 - 6x + 13 = 0$ and from this find the value of $x^4 - 4x^3 + 6x^2 - 4x + 72$

Solution

$$x = 3 + 2i$$
$$\Rightarrow (x-3) = 2i$$
$$\Rightarrow (x-3)^2 = (2i)^2$$
$$\Rightarrow x^2 - 6x + 9 = 4i^2$$
$$\Rightarrow x^2 - 6x + 9 = -4$$
$$\Rightarrow x^2 - 6x + 13 = 0$$
$$\text{Now } x^4 - 4x^3 + 6x^2 - 4x + 72$$
$$= x^2(x^2 - 6x + 13) + 2x^3 - 7x^2 - 4x + 72$$
$$= x^2(x^2 - 6x + 13) + 2x(x^2 - 6x + 13) + 5(x^2 - 6x + 13) + 7$$
$$= x^2(0) + 2x(0) + 5(0) + 7$$
$$= 7$$

ANALYTICAL EXAMPLES

1. Find the complex number β that satisfies the following equations

(i) $(7, 3) + \beta = (0, 0)$

(ii) $(2, -3) + \beta = (1, 1)$

(iii) $(3, 2) \cdot \beta = (-18, 14)$

(iv) $[(6, 5)(6, 5)] \cdot \beta = (1, 0)$

2. Find a, b satisfying $(3b - 6, 8 - 4a) = (0, 0)$

3. Find x, y satisfying $(a, b)(x, y) = (1, 0)$ where $(a, b) \neq (0, 0)$

4. Find the multiplicative inverse of the following complex numbers

(i) $(3, 0)$ (ii) $(3, -4)$

5. Express $\dfrac{2 + \sqrt{2}i}{2 - \sqrt{2}i}$ in the form of $x + iy$

6. Express $(1 + 2i)(-1 + 3i)$ in the form of $x + iy$

7. Express $\dfrac{1+i}{1-i} - \dfrac{1-i}{1+i}$ in the form of $x + iy$

8. Express in the form of $x + iy$

9. Express $\dfrac{1}{2 - \sqrt{3}i}$ in the form of $x + iy$

10. Express $2i^2 + 6i^3 + 3i^{16} - 6i^{19} + 4i^{25}$ in the form of $x + iy$

11. Express $\frac{6}{3+2i}$ in the form of $x + iy$

12. Express $\left(\frac{1-i}{1+i}\right)^{4m+3}$ $(\because m \in N)$ in the form of $x + iy$

13. Express $\left(\frac{1}{1-2i} + \frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$ in the form of $x + iy$

14. Express $\left(\frac{i+\sqrt{3}}{2}\right)^6 + \left(\frac{i-\sqrt{3}}{2}\right)^6$ in the form of $x + iy$

15. If $(3a + 2ib)(2 + i)^2 = 10(1 + i)$ then find the real numbers a and b

16. If $(a + ib)(2 - 3i) = 4 + i$ then find the real number a and b

17. If $(x^4 + 2xi) - (3x^2 + iy) = (3 - 5i) + (2iy + 1)$ then find the value of real number x and y

18. If $z = (1 + i)^7$ then find real and imaginary parts

19. Find the magnitude of $\sqrt{3} - i$

20. Find the magnitude of $\frac{1-3i}{3-i}$

21. Find the magnitude of $\left(\frac{1}{i+1/\left[i+(1/i+1)\right]}\right)$

22. Find the magnitude of $\frac{1}{i} + \frac{2}{i^2} + \frac{3}{i^3} + \frac{5}{i^4}$

23. Find the magnitude of $\frac{(3-4i)^3}{(2+5i)^4}$

24. Find the conjugate complex numbers of the following

(i) $\frac{1}{1-i}$ (ii) i^2 (iii) i (iv) $2 + i$ (v) $\left(\frac{2+i}{1+i}\right)^2$

25. In Argand diagram taking $z_1 = 3 + 4i$ and $z_2 = 5 - 12i$ represent $z_1 + z_2$ and $z_1 - z_2$

26. In Argand diagram show that if $z \neq 0$ then, 0, z and iz represent the vertices of an isosceles triangle

27. In Argand diagram show that the complex numbers $2 - 3i$, $5 - i$; $3 + 2i$ and 0 represent the vertices of a square

28. Obtain the solution in c ($\because$ c is a field)

$p(x) = x^2 + (i - 2)x - 2i = 0$

29. Obtain the solution in c ($\because$ c is a field)

$p(x) = ix^2 + x + 6i = 0$

30. Obtain the solution in c ($\because$ c is a field)
$p(x) = x^2 + (1 - 2i)x - 7 - i = 0$

31. Obtain the solution in c ($\because$ c is a field)
$p(x) = x^2 - x + 1 = 0$

32. Prove that $\dfrac{1+i}{1-i} + \dfrac{1-i}{1+i} \in Q$

33. Prove that $\dfrac{2+\sqrt{3}i}{2-\sqrt{3}i} + \dfrac{2-\sqrt{3}i}{2+\sqrt{3}i} \in Q$

34. If $z = a + ib$ and $|3z| = |z - 3|$ then prove that $8(a^2 + b^2) + 6x = 9$

35. If $z = a + ib$ and if $|z - 3| = |3z - 1|$ then prove that $a^2 + b^2 = 1$

36. If $a, b \in R;\ z = a + ib$ and $2|z - i| = |z + 1|$ then prove that $3a^2 + 3b^2 - 2a - 8b + 3 = 0$

37. If $a, b \in R;\ z = a + ib$ and $|z| = 1$ then prove that $1 + b + ia = (1 + b - ia)(b + ia)$

38. For the complex numbers z_1 and z_2 prove that if $z_1\overline{z_2}$ is real then $\overline{z_1}z_2$ is also real

39. Prove that if $z_1 + z_2$ and z_1z_2 are real numbers then $\overline{z_1} = z_2$

40. If $|z| \le 3$ then find maximum and minimum value of $|z - 4|$

41. If $|z - 2| \le 2$ then find the maximum and minimum value of $|z + 1|$

42. If $|2z - 1| \le 3$ then find the maximum and minimum value of $|4z - 3|$

43. If $|z - 1| \le 5$ then find the maximum and minimum value of $|z - 4 + 4i|$

44. If $z = a + ib$ where a and b are real numbers then prove that $|a| + |b| \le \sqrt{2}|z|$

45. If $(x + iy)^{1/3} = a + ib$ then check that $4(a^2 - b^2) = \dfrac{x}{a} + \dfrac{y}{b}$

46. Evaluate $(4 + 3\sqrt{20i})^{1/2} + (4 - 3\sqrt{20i})^{1/2}$

47. If $\dfrac{(a+i)^2}{2a-1} = p + qi$ then check that $p^2 + q^2 = \dfrac{(a^2+1)^2}{4a^2+1}$

48. If $x + iy = \dfrac{a+ib}{a-ib}$ then check that $x^2 + y^2 = 1$

49. If $z_1, z_2 \in c$ and $a, b \in R$ then check that
$|az_1 + bz_2|^2 + |bz_1 - az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

50. If $x + iy = (1 - \sqrt{3}i)^{100}$ then find the value of x and y $a, b \in R$ $()$

51. If $x = -5 + 4i$ then find the value of $x^4 + 9x^3 + 35x^2 - x + 4$

52. If $z = k + 3i$ is one root of the equation $z^2 + 4z + 13 = 0$ then find the value of k

53. If k is a real number then find the value of k for which $\left|k^2(1+i) - k(1+4i) - 3(2-i)\right| = 0$

54. If $\dfrac{1-iy}{1+iy} = \dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}$ then find the value of y

55. If $\dfrac{(1+i)a-2i}{3+i} + \dfrac{(2-3i)b+i}{3-i} = i$ then find the value of a and b

56. If $|z_1| = |z_2| = \ldots |z_n| = 1$ then show that $|z_1 + z_2 + \ldots z_n| = \left|\dfrac{1}{z_1} + \dfrac{1}{z_2} + \ldots + \dfrac{1}{z_n}\right|$

57. If $x = \dfrac{-1-\sqrt{3}i}{2}$ then prove that $\left(1-x+x^2\right)\left(1-x^2+x^4\right)\left(1-x^4+x^8\right)\left(1-x^8+x^{16}\right)$ $= 16$

58. If $x = a + b;\ y = a + bw;\ z = a + bw^2$ then show that $x^3 + y^3 + z^3 = 3\left(a^3 + b^3\right)$

59. If $(a + ib)(c + id) = x + iy$ then prove that $(a - ib)(c - id) = x - iy$

60. If $a = \dfrac{1+i}{\sqrt{2}}$ then prove that $a^6 + a^4 + a^2 + 1 = 0$

61. Prove that $\left(2 + 3\sqrt{5}i\right)^{1/2} + \left(2 - 3\sqrt{5}i\right)^{1/2} = 3\sqrt{2}$

62. If $x + iy = (a + ib)(c + id)$ where a, b, c, d, x, y are all real numbers and $i = \sqrt{+1}$ then prove that $x^2 + y^2 = (ac - bd)^2 + (ad + bc)^2$

63. If $i^2 = -1$ then prove that $\dfrac{3+2i}{2-5i} + \dfrac{3-2i}{2+5i} = \dfrac{-8}{29}$

64. If $a + b + c = 0$ and w be one imaginary cube root of unity then prove that $\left(a + bw + cw^2\right)^3 + \left(a + bw^2 + cw\right)^3 = 27abc$

65. If $x = a + b$; $y = aw + bw^2$; $z = aw^2 + bw$ then prove that
(i) $xyz = a^3 + b^3$ (ii) $x^2 + y^2 + z^2 = 6ab$

ANSWERS

(1) (i) $\beta = (-7, -3)$ (ii) $\beta = (-1, 4)$
(iii) $\beta = (-2, 6)$ (iv) $\beta = \left(\dfrac{11}{3721}, \dfrac{-60}{3721}\right)$

(2) $a = 2, b = 2$

(3) $x = \dfrac{a}{a^2 + b^2}$; $y = \dfrac{-b}{a^2 + b^2}$

(4) (i) $\left(\dfrac{1}{3}, 0\right)$ (ii) $\left(\dfrac{3}{25}, \dfrac{4}{25}\right)$

(5) $\dfrac{1}{3} + \dfrac{2\sqrt{2}i}{3}$

(6) $-7 + i$

(7) $0 - 2i$

(8) $-1 + 0i$

(9) $\dfrac{2}{7} + \dfrac{\sqrt{3}}{7}i$

(10) $1 + 4i$

(11) $\dfrac{18}{13} - \dfrac{12}{13}i$

(12) $0 + i$

(13) $\dfrac{1}{4} + \dfrac{9}{4}i$

(14) $-2 + 0i$

(15) $a = \dfrac{14}{15}$; $b = \dfrac{-1}{5}$

(16) $a = \dfrac{5}{13}$; $b = \dfrac{14}{13}$

(17) $x = 2$ then $y = 3$ or $x = -2$ then $y = \frac{1}{3}$

(18) $Re_{(z)} = 8, I_{m(z)} = -8$

(19) 2

(20) 1

(21) 1

(22) $\sqrt{13}$

(23) $\frac{125}{841}$

(24) (i) $\frac{1-i}{2}$ (ii) -1 (iii) $-i$

(iv) $2 - i$ (v) $2 + \frac{3i}{2}$

(28) $2, -i$

(29) $3i, -2i$

(30) $2 + i, i - 3$

(31) $\frac{1 \pm \sqrt{3}i}{2}$

(40) 7, 1

(41) 5, 1

(42) 7, 0

(43) 10, 0

(46) 6

(50) $x = -2^{99}$; $y = -2^{99}\sqrt{3}$

(51) -160

(52) $k = -2$ or $-6i - 2$

(53) $k = 3$

(54) $x = 0$

(55) $a = 3, b = -1$

5

Set, Relation and Function

LEARNING OBJECTIVES

This chapter will enable you to learn the concepts and application of:

- The concept of set, relation and function
- The definition of set, operation of sets and relation
- The definition of function and its types
- The concept of one-one, onto and inverse function
- The notation of set theory, relation and function and their related formulae

INTRODUCTION

Set, Relation and Function is a mathematical concept that draws majorily from the Venn-diagram and simple concepts of logic, this reason makes it an important topic. The applications of logic are some very simple concepts of arithmetic and many of them depend on the Venn-diagram.

SET THEORY AND RELATIONS

SET

A well-defined collection of objects is called a *set.*

Note: A set is denoted by A, B, C, etc. and it is expressed by { }.

Finite Set

Any set having certain numbers of elements (elements are countable) is called a *finite set*.

For example $A = \{a, b, c\}$; $B = \{1, 2, 3, 4, 5\}$

Infinite Set

If a set is not finite then it is called an *infinite set*.

For example $A = \{1, 2, \ . \ . \ .\}$

Subset

Let A and B be two sets. If all the elements of A are the elements of B then A is called *subset B* and it is denoted by $A \subset B$.

For example $A = \{a, b, c\}, B = \{a, b, c, d, e\}$

$$\therefore A \subset B$$

Note: Set A is subset of itself and *null* set is subset of every set.

Equal Sets

Let A and B be two sets. If two sets are subsets of each other then they are said to be equal sets.

For example $A = \{a, b, c\}, B = \{a, b, c\}$

$$\therefore A = B$$

Equivalent Set

Let A and B be two sets. If two sets have same numbers of elements they are said to be *equivalent sets*.

For example $A = \{1, 2, 3\}, B = \{x, y, z\}$

In notation $A \approx B$

Proper Subset

If $A \subseteq B$ and $A \neq B$ then A is said to be the *proper subset* of B and we write

$$A \subset B$$

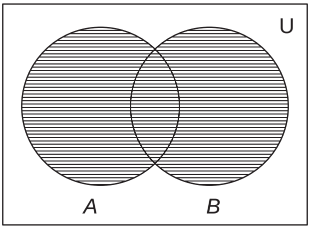
Power Set

The set of all possible subsets of the given set A is called the *power set* of A and is denoted by $P(A)$.

If A contains n elements then the number of possible subsets of A are $= 2^n$ and numbers of all possible proper subsets of A are $(2^n - 1)$.

Note: Null set $(\varnothing)$ is not a proper subset.

Union Set



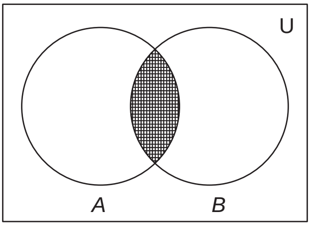
$A \cup B$

Let A and B be two sets. Then the set of elements, which are in A or in B or in both, is said to be *union set* of A and B. It is denoted by $A \cup B$.

For example $A = \{a, b\}, B = \{a, b, c, d\}$

$$\therefore A \cup B = \{a, b, c, d\}$$

Intersection Set



$A \cap B$

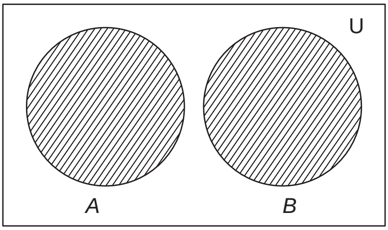
Let A and B be two sets. Then set of elements which are in A as well as in B is said to be *intersection set* of A and B and is denoted by

$$A \cap B$$

For example $A = \{a, b, c, d\}, B = \{c, d, e\}$

$$\therefore A \cap B = \{c, d\}$$

Disjoint Set



$A \cap B = \varnothing$

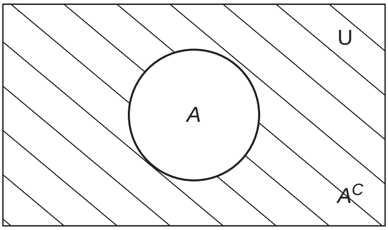
Let A and B be two sets. If there is no common element between A and B then the sets are said to be *disjoint sets*.

For example $A = \{1, 2\}, B = \{x, y\}$

$$\therefore A \cap B = \{\ \} = \varnothing$$

Complementary Set

Let U be a universal set and A be any set. If elements are in U but not in A, then it is said to be *complementary set* of A and is denoted by A' or A^C or $\bar{A}$ or A.



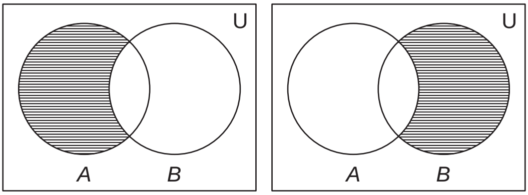
For example $U = \{a, b, c, d, e, f\}, A = \{c, d, e\}$

$$\therefore A' = U - A = \{a, b, f\}$$

Note: $A \cup A' = \text{U}$ and $A \cap A' = \varnothing$

Difference Set

Let A and B be two sets. If elements are in A but not in B, then it is called *difference set* of A and B and it is denoted by $A - B$.

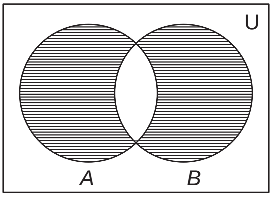


For example $A = \{1,2,3,4,5,6\}$; $B = \{5,6,7,8,9\}$

$$\therefore A - B = \{1,2,3,4\} \text{ and } \therefore B - A = \{7,8,9\}$$

Symmetric Difference Set

Let A and B be two sets. If elements are in A or in B but not in both, then it is said to be a *symmetric difference set* of A and B and is denoted by $A\ \Delta\ B$. It is defined as under:



For example $A\ \Delta\ B = \begin{cases} (A \cup B) - (A \cap B) \\ (A - B) \cup (B - A) \end{cases}$

$$\text{and } A=\{a,b,c,d\},\ B=\{c,d,e,f,g\}$$
$$\therefore A\cup B=\{a,b,c,d,e,f,g\}$$
$$\text{and } \therefore A\cap B=\{c,d\}$$
$$\therefore A\ \Delta\ B=(A\cup B)-(A\cap B)=\{a,b,e,f,g\}$$

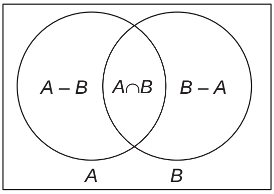
Laws of Set Operations

1. $(A\cup B)=(B\cup A)$ *and* $(A\cap B)=(B\cap A)$
2. $(A\cup B)\cup C=A\cup(B\cup C)$
3. $(A\cap B)\cap C=A\cap(B\cap C)$
4. $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$
5. $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$
6. $A\cap(B-C)=(A\cap B)-(A\cap C)$
7. $A\cup B=A\cap B\Leftrightarrow A=B$
8. $(A-B)\cup(B-A)=(A\cup B)-(A\cap B)=A\ \Delta\ B$
9. $(A-B)\cap B=\varnothing$
10. $A-(B\cup C)=(A-B)\cap(A-C)$
11. $A-(B\cap C)=(A-B)\cup(A-C)$
12. $A-B=A\cap B'$ *and* $B-A=B\cap A'$

De' Morgan's Laws

1. $(A\cup B)'=A'\cap B'$
2. $(A\cap B)'=A'\cup B'$

Very Important Results



1. $n(A\cup B)=n(A)+n(B)-n(A\cap B)$
2. If $A\cap B=\varnothing$ then $n(A\cup B)=n(A)+n(B)$
3. $n(A-B)+n(A\cap B)=n(A)$
4. $n(B-A)+n(A\cap B)=n(B)$
5. $n(A-B)+n(A\cap B)+n(B-A)=n(A\cup B)$
6. $n(A\cup B\cup C)=n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)$
   $-n(C\cap A)+n(A\cap B\cap C)$
7. If $n(A\cap B\cap C)=\varnothing$ then $n(A\cup B\cup C)=n(A)+n(B)+n(C)$
8. $n(A\cap B)=n(A\cup B)-n(A\cap B)$

CARTESIAN PRODUCT OF SETS

Let A and B be two sets. Then Cartesian product of A and B is denoted by $A \times B$ and is defined as under:

$$A \times B = \{(a,b) / a \in A \; and \; b \in B\}$$

$$\text{For example, } A = \{a,b,c\},\ B = \{x,y\}$$

$$\therefore A \times B = \{(a,x)(a,y),(b,x),(b,y),(c,y),(c,z)\}$$

Results on Cartesian Product of Sets

(1) $A \times B \neq B \times A$
(2) $A \times \varnothing = \varnothing \times A$
(3) $\varnothing \times \varnothing = \varnothing$
(4) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
(5) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
(6) $A \times (B - C) = (A \times B) - (A \times C)$
(7) $n(A \times B) = n(A)\, n(B)$

Binary Relation on a Set

Every subset of $A \times A$ is called a binary relation on A.

(1) Identity relation

$I_A = \{(a,a) / a \in A\}$

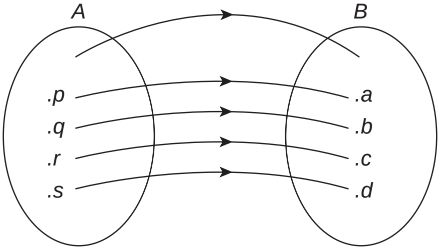
(2) Universal relation

$A \times A$ is called the universal relation on A

A Relation R on A is called

(1) ***Reflexive*** if $a\,R\,a\ \forall\ a \in A$
(2) ***Symmetric*** if $a\,R\,b \Rightarrow b\,R\,a$
(3) ***Transitive*** if $a\,R\,b$; $b\,R\,c \Rightarrow a\,R\,c$
(4) ***Anti-symmetric*** if $a\,R\,b$, $b\,R\,a \Rightarrow a = b$

FUNCTION OR MAPPING



Let A and B be two nonempty sets. When all the elements of A are uniquely related with the elements of B, it is called function from A to B and is denoted by

$$f : A \rightarrow B$$

where the set of the elements of A is said to be domain of function f and is denoted by D_f and the set of the elements of B is called codomain of the function.

Range

The set of the elements of B, which are associated with A, is called range of function and it is denoted by R_f.

VARIOUS TYPES OF FUNCTIONS

1. *One-one Function*

If distinct elements in A have distinct images in B, that is, f is one-one if

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

2. *Many-one Function*

If two or more than two elements of set A have the same image in B, then it is called many-one function.

3. *Onto Function*

For function f: $A \rightarrow B$ if range of function f is proper subset of B then it is called onto function.

4. *Into Function*

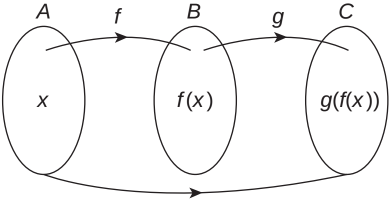
If $\exists$ at least one element in B has no pre-image in A, it is called an into function.

DO YOU KNOW?

(1) One-one mapping is called injective function.
(2) Onto mapping is called surjective function.
(3) One-one and onto mapping is called objective function.

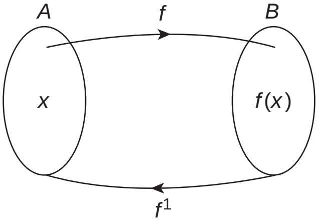
5. *Composite Function*

Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be given two functions. If $R_f \subset D_g$ is satisfied, then we can say that $gof : A \rightarrow C$ exists and is called composite function.



6. *Inverse Function*

If f: $A \rightarrow B$ is one-one and onto function the function $f: B \rightarrow A$ is called inverse function of f. Here f^{-1} of $f\left(x\right) = x$ and f of $f^{-1}\left(x\right) = x$



Note: If any function is one-one and onto then we can say that inverse function exists.

That is,	$f\left(x\right) = y$ (one-one and onto)	$x = f^{-1}\left(y\right)$ (Inverse)

ILLUSTRATIONS

Illustration 1 Find the number of subsets of $\left\{p, q, r, s\right\}$

Solution

$A = \left\{p, q, r, s\right\}$

$\therefore P\left(n\right) = 2^n = 2^4 = 16$

Illustration 2 Find the number of proper subsets of $\left\{a, b, c, d\right\}$

Solution

$A = \left\{a, b, c, d\right\}$

$\therefore P\left(n\right) = 2^n - 1$

$= 2^4 - 1 = 16 - 1 = 15$

($\because \varnothing$ *is not a proper subset*)

Illustration 3 Find the number of subsets of $\left\{a, b, \left\{c, d\right\}\right\}$

Solution

$B = \left\{a, b, \left\{c, d\right\}\right\}$

$\therefore P\left(B\right) = 2^n = 2^3 = 8$

Illustration 4 If $x \in n$ then what is the solution set of the equation $x + 2 = 0$?

Solution

$x + 2 = 0$

$\therefore x = -2 \notin N$

$\therefore \varnothing$

Illustration 5 If $A = \{a, b\}$ then find $P(A)$

Solution

$A = \{a, b\}$

$\therefore P(A) = \{\{a\}, \{b\}, \{a, b\}, \varnothing\}$

$P(A) = 2^n$

$P(2) = 2^2 = 4$

Illustration 6 If M and N are two sets in such a way that $n(M) = 70$, $n(N) = 60$ and $n(M \cup N) = 110$, then find $n(M \cap N)$

Solution

$n(M \cup N) = n(M) + n(N) - n(M \cap N)$

$\therefore 110 = 70 + 60 - n(M \cap N)$

$\therefore n(M \cap N) = 130 - 110 = 20$

Illustration 7 If $f : R \to R$, $f(x) = x + 1$ and $g : R \to R$; $g(x) = x^3$ then find $gof\ (x)$

Solution

$f(x) = x + 1$ *and* $g(x)\ x^3$

then $gof\ \ f(x) = g\,[f(x)]$

$= g(x + 1)$

$= (x + 1)^3$

Illustration 8 If $f : R^+ \to R^+$; $f(x) = \log_e^x$ and $g : R \to R$; $g(x) = e^x$ then find $fog\ (x)$

Solution

Here $f(x) = \log_e^x$; $g(x) = e^x$

then $fog(x) = f\,[g(x)]$

$= f(e^x)$

$= \log_e^{e^x}$

$= x \log_e^e = x(\because \log_e^e = 1)$

Illustration 9 If $f : R^+ \to R^+ ; f(x) = x^2 + 1$ then find the value of $f^{-1}(x)$

Solution

$$f(x) = x^2 + 1$$

$$\left[\text{Hint: } f(x) = y \Rightarrow x = f^{-1}(y)\right]$$

$$f(x) = x^2 + 1 = y$$

$$\therefore x^2 = y - 1$$

$$x = \sqrt{y - 1}$$

$$\therefore f^{-1}(y) = \sqrt{y - 1}$$

$$\therefore f^{-1}(x) = \sqrt{x - 1}$$

Illustration 10 If $f : R \to R, f(x) = e^x$ then find the value of $f^{-1}(x)$

Solution

$$f(x) = e^x$$

$$\left[\text{Hint: } f(x) = y \Rightarrow x = f^{-1}(y)\right]$$

$$f(x) = e^x = y$$

$$\therefore x = \log_e^y$$

$$\therefore f^{-1}(y) = \log_e^y$$

$$\therefore f^{-1}(x) = \log_e^x$$

Illustration 11 Suppose $f : R \to R$ be such that $f(x) = 2^x$ then find the value of $f(x + y)$

Solution

$$f(x) = 2^x$$

$$\therefore f(x + y) = 2^{x + y}$$

$$= 2^x 2^y$$

$$= f(x)\, f(y)$$

Illustration 12 If $f : R \to R; f(x) = x^2 + 1$ then find $fof(1)$

Solution

$$f(x) = x^2 + 1$$

$$\therefore fof(x) = f\left[f(x)\right]$$

$$= f(x^2 + 1)$$

$$= (x^2 + 1)^2 + 1$$

$$\therefore f \text{ of } (1) = (1^2 + 1)^2 + 1$$

$$= 4 + 1$$

$$= 5$$

Illustration 13 If $U = \{1,2. . . 10\}$ and $A = \{3,4,5,7,9\}$ then find A^c

Solution

$$A^c = U - A$$
$$= \{1,2\ldots0\} - \{3,4,5,7,9\} = \{ 1,2,6,8,10\}$$

Illustration 14 If $f(x) = x + 2$; $g(x) = x^2$ then find the value $f(x)g(x)$

Solution

$$f(x) = x + 2;\ g(x) = x^2$$
$$\therefore f(x)g(x)$$
$$= (x + 2)x^2$$
$$= x^3 + 2x^2$$

Illustration 15 If $f + g = \{2 , 3\} \rightarrow R$ then find $A \cup B$

Solution

$$A = \{x^2 = x / x \in R\}$$
$$\therefore x^2 = x$$
$$\therefore x^2 - x = 0$$
$$\therefore x(x - 1) = 0$$
$$\therefore x = 0 \text{ or } x = 1$$

$$B = \{x^3 = x / x \in N\}$$
$$\therefore x^3 = x$$
$$\therefore x^3 - x = 0$$
$$\therefore x(x^2 - 1) = 0$$
$$\therefore x(x - 1)(x + 1) = 0$$
$$\therefore x = 0 \text{ or } x = 1 \text{ or } x = -1$$

$$\therefore A = \{0,1\}; x \in R \qquad \therefore B = \{1\}\ (\because x \in N)$$
$$\therefore A \cup B = \{0,1\} \cup \{1\} = \{0,1\}$$

Illustration 16 If $f : R \rightarrow R\ \ f(x) = e^{-x}$ then find $fof(-1)$

Solution

$$f(x) = e^{-x}$$
$$\therefore fof(x) = f(f(x))$$
$$= f(e^{-x})$$
$$= e^{(e^{-x})}$$
$$\therefore fof(-1) = e^{(e^{-1})} = e^{-1/e}$$

Illustration 17 $f : R \rightarrow R$; $f(x) = \dfrac{1}{x^2 + 1}$ then find $f^{-1}(x)$

Solution

$$f(x) = \frac{1}{x^2 + 1}$$
$$\left[\text{Hint: } f(x) = y\ x = f^{-1}(y)\right]$$

$$\therefore f(x) = y = \frac{1}{x^2 + 1}$$

$$\therefore x^2 + 1 = \frac{1}{y}$$

$$\therefore x^2 = \frac{1}{y} - 1$$

$$\therefore x^2 = \frac{1 - y}{y}$$

$$\therefore x = \sqrt{\frac{1 - y}{y}}$$

$$\therefore f^{-1}(y) = \sqrt{\frac{1 - y}{y}}$$

$$\therefore f^{-1}(x) = \sqrt{\frac{1 - x}{x}}$$

Illustration 18 If $A = \{x / x \text{ is a multiple of } 2 ; x \in N\}$
$B = \{x / x \text{ is a multiple of } 3 ; x \in N\}$ then find $A \cap B$

Solution

$A = \{2,4,6,8,10. . . . .\}$
$B = \{3,6,9,12,15. . .\}$
$A \cap B = \{\ \} = \varnothing$

Illustration 19 In a survey it was found that 63% like Gujarati Thali and 76% like South food. How many like both?

Solution

Suppose Gujarati Thali = A
Suppose South food = B
$\therefore n(A) = 63\% \ ; \ n(B) = 76\%$
$n(A \cap B) = ?; \ n(A \cup B) = 100\%$
$\therefore n(A \cup B) = n(A) + n(B) - n(A \cap B)$
$\therefore 100\% = 63\% + 76\% - n(A \cap B)$
$\therefore n(A \cap B) = 139\% - 100\%$
$= 39\%$

Illustration 20 Let $A = \{1,3,5,7,9,11\}$ and $B = \{7,9,11\}$ then find $A - B$

Solution

$A - B$
$= \{1,3,5,7,9,11\} - \{7,9,11\}$
$= \{1,3,5\}$

Illustration 21 If $f: R \to R$; $f(x) = (-1)^x$ then find R_f

Solution

$$Df = R$$
$$f(x) = (-1)^x$$
$$f(0) = (-1)^0 = 1$$
$$f(1) = (-1)^2 = -1$$
$$f(2) = (-1)^2 = 1$$
$$\therefore Rf = \{-1, 1\}$$

Illustration 22 If $U = \{1, 2, 3, ... 10\}$; $A = \{2, 3, 5\}$; $B = \{1, 4, 6, 7\}$ then find $(A \cup B)'$

Solution

$$A \cup B = \{2, 3, 5\} \cup \{1, 4, 6, 7\}$$
$$= \{1, 2, 3, 4, 5, 7\}$$
$$\therefore (A \cup B)' = U - (A \cup B)$$
$$= \{1, 2, ... 10\} - \{1, 2, 3, 4, 5, 7\}$$
$$= \{6, 8, 9, 10\}$$

Illustration 23 If $f(x) = \dfrac{x}{x^2 - 3x + 2}$ then find Df

Solution

$$f(x) = \frac{x}{x^2 - 3x + 2}$$

Here $x^2 - 3x + 2 = 0$

$$(x - 1)(x - 2) = 0$$
$$\therefore x = 1 \text{ or } x = 2$$
$$\therefore x = \{1, 2\}$$

$\therefore Df = R - \{1, 2\}$ [If denominator is 0 then $f(x)$ does not exist]

Illustration 24 If $f(x) = \sqrt{\dfrac{x-1}{x-4}}$ then find Df

Solution

Here $f(x) = \sqrt{\dfrac{x - 1}{x - 4}}$

If in above function $x - 4 \neq 0$ then $f(x)$ exists

$$\therefore x - 4 > 0$$
$$\therefore x > 4$$
$$\therefore Df = (4, \infty)$$

Illustration 25 Suppose $f(x) = \frac{1}{1-x}$ then find $(fofof)(x)$

Solution

$$\therefore (fofof)(x) = fof\left[f(x)\right]$$

$$= fof\left(\frac{1}{1-x}\right)$$

$$= fo\left[f\left(\frac{1}{1-x}\right)\right]$$

$$= fo\left[\frac{1}{\{1-[1/(1-x)]\}}\right]$$

$$= f\left(\frac{1-x}{1-x-1}\right)$$

$$= f\left(\frac{1-x}{-x}\right)$$

$$= \frac{1}{1-[(x-1)/x]}$$

$$= \frac{x}{x-(x-1)}$$

$$= \frac{x}{x-x+1} = x$$

Illustration 26 If $A = \{p,q,r,s,t\}$, $B = \{a,b,c,d,e\}$ and $C = \{d,e,f\}$ then find $n\left[A \times (B \cap c)'\right]$

Solution

$$A = \{p,q,r,s,t\}$$

$$\therefore n(A) = 5 \qquad (1)$$

and $B \cap C' = B - (B \cap C)$

$$= \{a,b,c,d,e\} - \{d,e\}$$

$$= \{a,b,c\}$$

$$\therefore n(B \cap C') = 3 \qquad (2)$$

$$\therefore n(A \times B \cap C')$$

$$= n(A)n(B \cap C')$$

$= 5 \times 3$

$= 15$

Illustration 27 Prove that $A \times B = B \times A \Rightarrow A = B$

Solution

Suppose $a \in A$ and $b \in B$

Now $a \in A$ and $b \in B \Leftrightarrow (a , b) \in (A \times B)$

$\Leftrightarrow (a , b) \in (B \times A) \quad (\because A \times B = B \times A)$

$\Leftrightarrow a \in B$ and $b \in A$

$\therefore a \in A \Rightarrow a \in B$ and $b \in B \Rightarrow b \in A$

$\therefore A \subset B$ and $B \subset A$

$\therefore A = B$

Illustration 28 Prove that $A \subset B \Rightarrow B' \subset A'$

Solution

Here $a \in B' \Rightarrow a \in U$ and $A \notin B$

$\Rightarrow a \in U$ and $a \notin A$

$\Rightarrow a \in A'$

$\therefore B' \subset A'$

Illustration 29 Prove that $C \subset (A - B)$ and $D \subset (B - A)$ then $C \cap D = \varnothing$

Solution

$C \subset (A - B)$ and $D \subset (B - A)$

$\Rightarrow (C \cap D) \subset (A - B) \cap (B - A)$

$\Rightarrow (C \cap D) \subset (A - B) \cap (B - A)$

Also $\varnothing \subset (C \cap D)$ is obvious

Thus $(C \cap D) = \varnothing$ and $\varnothing \subset (C \cap D) \Rightarrow C \cap D = \varnothing$

Illustration 30 If $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ and $f = A \to R f = \{(1,1), (2,3), (3,5)\}$; $g = B \to R \ g = \{(2,1), (3,0), (4,1)\}$ then find $f + g$, $f - g$, fg and f/g

Solution

Here $A \cap B = \{2 , 3\}$ and $g(3) = 0$

Now

(1) $f + g = \{2 , 3\} \to R$

Also $(f + g)_{(2)} = f(2) + g(2) = 3 + 1 = 4$

$(f+g)_{(3)} = f(3) + g(3) = 5 + 0 = 5$

$\therefore (f+g) = \{(2,4),(3,5)\}$

(2) $f - g = \{2,3\} \to R$

Also $(f-g)_{(2)} = f(2) - g(2) = 3 - 1 = 2$

$(f-g)_{(3)} = f(3) - g(3) = 5 - 0 = 5$

$\therefore (f-g) = \{(2,2),(3,5)\}$

(3) $fg = \{2,3\} \to R$

Also $(fg)_{(2)} = f(2)g(2) = 3 \times 1 = 3$

$(fg) = f(3)g(3) = 5 \times 0 = 0$

$\therefore (fg) = \{(2,3),(3,0)\}$

(4) $(A \cap B) - \{x / g(x) = 0\} = \{2,3\} - \{3\} = \{2\}$

$\therefore \frac{f}{g} = \{2\} \to R$

$\frac{f}{g}(2) = \frac{f(2)}{g(2)} = \frac{3}{1} = 3$

$\frac{f}{g} = \{(2,3)\}$

Illustration 31 If $f : R^+ \to R$ $f(x) = \log_e^x$ and $g : R^+ \to R$ $g(x) = e^x$ then find *fog* and *gof*

Solution

Here $Rf \subset Dg$ $\therefore$ $gof = R^+ \to R^+$ exists

Also $gof(x) = g(f(x))$

$= g(\log_e^x)$

$= e^{\log_e^x} = x$

Also $Rg \subset Df$ $\therefore$ $fog : R \to R$ exists

and $fog(x) = f[g(x)]$

$= f(e^x)$

$= \log_e^{e^x}$

$= x\log_e^e \quad (\because \log_e^e = 1)$

$= x$

Here $gof(x) = fog(x)$

But $Dgof = R^+$ and $Dfog = R$

which are not equal

$\therefore gof \neq fog$

Illustration 32 If $f : R \to R$; $f(x) = \dfrac{x+1}{2}$; $g : R \to R$, $g(x) = 2x - 1$; $h : R \to R$; $h(x) = x^2$ then prove that $ho(gof) = (hog)of$

Solution

$Rf \subset Dg \;\therefore gof : R \to R$ exists

$$\text{Now } gof(x) = g[f(x)] = g\left(\frac{x+1}{2}\right) = 2\left(\frac{x+1}{2}\right) - 1 = x + 1 - 1 = x$$

Now $R_{gof} \subset Dh$

$\therefore ho(gof) : R \to R$ exists

$$\text{Now } ho(gof)(x) = h[gof(x)] = h(x) = x^2$$

Now $Rg \subset Dh \therefore hog : R \to R$ exists

$$\text{Now } hog(x) = h[g(x)] = h(2x-1) = (2x-1)^2$$

$Rf \subset Dhog$

$\therefore (hog)of : R \to R$ exists

Now $(hog)of(x)$

$$= hog[f(x)]$$

$$= hog\left(\frac{x+1}{2}\right)$$

$$= \left[2\left(\frac{x+1}{2}\right) - 1\right]^2$$

$$= (x + 1 - 1)^2$$

$$= x^2$$

$\therefore ho(gof) = (hog)of$

Illustration 33 $A = \{1,2,3\}, f = \{(1,2), (2,3), (3,1)\}; g = \{(1,1), (2,3), (3,2)\}$ and if f and g are functions from A to A then prove that $(fog)^{-1} = g^{-1}of^{-1}$

Solution

$Rf = A \subset Df$

$\therefore\ fog = A \to A$ exists

Here $fog = \{(1,2), (2,1), (3,3)\}$

and fog is one-one and onto function

$(fog)^{-1} = A \to A$ exists

$\therefore (fog) = \{(1,2), (2,1), (3,3)\}$ (1)

Here f and g are one-one and onto functions

$\therefore\ f^{-1}$ and g^{-1} exist

Now $f^{-1} = \{(2,1), (3,2), (1,3)\}$

and $g^{-1} = \{(1,1), (3,2), (2,3)\}$

and $Rf^{-1} \subset Dg^{-1}$

$\therefore g^{-1}of^{-1} = A \to A$ exists

$\therefore g^{-1}of^{-1} = \{(1,2), (2,1), (3,3)\}$ (2)

Thus from (1) and (2) we can say that

$(fog)^{-1} = g^{-1}of^{-1}$

Illustration 34 If $f = R \to R$ $f(x) = x + 2$ and $g = R \to R$ is a function such that $gof(x) = 2x + 5$ then find the function $g(x)$

Solution

Here $(gof)of^{-1} = go(fof^{-1}) = goIR = g(x)$

Now first we will find f^{-1}

$\forall x_1, x_2 \in R$

$f(x_1) = f(x_2)$

$\Rightarrow x_1 + 2 = x_2 + 2$

$\Rightarrow x_1 = x_2$

f is a one-one function

Here $\forall y \in R \exists (y - 2) \in R$

Such that $f(y - z) = y - 2 + 2 = y$

$\therefore f$ is an onto function

Now f being one-one and onto function

=>f^{-1} exists

Here $y = x + 2$

=> $x = y - 2$

=> $f^{-1}(y) = y - 2$

=> $f^{-1}(x) = x - 2$

Now $\forall\ y \in R$

$$\begin{aligned} g(y) &= \left[(gof)\, of^{-1}\right](y) \\ &= (gof)\left[f^{-1}(y)\right] \\ &= 2(y-2)+5 \\ &= 2y-4+5 \\ &= 2y+1 \end{aligned}$$

$\therefore g(y) = 2y + 1$

$\therefore g(x) = 2x + 1$

Illustration 35 If $gof = IA$; for $f = A \to B$ and $g = B \to A$ then prove that f is one-one and g is onto function

Solution

We know that if gof is one-one and onto function then f is one-one function and g is onto function

Now $gof = IA$ and $IA = A \to A$ is a one-one and onto function

$\therefore$ $gof = A \to A$ is also a one-one and onto function

$\therefore$ f is one-one function and g is onto function

Illustration 36 If $f = A \to B$ is a one-one function and $g = C \to A$ and $h = C \to A$ are functions and $fog = foh$ then prove that $g = h$

Solution

Here $fog = C \to B$ and $foh = C \to B$ exist

Now $fog = foh$

$\therefore fog(x) = foh(x)$

$\therefore f\left[g(x)\right] = f\left[h(x)\right]$ let $m = g(x)$

$\therefore f(m) = f(n)$ and $n = h(x)$

$\therefore m = n$ (f is a one-one function)

$\therefore g(x) = h(x)$ exits

Also $Dg = Dh = C$

$\therefore g = h$

ANALYTICAL EXERCISES

1. If $p = \left\{3n/n \in N\right\}$; $Q = \left\{3n - 1/ nN\right\}$ and $R\ \left\{3n - 2/nN\right\}$ then obtain (1) $P \cup Q \cup R$ $P \cap Q \cap R$ (3) $N - Q \cup R$

2. If $A = \left\{4k + 3 / k \in z\right\}$ and $B = \left\{6k + 1 / k \in z\right\}$ obtain (1) $A \cup B$ and (2) $A \cap B$

3. If $\left\{4n / n \in z\right\} n \left\{6n / n \in z\right\} = \left\{kn / n \in z\right\}$ then find the value of k

4. Show that the sets $M = \left\{2n / n \in N\right\}$ and $N = \left\{2n - 1 / n \in N\right\}$ are naturally disjoint also find $M \cup N$

5. If $P = \left\{x \in z / x^2 - 2x - 3 = 0\right\}$ and $Q = \left\{x \in z / x^2 + 3x + 2 = 0\right\}$ then find (1) $P \cup Q$ and (2) $P \cap Q$

6. For $P = \left\{x \in N / x^3 = x\right\}$ $Q = \left\{x \in Z / x^2 = x\right\}$ and $R = \left\{x \in Z / x^3 = x\right\}$ then prove that $P \in P(Q) \cap P(R)$

7. If $M = \left\{x / x = 3k - 1, k \in z\right\}$ and $N = \left\{x / x = 2k - 1, k \in z\right\}$ then find (1) $M \cup N$ and (2) $M \cap N$

8. If $U = \left\{x / 1 \le x \le 10 , x \in N\right\}$ and $P = \left\{1,2,3,4,5,6\right\}$ and $Q = \left\{2,3,5,7\right\}$ then find P' , Q' ,$P \cup Q$ $P \cap Q$, $P - Q$, $Q - P$ and $P \Delta Q$

9. If $P = \left\{n / n \in N , n < 10\right\}$; $Q = \left\{n / n \in N , n > 5\right\}$ and $R = \left\{3n / n \in N\right\}$ then obtain $P \cap Q$ and $P \cap R$

10. If A = a set of the letters of the word *SCIENCE* and B = a set of the letters the word *RELIANCE* then evaluate $A \Delta B$

11. If $U = \left\{a,b,c,d,e,f\right\}$; $A = \left\{a,b,c,d\right\}$, $B = \left\{b,c,d,e\right\}$ and $C = \left\{c,d,e,f\right\}$ then check that

(1) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(2) $(A \cap B)' = A' \cup B'$

(3) $(A \cup B)' = A' \cap B'$

(4) $(A \cap B)' = A' \cup B'$

12. If $U = \left\{x / 3 \le x \le 12, x \in N\right\}$, $P = \left\{x / 3 < x < 7, x \in N\right\}$ $Q = \left\{x / 5 < x < 12, x \in N\right\}$ and $R = U$ then check that

(1) $(A \cup B)' = A' \cap B'$

(2) $(A \cap B)' = A' \cup B'$

(3) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(4) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

13. If $U = \left\{p,q,r,s\right\}$, $M = \left\{p,q,r\right\}$ and $Q = \left\{q,r,s\right\}$ then verify that

(1) $(M \cup N) - N = M \cap N'$

(2) $(M \cup N)' = M' \cap N'$
(3) $(M \cap N)' = M' \cup N'$

14. If $U = \{1,2,\ldots10\}$; $P = \{1,3,5,7,9\}$ $Q = \{1,4,5,8\}$ and $R = \{2,5,6,9\}$ then check that
(1) $P \cup (Q - R) = (P \cup Q) - (P \cup R)$
(2) $P \cap (Q - R) = (P \cap Q) - (P \cap R)$

15. If $X = \{1,2,3\}$, $Y = \{3,4\}$ and $Z = \{1,4\}$ then check that
(1) $X \times (Y \cap Z) = (X \times Y) \cap (X \times Z)$
(2) $X \times (Y - Z) = (X \times Y) - (X \times Z)$
(3) $X \times (Y \,\Delta\, Z) = (X \times Y) \,\Delta\, (X \times Z)$

16. If $P = \{a, b, c\}$; $Q = \{b, d\}$ and $R = \{c, d\}$ then verify that
(1) $P \times (Q \cap R) = (P \times Q) \cap (P \times R)$
(2) $P \times (Q \cup R) = (P \times Q) \cup (P \times R)$
(3) $P \times (Q - R) = (P \times Q) - (P \times R)$
(4) $P \times (Q \,\Delta\, R) = (P \times Q) \,\Delta\, (P \times R)$

17. If $P = \{a, b\}$, $Q = \{b, c\}$, $R = \{c, d\}$, and $S = \{d, e\}$ then verify that $(P \cap Q) \times (R \cap S) = (P \times R) \cap (Q \times S)$

18. $"(A \cap B)' = (A' \cup B')"$ verify by Venn-diagram

19. $"(A \cap B)' = (A' \cap B')"$ verify by Venn-diagram

20. $"A \cap (B \cup C) = (A \cap B) \cup (A \cap C)"$ verify by Venn-diagram

21. $"(A - B) - C = A - (B \cup C)"$ verify by Venn-diagram

22. $"(A \cup B')' = A' \cap B"$ verify by Venn-diagram

23. $"A \cup (B \cap C) = (A \cup B) \cap (A \cup C)"$ verify by Venn-diagram

24. $"A \cup B = (A - B) \cup (B - A)"$ verify by Venn-diagram

25. $"(A \cap B) \cup (A \cap B') = A"$ verify by Venn-diagram

26. $"A \cap (B - C) = (A - C) \cap (B - C)"$ verify by Venn-diagram

27. In a class of 60 students of B.B.A. (1) 24 students have failed in Mathematics (2) 18 students have failed in English (3) 6 students have failed in both subjects. How many students have failed in both subjects?

28. There are 80 students in a class. The set of students playing cricket, hockey and football are P, Q and R, respectively. The following is the information regarding these sets:
$n(P) = 30$, $n(Q) = 40$; $n(R) = 50$,
$n(Q \cap R) = 17$, $n(P \cap R) = 23$ and $n(P \cap Q \cap R) = 10$

then (1) How many students play only one game? (2) How many students play exactly two games?

29. In a class of 70 students
 (1) 30 students play cricket
 (2) 25 students play hockey
 (3) 20 students do not play cricket or hockey
  then
  (1) How many students play both the games?
  (2) How many students play only cricket?
  (3) How many students play only hockey?

30. 150 students appeared in an examination of three subjects physics, chemistry and biology. Out of these, 100 students passed in physics, 120 in chemistry and 80 in biology. 85 students passed in both physics and chemistry, 70 in both chemistry and biology while 60 passed in both biology and physics, whereas 50 students passed in all the three subjects. Then find (1) The number of students failing in all the three subjects, (2) the number of students passing in exactly two subjects.

31. If $n(A \times B) = 2, P \in B, (a, q) \in A \times B$ then find A and B.

32. For the real numbers x and y, if two ordered pairs $(x + 2y, 2y - 1)$ and $(2x + y, y + 2)$ are identical then find x and y.

33. Using the De' Morgan's law $(A \cup B)' = A' \cap B'$ and $(A')' = A$ prove the second law of De' Morgan's $(A \cap B)' = A' \cup B'$ where $A, B \in P(\cup)$

34. By usual notation prove that $A \times A = B \times B => A = B$

35. By usual notation prove that $A = B$ *then* $A' = B'$

36. By usual notation prove that $A \cup B = A \cap B <=> A = B$

37. Obtain $(A \cup B)' = A' \cap B'$ from $(A \cap B)' = A' \cup B'$ and $(A')' = A$

38. By usual notation prove that $(A \cup B) \cap (A \cup B') = A$

39. If $A \cap B = \phi$ and $A \cup B = \cup$ prove that $A = B'$ where $\cup$ is a universal set

40. If $= A \cap B = A \cap C , B - A = C - A$ then $B = C'$ prove that

41. By usual notation prove that if $A \cap B = \phi$ then $B \subset A'$

42. If $A \cup B = A \cup C$ and $A - B = A - C$ then prove that $B = C$

43. For $A, B \in P(\cup)$ prove that (1) $A \subset B$ (2) $A \cup B = B$ are equivalent (3) $A \cap B = A$

44. If $A \subset B$ then by usual notation prove that $A \times C \subset B \times C$

45. For any sets A, B, C, D prove that $(A \times B) \cap (C \times D) = (A \times C) \cap (B \times D)$

46. For any nonempty sets if A, B, C *if* $(A \times B) \cup (B \times A) = C \times C$ then prove that $A = B = C$

47. For any nonempty sets if $A \cup B = A \cup C$, $A \cap B = A \cap C$ then prove that $B = C$

48. For any nonempty sets if $A - B = A - C; B - A = C - A$ then prove that $B = C$

49. Verify the following results taking $U = \{-2, -1, 0, 1, 2\}$, $A = \{-2, -1, 0\}$, $B = \{0, 1, 2\}$ and $C = \{-1, 0, 1\}$:

(1) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(2) $A \cup B = (A - B) \cup (B - A)$

(3) $(A \cup B)' = (A' \cap B')$

(4) $(A \cap B)' = (A' \cup B')$

50. Using Venn-diagram verify that

$P \Delta (Q \Delta R) = (P \Delta R) \Delta R$

51. If $M = \{3K + 1 / K \in Z\}$ and $N = \{5K + 3 / K \in Z\}$ then find

$M \cap N$ *and* $M \cup N$

52. In an examination of accounts and statistics, the result obtained by 100 students is as follows:

(1) The number of students passing in accounts is 50
(2) The number of students passing in statistics is 55
(3) The number of students passing in both the subjects is 20

then

(1) What is the percentage of students failing in both subjects?
(2) How many students passed only in accounts?
(3) How many students passed only in statistics?

53. Complete the following table by taking $P = \{a,b\}$, $Q = \{a\}$, $R = \{b\}$ and $S = \phi$ and $A \Delta B = (A \cup B) - (A \cap B)$ Using this prove that power set of $P\ \{P(P)\}$ is closed under the operation

Δ	P	Q	R	S
P				
Q				
R				
S				

54. In a school, 70% of students play cricket, 50% of students play hockey and 40% of students play volleyball. Also 19% of play both cricket and hockey. The percentage of students playing cricket and volleyball is 13 while the percentage of students playing hockey and volleyball is 15. All the three games are played by 7% of students. Prove that this information is incorrect.

55. If $f = R - \{-1\} \to R;\ f(x) = \left(\dfrac{1 - x}{1 + x}\right)$ then prove that

(1) $f(y) + f\left(\dfrac{1}{y}\right) = 0\ (y \neq 0)$

(2) $f(x)\, f(y) = f\left(\dfrac{x + y}{1 + xy}\right)$

56. If $R - \{-1\} \to R;\ f(x) = \left(\frac{1-x}{1+x}\right)$ then prove that

(1) $f\left[f(x)\right] = x$

(2) $f(x) - f\left(\frac{1}{x}\right) = 2f(x);\ x \neq 0$

57. If $A = \{0, 2, 4\};\ B = \{1, 3, 5\}$ and $f = A \to B;\ f = \{(0, 1), (2, 3), (4, 5)\};$ $g = \{(1, 0), (3, 2), (5, 4)\};\ g = B \to A$ then find *fog* and *gof* respectively

58. $A = \{a, b, c\}$ and f and g are function from A to A. If $f = \{(a, b), (b, c), (c, a)\}$ and $g = \{(a, b), (b, a), (c, c)\}$ then show that $fog \neq gof$

59. If $f = R^+ \to R;\ f(x) = \log_e^x$ and $g = R \to R^+;\ g(x) = e^x$ then prove that $fog \neq gof$

60. If $f = R \to R;\ f(x) = x - 1$ and $g = R \to R_1;\ g(x) = x + 2$ then is $gof = fog$?

61. If $R_1 = \{x \in R \mid x \geq 0\}$ and $f = R \to R_1;\ f(x) = x^2$ and $g = R_1 \to R;\ g(x) = \sqrt{x}$ then prove that $fog = IR$ and $gof = IR$? Give reason.

62. If $f = R \to R;\ f(x) = 2x - 1,\ g = R \to R\ g(x) = x + k$ and $fog = gof$ then find k

63. If $f = R \to R;\ f(x) = 2x;\ g = R \to R\ g(x) = x + 2$ and $h = R \to R;\ h(x) = \frac{x}{2}$ then prove that $ho(gof) = (hog)of$

64. If $f = A \to A, A = R - \{0,1\}$ and $f(x) = 1 - \frac{1}{x}$ then prove that $f(f(f(x))) = x$ and $f^{-1}(x) = \frac{1}{1-x}$

65. For $f = A \to R$; if $f(x) = \left(\frac{1-x}{1+x}\right);\ A = R - \{-1\}$ then prove that $fof = IA$

66. If $f(x) = |x|;\ x \in R$ and $g(x) = (-1)^x;\ x \in N$ then show that *fog* is a constant function

67. If $f = R \to R;\ f(x) = 2x - 1$ and $g = R \to R, g(x) =$ then prove that $(gof)^{-1} = f^{-1}og^{-1}$

68. If $f = R \to R;\ f(x) = 3x + 5$ and $g = R \to R$ is a function such that $fog(x) = 6x + 14$ then find the formula of g

69. For $A = \{1, 2, 3\}$ and $B = \{2, 4, 5\}$ if $f = A\ B, f = \{(1,2)\ (3,4)\ (4,5)\}$ and $g = B\ A\ g = \{(2,1)\ (4,3)\ (5,4)\}$ then show that $fog = gof$

70. If $R \to R;\ f(x) = x - 2;\ g = R\ R\ g(x) = x + 2$ then prove that $gof = fog$

71. If $f = N \to N f(x) = 2x + 1$ and $g = N \to N;\ g(x) = x + 2$ then prove that $fog \neq gof$

72. If $f = R \to R \times R;\ f(x) = (x, x)$ and $g = R \times R \to R\ \ g(x, y) = \frac{x+y}{2}$ then find *fog* and *gof*

73. If $f = R \to R$ and $g = R \to R$ and if $f(x) = 2x + 1,\ g(x) = \frac{x-3}{2}$ then prove that $(gof)^{-1} = f^{-1}og^{-1}$

74. For the functions f, g and h from R to R, if $f(x) = x^2;\ g(x) = -x^3$ and $h(x) = x^3$ then show that $fog = foh$

75. For the functions f and g from R to R, if $f\left(x\right) = \frac{x}{2} + 3$; $g\left(x\right) = 2x - k$ and $fog = gof$ then find k

76. For the functions f, g, h from R to R, if $f\left(x\right) = 2x - 3$; $g\left(x\right) = x^3$ and $h\left(x\right) = x^2$ then show that $\left(hog\right)of = ho\left(gof\right)$

77. If $f = R \rightarrow R$; $f\left(x\right) = 2x - 1$; $g = R \rightarrow R\ g\left(x\right) = \frac{x+1}{2}$ then show that f and g are inverses of each other

78. If $f = R \rightarrow R$; $f\left(x\right) = x + 5$ and $g = R \rightarrow R$ is a function such that $gof\left(x\right) = 2x + 3$ then find $g\left(x\right)$

79. If $f = A \rightarrow B$ is a one-one and onto function then prove that $\left(f^{-1}\right)^{-1} = f$

ANSWERS

(1) (1) N (2) $\varnothing$ (3) A

(2) (1) $\left\{x / x = 4k + 3 \text{ or } x = 6k + 1 / k \in z\right\}$

(2) $\left\{x / x = 12k - 5, k \in z\right\}$

(3) $k = 12$

(4) N

(5) (1) $\left\{-2, -1, 3\right\}$ (2) $\left\{-1\right\}$

(7) (1) $\left\{x/x = 3k - 1 \text{ or } x = 2k - 1, kz\right\}$

(2) $\left\{x/x = 6k - 1, kz\right\}$

(8) $P' = \left\{7,8,9,10\right\}$; $Q' = \left\{1,4,6,8,9,10\right\}$;

$P \cup Q = \left\{1,2,\ldots 7\right\}$; $P \cap Q = \left\{2,3,5\right\}$;

$P - Q = \left\{1,4,6\right\}$; $Q - P = \left\{7\right\}$

$P \,\Delta\, Q = \left\{1,4,6,7\right\}$

(9) (1) $P \cap Q = \left\{6,7,8,9\right\}$

(2) $P \cap R = \left\{3,6,9\right\}$

(10) $A \,\Delta\, B = \left\{\text{S, R, L, A}\right\}$

(27) 24

(28) (1) 40, (2) 25

(29) (1) 5, (2) 25, (3) 20

(30) (1) 15; (2) 65

(31) $A = \left\{a\right\}, B = \left\{p, q\right\}$

(32) $x = y = 3$

(51) $M \cap N = \left\{15\text{K} + 13 \mid \text{K} \in \text{Z}\right\}$

$M \cup N = \left\{3\text{K} + 1 \text{ or } 5\text{K} + 3 \mid \text{K} \in \text{Z}\right\}$

(52) (1) 15, (2) 30, (3) 35

(57) $fog = \left\{(1,1), (3,3), (5,5)\right\}$

$gof = \left\{(0,0), (2,2), (4,4)\right\}$

(61) No. $I_R(x) = x$, $gof\left(x\right) = |x|$

(62) $k = 0$

(68) $g(x) = 2x + 3$

(72) $gof(x) = x\ fog\left(x, y\right) = \left(\frac{x+y}{2}, \frac{x+y}{2}\right)$

(75) $k = 6$

(78) $g(x) = 2x - 7;\ x \in R$

6

Profit–Loss, Discount, Commission and Brokerage

LEARNING OBJECTIVES

Systematic study of this chapter will help the student to understand:

- Basic terminology of profit and loss
- Meaning of discount and its different types
- Concepts of commission, brokerage and bonus and to calculate all these under practical situations

INTRODUCTION

In today's business world the aim of every businessman is not only to earn profit but also to enhance their profit. The purchase and sale of items is a common practice for the producer, or for the seller of the item Hence the knowledge of profit–loss, discount, commission and brokerage is very essential.

PROFIT AND LOSS

The price at which an item is purchased or the cost of production of an item is called as *purchase price* or *cost price* (*C.P.*) and the price at which an item is sold is called *selling price* (*S.P.*) of the item. If the cost price of an item is more than the selling price of the item then the item is sold at profit or gain. The difference between selling price and cost price is called *profit.*

That is, profit = S.P. − C.P. (S.P. > C.P.)

Now if the cost price of an item is more than the selling price of an item then the item is sold at loss. The difference between cost price and selling price is called *loss.*

That is, loss = C.P. − S.P. (C.P. > S.P.)

The costs which are directly connected with the manufacturing or purchasing the items are called *direct costs* or *prime costs*. For example, wages of laborers, price of machinery, cost of production, cost of raw material, etc. are called direct costs. The costs which are not directly connected with the manufacturing or purchasing the items are called *indirect costs* or *overhead costs.* For example, house rent, advertisement cost, maintenance charges, etc. are called indirect costs. Now when direct costs are subtracted from the selling price of an item then it is called *gross profit* and when indirect costs are subtracted from gross profit then it is called *net profit.* The following are some important formulas.

Gross profit = Selling price − Direct costs
Net profit = Gross Profit − Indirect costs

$$\text{Gain percentage} = \frac{\text{Gain}}{\text{Cost price}} \times 100$$

$$= \frac{\text{Gain}}{\text{Selling price}} \times 100$$

$$\text{Loss percentage} = \frac{\text{Loss}}{\text{Cost price}} \times 100$$

$$= \frac{\text{Loss}}{\text{Selling price}} \times 100$$

ILLUSTRATIONS

Illustration 1 Mustak purchases 200 dozen eggs at the rate of Rs. 18 per dozen. 72 eggs are broken in transportation. He sells remaining eggs at the rate of Rs. 24 per dozen. Calculate his profit.

Solution

Purchase price of 200 dozen eggs = 18 × 200 = Rs. 3,600
Number of broken eggs = 72 = 6 dozen
∴ No. of eggs remaining = 200 − 6 = 194 dozen
∴ Selling price of 194 dozen eggs = 194 × 24
= Rs. 4,656
∴ Profit = Selling price − Purchase price
= 4,656 − 3,600
= Rs. 1,056

Illustration 2 A trader purchases 100 units of an item at Rs. 630 per unit. He sells 45 units at the rate of Rs. 665 per unit, 35 units at the rate of Rs. 650 per unit and remaining at the rate of Rs. 635 per unit. Find his total gain in percentage.

Solution

Cost price of the item is Rs. 630 per unit
∴ Total cost of purchasing 100 items = 630 × 100
= Rs. 63,000
Now selling price of 45 items is Rs. 665 per unit
∴ Total selling price of 45 items = 665 × 45
= Rs. 29,925

Total selling price of 35 items = 650 × 35

= Rs. 22,750

Total selling price of remaining items = $[100-(45-35)]$

= 20 items at the rate of Rs. 635 per unit is

= 635 × 20

= Rs. 2,375

∴ Total selling price of 100 items = 29,925 + 22,750 + 12,700

= Rs. 65,375

∴ Profit = Selling price − Purchase price

= 65,375 − 63,000

= Rs. 2,375.

$$\therefore \text{Gain percentage} = \frac{\text{Gain}}{\text{Purchase price}} \times 100$$

$$= \frac{2,375}{63,000} \times 100$$

= 3.77

Illustration 3 By Selling an article for Rs. 2,000 a seller suffers a loss of 8 %. What is the cost price of the article?

Solution

Let the cost price of article be Rs. x. Now selling price at 8% loss is Rs. 2,000

∴ 2,000 = $(100 - 8)$% of x

2,000 = 92% x

$$\therefore x = \frac{2,000 \times 92}{100} = 1,840$$

∴ Cost price of the article is Rs. 1,840.

Illustration 4 A person sells 75 units of an item for Rs. 1,012.50 and earns 8% profit. Find cost price of the item per unit.

Solution

$$\text{Now cost price C.P.} = \frac{\text{S.P.} \times 100}{(100 + \text{Profit }\%)}$$

$$= \frac{1,012.5 \times 100}{100 + 8}$$

= 9,375

$$\therefore \text{Cost price per unit of the item} = \frac{9,375}{75}$$

= Rs. 12.50/unit

Illustration 5 A TV set is sold for Rs. 36,375 at a loss of 15%. Find the purchase price of the TV set

Solution

$$\text{Purchase price} = \frac{\text{S.P.} \times 100}{(100 - \text{Loss }\%)}$$

$$= \frac{36,375 \times 100}{(100 - 15)}$$
$$= \frac{36,375 \times 100}{85}$$
= Rs. 42,794.12

Illustration 6 A diamond merchant makes the same percentage profit on which he purchases a diamond by selling it for Rs. 1,515. Find the cost price of the diamond.

Solution

Selling price = Purchase price + Profit

$= x + \frac{x}{100}$ (let cost price = x)

$\therefore$ 1,515 = x + 0.01x

$\therefore$ 1,515 = 1.01x

$\therefore \ x = \frac{1,515}{1-01}$

$\therefore \ x$ = 1,500

$\therefore$ Cost price of the diamond is Rs. 1,500.

Illustration 7 Kesha sells two items for Rs. 4,800 each. In one deal she earns a profit of 25% and in the other she suffers a loss of 20%. Find the profit or loss percentage in the whole deal.

Solution

Selling price of first item = Rs. 4,800

Profit % = 25

$\therefore$ Cost price of first item = $\frac{\text{Selling price} \times 100}{100 + \text{Profit }\%}$

$= \frac{4,800 \times 100}{100 + 25}$

= 3,840

Selling price of second item = Rs. 4,800

Loss % = 20

$\therefore$ Cost price of the item = $\frac{\text{Selling price} \times 100}{(100 - \text{Loss }\%)}$

$\therefore$ Cost price of the item = $\frac{4,800 \times 100}{100 - 20}$

$\therefore$ Total cost price of both the items = 3,840 + 6,000

= 9,840

and selling price of both the items = 9,600

Since cost price is more than selling price so there is a loss in the business.

Loss = 9,840 − 9,600

= Rs. 240

$$\therefore \text{Loss \%} = \frac{\text{Loss} \times 100}{\text{Cost price}}$$

$$= \frac{240 \times 100}{9{,}840}$$

$$= 2.44$$

Illustration 8 Mr X purchased three items, each costing Rs. 2,500. He sold one item at a 10% loss. At what per cent should the other two items be sold to earn 20% profit on the whole deal.

Solution

Cost price of the item = Rs. 2,500

Loss % = 10

$$\text{Selling price of the item} = \frac{\text{Cost price} \times (100 - \text{Loss})}{100}$$

$$= \frac{2{,}500 \times (100 - 10)}{100}$$

$$= \text{Rs. } 2{,}250$$

Also cost price of 3 items = 3 × 2,500

= Rs. 7,500

Now profit on the whole deal = 20%

$$\therefore \text{Selling price of the item} = \frac{\text{Cost price} \times (100 + \text{Profit})}{100}$$

$$= \frac{7{,}500 \times (100 + 20)}{100}$$

$$= \text{Rs. } 9{,}000$$

Now selling price of one item = Rs. 2,250

∴ Selling price of remaining two items = 9,000 − 2,250

= Rs. 6,750

∴ Selling price of each of the items = Rs. 3,375

Now the cost price of two items = 2,500 × 2

= 5,000

∴ Profit = Selling price − Cost price

= 6,750 − 5,000

= Rs. 1,750

$$\therefore \text{Profit \%} = \frac{\text{Profit} \times 100}{\text{Cost price}}$$

$$= \frac{7{,}500 \times 100}{5{,}000}$$

$$= 35$$

Illustration 9 A retailer sells his articles at 20% profit. If he adds Rs. 16 more to the selling price of the item then the retailer has a profit of 30%. Find the purchase price of the article.

Solution

Let the purchase price be x

$$\therefore \text{Selling price} = \frac{\text{Cost price} \times (100 + \text{Proft})}{100}$$

$$= \frac{x(100+20)}{100}$$

$$= 1.20x$$

Now if selling price of the item is = $1.20x + 16$

then profit = 30%

$$\therefore \text{Cost of article} = \frac{\text{Selling price} \times 100}{100 + \text{Profit}\ \%}$$

$$\therefore\ x = \frac{(1.20x + 16) \times 100}{100 + 30}$$

$$x = \frac{120x + 1{,}600}{130}$$

$\therefore$ $130x = 120x + 1{,}600$

$\therefore$ $10x = 1{,}600$

$\therefore$ x = Rs. 160

Illustration 10 A grocer sells a particular variety of sugar at 4% loss at Rs. 24 per kg. He also sells another variety of sugar at 8% gain at Rs. 34.56 per kg. He mixes the two varieties of sugar and sells at Rs. 32 per kg. Find the profit or loss per cent.

Solution

Let the merchant mix y kg of each variety

$\therefore$ Selling price of y kg of sugar of 1st variety = $24y$

Loss % = 4

$$\therefore \text{Cost price of } y \text{ kg of sugar of 1st variety} = \frac{\text{Selling price} \times 100}{100 + \text{Loss}\ \%}$$

$$= \frac{24y \times 100}{100 - 4}$$

$$= 25y$$

Now selling price of y kg of sugar of 2nd variety = $34.56y$

$$\therefore \text{Cost price of } y \text{ kg of sugar of 2nd variety} = \frac{34.56y \times 100}{100 + 8}$$

$$= 32y$$

$\therefore$ Total cost of $2y$ kg of sugar = $25y + 32y$

= $57y$

$\therefore$ Selling price per kg of mixed sugar = Rs. 32

$\therefore$ Selling price of $2y$ kg of mixed sugar = $64y$

$\therefore$ Profit = $64y - 57y = 7y$

$\therefore$ Profit per 2 kg of mixed sugar = Rs. 7

$$\therefore \text{Profit} = \frac{\text{Profit} \times 100}{\text{Cost price}}$$

$$= \frac{7 \times 100}{57}$$

$$= 12.28\%$$

Illustration 11 A fruit merchant purchased apples worth Rs. 9,000. He sold 70% of apples at 40% profit, 20% apples at 20% profit and remaining apples at 30% loss. Find the profit in percentage.

Solution

Cost price of apples purchased = Rs. 9,000

Cost price of 70% of apples $\frac{70 \times 9{,}000}{100} = 3{,}600$

Profit % = 40

$$\text{Selling price of those apples} = \frac{\text{Cost price} \times (100 + \text{Profit \%})}{100}$$

$$= \frac{6{,}300 \times 140}{100}$$

$$= \text{Rs. } 8{,}820$$

Cost price of 20% of apples $= \frac{20 \times 9{,}000}{100} = \text{Rs. } 1{,}800$

Profit % = 20

Selling price $= \frac{1{,}800 \times 120}{100} = 2{,}160$

Lastly cost price of 10% of apples $= \frac{10 \times 9{,}000}{100} = 900$

$$\therefore \text{Selling price} = \frac{\text{Cost price} \times (100 - \text{Loss})}{100}$$

$$= \frac{900 \times (100 - 30)}{100}$$

$$= \text{Rs. } 630$$

$$\therefore \text{Total selling price} = \text{Rs. } (8{,}820 + 2{,}160 + 630)$$

$$= \text{Rs. } 11{,}610$$

$$\therefore \text{Profit} = \text{Selling price} - \text{Cost price}$$

$$= 11{,}610 - 9{,}000$$

$$= 2{,}610$$

$$\therefore \text{Profit \%} = \frac{\text{Profit} \times 100}{\text{Cost price}}$$

$$= \frac{2{,}610 \times 100}{9{,}000}.$$

$$= 29\%$$

Illustration 12 Mr P sells his house to Mr Q at 15% profit. Mr Q sells it to Mr R at 20% gain and Mr R sells it to Mr S at 10% loss. for Rs. 3,72,600. Find the cost price of the house for Mr P

Solution

Let the cost price of house for Mr P = Rs. x

Gain % = 15

$$\text{Selling price of house to Mr Q} = \frac{\text{Cost price} \times (100 + \text{Gain\%})}{100}$$

$$= \frac{x \times (100 + 15)}{100}$$

$$= 1.15x$$

Purchase price for Mr Q = $1.15x$

Now for Mr Q gain % = 20

$$\therefore \text{Selling price of house to Mr R} = \frac{1.15x(100 + 20)}{100}$$

$\therefore$ Purchase price of house for Mr R = $1.38x$

Now Mr R sells the house to Mr S at a loss of 10%

$$\therefore \text{Selling price of house to Mr S} = \frac{\text{Cost price} \times (100 - \text{Loss})}{100}$$

$$= \frac{1.38x(100 - 10)}{100}$$

$$= 1.38x\ 0.90$$

$$= 1.242x$$

$\therefore$ Purchase price of house for Mr S = Rs. $1.242x$

But the purchase price for Mr S = 3,72,600

$\therefore 1.242x = 3{,}72{,}600$

$$\therefore x = \frac{3{,}72{,}600}{1.242}$$

x = Rs. 3,00,000

$\therefore$ Cost price of the house for Mr P = Rs. 3,00,000

Illustration 13 A person suffers a loss of 15% by selling an item at Rs. 552.5 per unit. If the item is sold for Rs. 700 then find out the profit percentage.

Solution

Selling price of the item = Rs. 552.5

Loss % = 15

$$\therefore \text{Cost price of the item} = \frac{\text{Selling price} \times 100}{(100 - \text{Loss})}$$

$$= \frac{552.5 \times 100}{85}$$

$\therefore$ Cost price of the item = Rs. 650

Selling price of the item = Rs. 700

$\therefore$ Profit = Selling price − Cost price

= 700 − 650

= Rs. 50

$$\therefore \text{Profit \%} = \frac{\text{Profit} \times 100}{\text{Cost price}}$$

$$= \frac{50 \times 100}{650}$$

= 7.69

Illustration 14 Jainil sells an item at 10% profit instead of 5% loss and gets Rs. 50 more. What is the cost price of the item?

Solution

Let the cost price of the item = Rs. 100

Profit % = 10

$$\therefore \text{Selling price} = \frac{\text{Cost price} \times (100 + \text{Profit \%})}{100}$$

$$= \frac{100 \times (100 + 10)}{100}$$

= Rs. 110

Again cost price of the item = Rs. 100

loss % = 5

$$\therefore \text{Selling price} = \frac{\text{Cost price} \times (100 - \text{Loss \%})}{100}$$

$$= \frac{100 \times 95}{100}$$

= Rs. 95

$\therefore$ Difference between selling prices = Rs. $(110 - 95)$

= Rs. 15

Now according to given data the difference of the selling prices = Rs. 50

Now if difference in selling price is Rs. 15 then cost price = Rs. 100

$\therefore$ When difference in selling price = Rs. 50 then

$$\text{the cost price} = \frac{50 \times 100}{15} = \frac{1{,}000}{3}$$

= Rs. 333.33

Illustration 15 A person sells a watch for Rs. 4,000 and loses some money. If he had sold it for Rs. 5,000, his profit would have been 2/3 of the former loss. Find the cost price of the watch.

Solution

Let the cost price of watch = Rs. x

Selling price of watch = Rs. 4,000

$\therefore$ Loss = Cost price − Selling price.
$= x - 4{,}000$
Also, the cost price of watch = Rs. x
and selling price of the watch = Rs. 5,000
$\therefore$ Profit $= 500 - x$
Now according to the given information

$$\text{Profit} = \frac{2}{3}\text{Loss}$$

$$\therefore 5{,}000 - x = \frac{2}{3}(x - 4{,}000)$$

$\therefore 15{,}000 - 3x = 2x - 8{,}000$
$\therefore 7{,}000 = x$
$\therefore$ Cost of watch is Rs. 7,000.

Illustration 16 A retailer sells 100 boxes of the items at the rate of Rs. 10 per item and suffers a loss of Rs. 800. He makes a profit of Rs. 1,200, if he sells items at the rate of Rs. 15 per unit. Find out the quantity of items and cost per item.

Solution

Suppose there are y number of units in a box
Selling price of the item per unit = Rs. 10
$\therefore$ Selling price of 100 boxes = Rs. $(100 \times 10 \times y)$
= Rs. 100 y
Loss = Rs. 800
$\therefore$ Cost price of 100 boxes = Selling price + Loss
$= 1{,}000y + 800$
Again selling price per unit = Rs. 15
$\therefore$ Selling price of 100 boxes $= 15 \times 100 \times y$
= Rs. $1{,}500y$
Profit = Rs. 1,200
$\therefore$ Cost price = Selling price − Profit
$= 1{,}500y - 1{,}200$
Now $1{,}000y + 800 = 1{,}500y - 1{,}200$
$\therefore 1{,}200 + 800 = 1{,}500y - 1{,}000$
$\therefore 500y = 2{,}000$
$\therefore\ y = 4$
$\therefore$ A box contains 4 items

$$\therefore \text{Cost price of each item} = \frac{1{,}000 \times 4 + 800}{100}$$

= Rs. 48

Illustration 17 Person A sold an article to person B at 10% profit. Person B sells back to person A at 10% loss. In this deal, person A gained Rs. 100. Find the original cost price of the article for person A.

Solution

Let the cost price of the item for A = 100 + profit 10%

$$\text{Selling price of the item} = \frac{\text{Cost price}(100 + \text{Profit})}{100}$$

$$= \frac{100(100+10)}{100}$$

$$= 110$$

Loss = 10%

$$\text{Selling price} = \frac{\text{Cost price}(100 - \text{Loss})}{100}$$

$$= \frac{100(100-10)}{10}$$

$$= 90$$

$\therefore$ Gain of A in the deal = Rs. $(110 - 90)$

Now when gain of A is Rs. 20, the cost price of item = Rs. 100

$\therefore$ When gain of A is Rs. 100, the cost price of the item

$$= \frac{100 \times 100}{20}$$

= Rs. 500

Illustration 18 A man purchases chairs and tables. When he sells the tables at 20% profit and changing at 10% loss, there is no profit or loss to him. If he sells chairs at 5% loss and tables at 5% profit then he has to bear a loss of Rs. 100. Find the purchase price of table and chair.

Solution

Let the purchase price of table = Rs. x

and the purchase price of chair = Rs. y

Now profit on table = 20%

$$\therefore \text{Selling price of table} = \frac{\text{Cost price}(100 + \text{Profit})}{100}$$

$$= \frac{x(100+20)}{100}$$

$$= 1.20x$$

and loss on chair = 10%

$$\therefore \text{Selling price of chair} = \frac{\text{Cost price}(100 - \text{Loss}\,\%)}{100}$$

$$= \frac{y(100-10)}{10}$$

$$= 0.90y$$

Now $x + y = 1.20x + 0.90y$ (no profit or loss)

$\therefore\ 0.20x = 0.10y$

$y = 2x$

Now selling chair at 5% loss

$\therefore$ the selling price of chair $= \dfrac{y(100-10)}{10}$

$= 0.95y$

and selling tables at 5% gain,

$\therefore$ the selling price of table $= \dfrac{x(100+150)}{10}$

$= 1.05y$

In this transaction he has a profit of Rs. 100

$\therefore\ x + y - (0.95y + 0.05x) = 100$

$\therefore\ 0.05y - 0.05x = 100$

$\therefore\ 5y - 5x = 1{,}000$

$\therefore\ y - x = 2{,}000$ but $y = 2x$

$\therefore\ 2x - x = 200$

$\therefore\ x = 2{,}000$ and $y = 4{,}000$

$\therefore$ Cost price of table = Rs. 2,000 and that of chair is Rs. 4,000.

Illustration 19 Two items A and B are bought for Rs. 45,000. Item A is sold at 16% gain. and item B is sold at 25% gain. Find the cost price of both the items if the net gain is 20%.

Solution

Let the cost price of item A = Rs. x

$\therefore$ Cost price of item B $= 45{,}000 - x$

Profit on the sale of item A = 16%

$\therefore$ Selling price of item A $= \dfrac{\text{Cost price}(100 + \text{Gain }\%)}{100}$

$= \dfrac{x(100+16)}{100}$

$= 1.16x$

Profit on the sale of item B = 25%

$\therefore$ Selling price of item B $= \dfrac{(45{,}000 - x)(100+25)}{100}$

$= 1.25\,(45{,}000 - x)$

Combined profit on sale of A and B both = 20%

$\therefore$ Combined selling price $= \dfrac{45{,}000(100+20)}{100}$

$= 54{,}000$

Thus, $1.16x + 1.25\,(45{,}000 - x) = 54{,}000$

$1.16x + 56{,}250 - 1.25x = 54{,}000$

$0.09x = 2{,}250x = 2{,}500$

$\therefore$ Selling price of item A = Rs. 25,000 and

Selling price of item B = 45,000 − 25,000 = Rs. 20,000

Illustration 20 A shopkeeper earns 30% profit on selling price. What would be this percentage of profit on cost price?

Solution

Let the selling price = Rs. 100

Profit on selling price = 30%

$$\text{Profit} = \frac{100 \times 30}{100}$$

$$= \text{Rs. } 30.$$

$$\text{Cost price} = \text{Selling price} - \text{Profit}$$

$$= 100 - 30$$

$$= 70$$

$$\therefore \text{Profit on cost price } \% = \frac{\text{Profit} \times 100}{\text{Cost price}}$$

$$= \frac{30 \times 100}{70}$$

$$= 42.86$$

Illustration 21 A tailor measures 96 cms for each meter. Find how much percentage he earns by incorrect measurement?

Solution

1 meter = 100 cm

Incorrect measurement 1 meter = 96 cm (cheating)

∴ Gain in cost due to cheating = Cost price of 4 cm

Let the cost price = Rs. 100

$$\therefore \text{Profit } \% = \frac{\text{Profit} \times 100}{\text{Cost price}}$$

$$= \frac{4 \times 100}{96}$$

$$= 4.17$$

Illustration 22 A dishonest shopkeeper professes to sell at cost price but uses a false kilogram weight of 975 grams. Find his gain percentage.

Solution

1 kilogram = 1,000 grams

Suppose the cost price of 1,000 grams = Rs. 100

Cost price of 975 grams = Rs. 97.5

$$\text{Gain} = 100 - 97.50$$

$$= \text{Rs. } 2.50$$

$$\therefore \text{Gain } \% = \frac{\text{Gain} \times 100}{\text{Cost price}}$$

$$= \frac{2.50 \times 100}{97.50}$$

$$= 2.56$$

Illustration 23 A retailer earns 10% profit after giving 20% discount on an article. If he earns Rs. 128 per article. Find the cost price of that article.

Solution

Let the cost price = Rs. x

Now a discount of 20% is given

$$\therefore \text{Discount} = \frac{20 \times x}{100} = 0.2x$$

$$\therefore \text{Selling price} = x - 0.2x = 0.8x$$

Now profit % = 10

$$\therefore \text{Cost price} = \frac{\text{Selling price} \times 100}{100 + \text{Profit } \%}$$

$$= \frac{0.8x \times 100}{(100 + 10)}$$

$$= \frac{8x}{11}$$

$$\therefore \text{Profit} = \text{Selling price} - \text{Cost price}$$

$$128 = 0.8x - \frac{8x}{11}$$

$$\therefore 128 = \frac{8{\cdot}8x - 8x}{11} = \frac{0.8x}{11}$$

$$\therefore x = \frac{128 \times x}{0.8} = 1{,}760$$

∴ Cost price of the article is Rs. 1,760

Illustration 24 The list price of an item is 25% above the selling price and the cost price 40% below the list price. Find percentage rate of profit and discount.

Solution

Let the selling price = Rs. 100

The list price of the item = Rs. 125

$$\therefore \text{The cost price} = \text{Rs. } 125\,(100 - 40)\%$$

$$= \text{Rs. } 75$$

$$\therefore \text{Amount of discount} = \text{List price} - \text{Selling price}$$

$$= 125 - 100 = \text{Rs. } 25$$

$$\therefore \% \text{ of discount} = \frac{\text{Discount} \times 100}{\text{List price}}$$

$$= \frac{25}{125} \times 100 = 25$$

$$\text{Now Profit} = \text{Selling price} - \text{Cost price}$$

$$= 100 - 75$$

$$= \text{Rs. } 25$$

$$\therefore \text{ \% Profit} = \frac{\text{Total Profit} \times 100}{\text{Cost price}}$$

$$= \frac{25 \times 100}{75}$$

$$= 33.33$$

Illustration 25 A merchant allows 25% discount on his advertised price and then makes profit of 20% on the cost price. What is the advertised price of the item on which he gains Rs. 300?

Solution

Let the cost price of the item be Rs. 100

∴ Selling price = 100 + 20 = Rs. 120

Let the advertised price = Rs. x

Now selling price at 25% discount = advertised cost (100 − 25)%

$\therefore 120 = x\ 75 = 0.75x$

$$\therefore x = \frac{120}{0.75}$$

$$\therefore x = 160$$

Now when profit is Rs. 20 then advertised price is Rs. 160

When profit is Rs. 300 then advertised price

$$= \frac{300 \times 160}{20}$$

= Rs. 2,400

ANALYTICAL EXERCISES

1. A shopkeeper purchased an article for Rs. 700 and sold it for Rs. 784. Find his percentage profit.
2. Mehul purchases 200 dozen notebooks at the rate of Rs. 240 per dozen. A bunch of 48 notebooks was wasted in transportation and became worthless. The remaining notebooks he has sold at Rs. 320 per dozen. Calculate his profit/loss.
3. A trader purchased 250 units at the rate of Rs. 60 per unit. He sold 125 units at the rate of Rs. 75 per unit, 75 units at the rate of Rs. 70 per unit and the remaining units at the rate of Rs. 55 per unit. Find his total profit/loss in percentage.
4. A person sells 100 units of an item for Rs. 10,125 and earns 10% profit. Find the cost price of item per unit.
5. A fridge is sold for Rs. 18,750 at a loss of 12%. Find the purchase price of the fridge.

6. A person makes the same percentage profit on which he purchases a used car by selling it for Rs. 75,000. Find the cost price of the car.
7. Kena sells two items for Rs. 6,000 each. In one deal she earns a profit of 40% and in other she suffers a loss of 20%. Find the profit/loss percentage in the whole deal.
8. A retailer sells his product at 25% profit. If he adds Rs. 20 to its selling price then the retailer has a profit of 35%. Find the purchase price of the article.
9. A grocer sells a particular variety of rice at 4% loss at Rs. 18/kg. He also sells another variety of rice at 10% gain at Rs. 33/kg. He mixes two varieties in equal proportion and sells at Rs. 30/kg. Find the profit or loss percentage.
10. A fruit merchant purchased mangoes worth Rs. 5,000. He sold 70% mangoes at 50% profit, 20% mangoes at 25% profit and remaining 10% mangoes at cost price. Find the profit percentage in the total selling of mangoes.
11. A person sold 50 articles for the same amount of money as he paid for 70. What was the percentage gain on his outlay?
12. By selling 200 units of an item, a retailer gains the selling price of 20 items. Find the gain percentage
13. While selling an item a retailer loses 6.5%. He would have gained 10% had he sold it for Rs. 660 more. Find the actual price of the item.
14. The producer, the wholesaler and the retailer gains 20%, 30% and 40%, respectively. If the retailer sells the item at the cost of Rs. 5,460 then, what is the actual cost of producing the item?
15. Krupa suffers a loss of 10% by selling an article at Rs. 540 per unit. If she sold the article at Rs. 650 then find out the profit percentage.
16. Mr X sells an item at 10% profit instead of 10% loss and gets Rs. 100 more. What is the cost price of the item?
17. An item was sold at a profit of 12%. If the cost price decreases by 10% and selling price increases by Rs. 40 then the profit is 100/3%. What will be the selling price of the item, if it is sold at 20% profit?
18. A person sells an item for Rs. 500 and loses some money. If he had sold it for Rs. 600, his profit would have been 1/3 of the former loss. Find the cost price of the item.
19. If 6% of selling price of an article is equal to 5% of its cost price and 10% of selling price exceeds 9% of cost price by Rs. 450, find the cost price and selling price.
20. A milkman purchased 40 liter of milk at the rate of Rs. 15/liter and 60 liter of milk at the rate of Rs. 18/liter. He sold the whole milk at the rate of Rs. 20/liter. Find profit and profit percentage.
21. A merchant earns 20% profit on selling price. What would be this percentage of profit on cost price?

22. The capital of a firm is Rs. 5,00,000 and its annual sale is Rs. 4,60,000. The cost of production is 55% of selling price, expenses on advertisement are 1.5% of sales, and salary of workers is Rs. 95,000, rent and other miscellaneous expenditure is Rs. 20,000 and sales tax is 4%. Find the percentage gain on capital. If on doubling the advertising expenses the sales increase by 12%, find the new gain.

23. A manufacturer wants to sell his goods at 20% of the cost price. When he gives X% discount he earns 10% profit. Find the value of X.

24. A manufacturer gives 25% discount and still he earns 10% profit. If he wishes to earn Rs. 1,100 per item then find the list price of the item.

25. A merchant earns a 100/9% profit by underweighting the goods he sells. For how much does he weigh in place of 1 kg?

ANSWERS

(1) 12%
(2) loss of Rs. 640
(3) 15.83%
(4) Rs. 92.05
(5) Rs. 21,306.82
(6) Rs. 74,257.43
(7) Profit 1.82%
(8) Rs. 200
(9) 23.08% profit
(10) Profit 40%
(11) 40%
(12) 11.11%
(13) Rs. 4,000
(14) Rs. 2,500
(15) 8.33%
(16) Rs. 500
(17) Rs. 600
(18) Rs. 575
(19) SP = Rs. 16,071.42, CP = Rs. 19,285.71
(20) Profit = Rs. 320, Profit % = 19.05
(21) 25%
(22) 13.34%, 30.36%
(23) $8\frac{1}{3}\%$
(24) Rs. 16,133.33
(25) 0.9 kg

DISCOUNT

It is very common that in day-to-day business, in order to retain customer ship or to encourage the customers to make early payments, some lower price than the printed price is offered to the customer. This reduction in the printed price is called *discount*, that is, the difference between the printed price and the selling price of an article is called discount,

$$\text{Discount} = \text{Printed price} - \text{Selling price}$$

$$\text{and Discount percentage } = \frac{\text{Discount} \times 100}{\text{Printed price}}$$

In practice, the manufacturer sells the item to the dealer and dealer to the retailer at a price lower than the printed price. This discount given on the printed price in the sale of the item is called *trade discount*. Sometimes the manufacturer gives some additional item(s) of the value equal to the trade discount to the purchaser. This discount is called *commodity discount*. In order to attract people to buy the items in large number of units, sometimes the seller gives some extra discount in addition to the said discount. This discount is called *successive discount*. The difference between the printed price and the price after successive discount is called *equivalent discount*. The discount given to the purchaser on cash payment or early payment is called *cash discount*.

Illustration 26 A manufacturer gives 25% trade discount and 10% cash discount. If the list price of that article is Rs. 4,000 then what will be its selling price?

Solution

The price of item at 25% discount

$$= \text{print price } (100 - 25)\% = 400\frac{75}{100}$$

= Rs. 3,000

and the actual selling price of the item at 10% cash discount

$$= 3{,}000 \times (100 - 10)\%$$

$$= 3{,}000 \times \frac{90}{100}$$

= Rs. 2,700

Illustration 27 A wholesaler gives 10% discount on the market price. What will be the market price of the item whose cost price is Rs. 750 in order to have 25% profit?

Solution

The cost price is Rs. 750

Profit % = 25%

$$\therefore \text{Selling price} = \frac{750 \times (100 + 25)}{100}$$

= Rs. 937.50

Now wholesaler must print the price of item in such a way that after giving 10% discount on market price the print price is 937.50

$$\therefore 937.50 = \text{marked price } (100 - 10)\%$$

$$\therefore 937.50 = x \times \frac{90}{100}$$

$$x = \frac{937.50 \times 100}{90}$$

$$\therefore x = \text{Rs. } 1{,}041.67$$

Illustration 28 The list price of an item is 30% above selling price and the cost price is 50% below the list price. Find the percentage of rate of discount and profit.

Solution

Suppose the selling price = Rs. 100

$\therefore$ List price of the item = Rs. $(100 + 30)$ = Rs. 130

and cost price is below 50% of list price

$\therefore$ Cost price = $130(100 - 50)\%$

= Rs. 65

$\therefore$ Amount of discount = List price − Selling price

= Rs. 130 − 100

= Rs. 30

$$\therefore \text{Percentage rate of discount} = \frac{\text{Discount}}{\text{List price}} \times 100$$

$$= \frac{30}{130} \times 100$$

= 23.08%

Again Profit = Selling price − Cost price

= Rs. 100 − Rs. 65

= Rs. 35

$$\therefore \text{Percentage profit} = \frac{\text{Total profit}}{\text{Cost price}}$$

$$= \frac{35}{65} \times 100$$

= 53.85%

Illustration 29 A profit of 40% was made on the selling of an item which is purchased at Rs. 500 after giving 20% trade discount and 20% cash discount. What is the print price of that item?

Solution

Let the print price = Rs. x

The price of the item at 20% trade discount = print price $(100 - 20)\%$

$$= x \times \frac{80}{100}$$

= $0.8x$

Again after giving 20% cash discount the selling price of the item

= $0.8x(100 - 20)\%$

= $0.8x \times 0.8$

= $0.64x$

Now a profit of 40% was made on the selling of the item

Hence Cost price = Selling price − Profit

= 100 − 40

= Rs. 60

Now Cost price = Rs. 60, Selling price = Rs. 100

$\therefore$ Cost price = Rs. 500, Selling price = ?

$$\therefore \text{Selling price} = \frac{500 \times 100}{60}$$

$$= \text{Rs. } 833.33$$

But selling price = $0.64x$

$\therefore$ 833.33 = $0.64x$

$x = 1302.08$

$\therefore$ The print price of the item = Rs. 1302.08

Illustration 30 A dealer allows 25% discount on his advertised price and makes a profit of 25% on the cost price of an item. What is the advertised price of the item on which he gains?

Solution

Let the cost price of an item be Rs. 100 and the advertised price be Rs. x

$\therefore$ Since the profit is 25% on the cost price

$\therefore$ Selling price = 100 + 25 = Rs. 125

Now selling price at 25% discount = advertised price × $(100 - 25)\%$

$125 = x \times 75\%$

$$\therefore x = \frac{125 \times 100}{75} = \text{Rs.}\, 166.67$$

Now if profit is Rs. 25 then advertised cost = Rs. 166.67

$\therefore$ If profit is Rs. 300 then advertised cost = ?

$$\therefore \text{Advertised cost} = \frac{300 \times 166.67}{25}$$

$$= \text{Rs. } 1{,}999.99$$

$$= \text{Rs. } 2{,}000$$

Illustration 31 The sale price of 125 pens is equal to the print price of 105 pens. Calculate the percentage discount.

Solution

Let the print price of one pen = Rs. x

$\therefore$ Print price of 105 pens = $105x$

Now sale price of 125 pens = Print price of 105 pens

= $105x$

$$\therefore \text{Sale price of one pen} = \frac{105x}{125}$$

$$\therefore \text{Discount per pen} = x - \frac{105x}{125}$$

$$= \frac{20x}{125} = \frac{4x}{25}$$

$$\therefore \text{Rate of discount} = \frac{\text{Discount} \times 100}{\text{Print price}} = \frac{4x \times 100}{25 \times x}$$

$$= 16\%$$

Illustration 32 A manufacturer allows two successive discounts on the sale of articles. Prove that the equivalent discount $D = \left(d_1 + d_2 - \frac{d_1 d_2}{100}\right)\%$.

Solution

Let d_1 and d_2 be two successive discounts and D% the equivalent discount. Also the print price of the item is say Rs. 100.

$$\therefore \text{Discount} = \frac{\text{Discount\%} \times \text{Print price}}{100}$$

$$= \frac{d_1 \times 100}{100} = d_1$$

$\therefore$ Amount payable after 1st discount = Rs. $(100 - d_1)$

Now second discount = $\frac{d_2 \times (100 - d_1)}{100}$

Now selling price after two successive discounts

$$= \left[(100 - d_1) - \frac{(100 - d_1)d_2}{100}\right]$$

$\therefore$ Equivalent discount = List price − Selling price after both discounts

$$= 100 - \left[100 - d_1 - \frac{(100 - d_1)d_2}{100}\right]$$

$$= d_1 + \frac{(100 - d_1)d_2}{100}$$

$$= \text{Rs.}\left(d_1 + d_2 - \frac{d_1 d_2}{100}\right)$$

Illustration 33 Which option is more profitable to the purchaser?
(i) 25% discount, (ii) 20% + 5% discount, (iii) 15% discount + 10% discount

Solution

Let the print price = Rs. 100
First option gives 25% discount
Second option gives 20% + 5% discount

$$\therefore \text{Equivalent discount} = \left[20 + 5 - \frac{20 \times 5}{100}\right] = 24\% \text{ discount}$$

Third option gives 15% + 10% discount

$$\therefore \text{Equivalent discount} = \left[15 + 10 - \frac{15 \times 10}{100}\right] = 23.50\% \text{ discount}$$

Hence the first option of 25% discount is profitable to the purchaser

Illustration 34 A trader allows 20% + 10% discount to his customer. A customer has placed the order amounting to Rs. 15,500. Calculate the amount due to the trader.

Solution

$d_1 = 20\%$, $d_2 = 10\%$, List price = Rs. 15,500

$$\text{Equivalent discount (D)} = \left(20 + 100\frac{20 \times 10}{100}\right) = 28\%$$

$$\text{Discount} = \frac{\text{Equivalent discount} \times \text{List price}}{100}$$

$$= \frac{28 \times 15{,}500}{100}$$

= Rs. 4,340

∴ Amount due to the trader = Rs. 15,500 − Rs. 4,340

= Rs. 11,160

Illustration 35 Price of a TV is Rs. 34,500. A dealer earns 20% profit after giving 10% discount. Find the cost price of TV.

Solution

$$\text{Selling price} = \text{List price}\left(1 - \frac{\text{Discount}}{100}\right)$$

$$= 34{,}500\left(1 - \frac{10}{100}\right) = \text{Rs. } 31{,}050$$

Now profit = 20%

$$\therefore \text{Cost price} = \frac{\text{Selling price} \times 100}{100 + \text{Gain}}$$

$$= \frac{31{,}050 \times 100}{100 + 20}$$

= Rs. 25,875

COMMISSION AND BROKERAGE

A person employed in buying or selling for another person is called an *agent*. The amount paid to the agent for his job is called *commission.*

A person who works as a mediator between a buyer and a seller is called a *broker.* The commission that a broker gets for his mediatorship is called *brokerage.* A buyer has to pay the actual cost price plus the brokerage and the seller gets the actual selling price minus the brokerage. The additional amount paid to the agent when the sales exceed the pre-determined sales is called *bonus*.

Following are some important formulae.

$$\text{Amount of commission} = \frac{\text{Rate of commission} \times \text{Amount of sale}}{100}$$

$$\text{Rate of commission} = \frac{\text{Amount of commission} \times 100}{\text{Amount of sale}}$$

$$\text{Amount of sale} = \frac{\text{Amount of commission} \times 100}{\text{Rate of commission}}$$

$$\text{Amount of del credere commission} = \frac{\text{Rate of del credere commission} \times \text{Credit sale}}{100}$$

Illustration 36 An agent receives Rs. 1,500 as commission at the rate of 7.5% on sales. Find the amount of sales.

Solution

$$\text{Amount of sales} = \frac{\text{Amount of commission}}{\text{Rate of commission}} \times 100$$

$$= \frac{1{,}500 \times 100}{7.5}$$

$$= \text{Rs. } 20{,}000$$

Illustration 37 By selling some items at the rate of Rs. 350 per unit, how many units does the agent have to sell so that his total commission is Rs. 4,500, if the rate of commission is 12.5%?

Solution

Suppose the agent sells x units of item, then total selling price = Rs. $350x$

$$\text{Now amount of commission} = \frac{\text{Amount of sales} \times \text{Rate of comission}}{100}$$

$$\therefore 4500 = \frac{350x \times 12.5}{100}$$

$$\therefore \frac{4500 \times 100}{350 \times 12.5} = x$$

$$\therefore x = 102.86$$

$$\therefore x \approx 103$$

Illustration 38 An agent receives del credere commission of 2.5% in addition to general commission of 6.5%. If he sells goods worth Rs. 25,000 on credit and Rs. 10,000 in cash, find his commission.

Solution

$$\text{Amount of general commission} = \frac{\text{Total sales} \times \text{Rate of commission}}{100}$$

$$= \frac{(25{,}000 + 10{,}000) \times 6.5}{100}$$

$$= 2{,}275$$

Rate of del credere commission = 2.5%

$\therefore$ Amount of del credere commission

$$= \frac{\text{Del credere sale} \times \text{Rate of commission}}{100}$$

$= \dfrac{25,000 \times 2.5}{100}$

= 625

Amount of total commission = Rs. 2,275 + Rs. 625

= Rs. 2,900

Illustration 39 An agent receives 6.25% commission on sales. He sends Rs. 1,20,000 after deducting his commission and Rs. 2,000 as miscellaneous expenses to the principal. Find the total sales and the commission of the agent.

Solution

Total amount sent to the principal after deducting commission.

= Rs. 1,20,000 + Rs. 2,000

= Rs. 1,22,000

Rate of commission = 6.25%

i.e. the amount of commission per Rs. 100 = Rs. 6.25

∴ Amount to be sent to employer on sale of goods worth

Rs. 100 = Rs. $(100 - 6.25)$ = Rs. 93.75

Now when amount sent is Rs. 93.75 the sale = Rs. 100

∴ When the amount sent is Rs. 1,22,000 the sale

$= \dfrac{12,000 \times 100}{93.75}$

= Rs. 1,30,133.33

∴ Commission of the agent = Sale − Amount sent

= 1,30,133.33 − 1,22,000

= Rs. 8,133.33

Illustration 40 A sales agent receives Rs. 12,000 as a total commission. He sells goods worth Rs. 1,00,000. If he receives general commission at 15%, find the rate of del credere commission.

Solution

Amount of general commission $= \dfrac{\text{Rate of commission} \times \text{Sale}}{100}$

$= \dfrac{1,00,000 \times 15}{100}$

= Rs. 15,000

∴ Amount of del credere commission = 15,000 − 12,000

= Rs. 3,000

∴ Rate of del credere commission $= \dfrac{3,000 \times 100}{1,00,000} = 3\%$

Illustration 41 An agent charges 5% commission on cash sales and 9% on credit sales. If his over all return is 7.5%, find the ratio between two sales.

Solution

Let the amount of cash sale be p and amount of credit sales q

$\therefore$ Amount of commission on sales = $\dfrac{\text{Rate of commission} \times \text{Sales}}{100}$

Amount of commission on cash sales = $\dfrac{5p}{100} = 0.05p$

and amount of commission on credit sales = $0.09q$

Rate of overall commission is 7.5%

$\therefore$ Amount of total sales = Rs. $(p + q)$

$\therefore$ Amount of commission on total sales = $\dfrac{(p+q)\times 7.5}{100}$

$$0.05p + 0.09q = 0.075(p + q)$$

$$5p + 9q = 7.5q + 7.5q$$

$$9q - 7.5q = 7.5p - 5p$$

$$\therefore 1.5q = 2.5p \frac{1.5}{2.5} = \frac{p}{q}$$

$$p : q = 3 : 5$$

Illustration 42 An agent receives Rs. 8,000 monthly salary and 5% general commission and 2% del credere commission on the sales exceeding Rs. 75,000. He is also entitled for creative bonus of 2% on the sales exceeding Rs. 2,00,000. If he has sold goods worth Rs. 3,20,000 in a month in which Rs. 25,000 worth credit sales were made, find his total income of the month.

Solution

Monthly salary of agent = Rs. 8,000

Amount of goods sold = Rs. 3,20,000

Amount of goods sold on credit = Rs. 3,20,000 − 75,000

= Rs. 2,45,000

Amount of sales exceeding Rs. 2,00,000 = 3,20,000 − 2,00,000

General commission = 5% = Rs. 1,20,000

Del credere commission = 2%

Rate of bonus = 2%

$\therefore$ Amount of commission

$$= \frac{\text{Amount of sales} \times \text{Rate of commission}}{100}$$

$$= \frac{2{,}45{,}000 \times 5}{100} = \text{Rs. } 12{,}250$$

Amount of del credere commission = $\dfrac{25{,}000 \times 2}{100} = 500$

Amount of bonus = $\dfrac{25{,}000 \times 2}{100} = 2{,}400$

∴ Total income of the month = 8,000 + 12,250 + 500 + 2,400
= Rs. 23,150

Illustration 43 A salesman has two options (1) Rs. 6,000 per month and 5% commission on total sales, (2) 8% commission on total sales. If his remunerations in both the cases are same, find the amount of the sales per year.

Solution

Let the amount of monthly sales = Rs. x
First option:
Amount of commission

$$= \frac{\text{Amount of sales} \times \text{Rate of commission}}{100}$$

$$= \frac{x \times 5}{100}$$

$= 0.05x$

Total monthly income = $0.05x + 6{,}000$
Second option:

Amount of commission = $\dfrac{x \times 8}{100}$
= $0.08x$

Now he gets same remuneration
∴ $0.05x + 6{,}000 = 0.08x$
∴ $0.03x = 6{,}000$
∴ $x = 2{,}00{,}000$

Illustration 44 An agent remits Rs. 50,000 to his employer after deducting 5% commission on first 15,000 sales and 3.5% commission on the remaining sales. Calculate the value of the goods sold and his commission.

Solution

Suppose the value of goods sold = Rs. x
Rate of commission on 1st sale of Rs. 15,000 = 5%
∴ Amount of commission

$$= \frac{\text{Rate of commission} \times \text{Amount of sale}}{100}$$

$$= \frac{5 \times 15{,}000}{100}$$

= Rs. 750

Amount exceeding Rs. 15,000 = Rs. $(x - 15{,}000)$ Rate of commission = 3.5%

$$\therefore \text{Amount of commission} = \frac{3.5 \times (x - 15{,}000)}{100}$$

$$= 0.035x - 525$$

Amount remitted to employer = Rs. 50,000

$\therefore x = 750 + 0.035x - 525 + 50{,}000$

$\therefore 0.965x = 50{,}225$

$\therefore\ x = 52{,}046.63$

$\therefore$ The Amount of commission = 52,046.63 − 50,000

= Rs. 2,046.63

Illustration 45 An agent sells a house on the advice of owner. After deducting 2.5% brokerage, he sends Rs. 5,50,000 to the owner of the house. Find the cost of the house.

Solution

Let the value of house = Rs. 100

Brokerage = 2.5%

$$\text{Amount of brokerage} = \frac{2.5 \times 100}{100} = \text{Rs. } 2.5$$

$\therefore$ Amount received by owner = 100 − 2.5 = Rs. 97.5

Now if amount received by owner = 97.5 then value of house = Rs. 100

$\therefore$ If amount received by owner = 5,50,000 then value of house.

$$= \frac{5{,}50{,}000 \times 100}{97.5}$$

= Rs. 5,64,102.56

Illustration 46 A broker charges 5% commission on each transaction. He settled a transaction at Rs. 50,000 and immediately sold it at a loss of 10%. Calculate how much the broker earned on the said transaction.

Solution

$$\text{Amount of brokerage} = \frac{\text{Rate of brokerage} \times \text{Amount of transaction}}{100}$$

$$= \frac{5 \times 50{,}000}{100} = 2{,}500$$

Amount of transaction (cost price) = Rs. 50,000

$\therefore$ Loss = 15%

$$\text{Selling price} = \frac{\text{Cost price}(100 - \text{Loss }\%)}{100}$$

$$= \frac{50{,}000(100 - 10)}{100}$$

= Rs. 45,000

$\therefore$ Brokerage of agent $= \dfrac{5 \times 45{,}000}{100} = 2{,}250$

$\therefore$ Total brokerage = 2,500 + 2,250

= Rs. 4,750

Illustration 47 A person gets Rs. 8,10,000 after deduction of brokerage of 3% on selling a house. Find brokerage and cost of the house.

Solution

Let the value of the house = Rs. x

$$\text{Amount of brokerage} = \frac{\text{Rate of brokerage} \times \text{Cost of house}}{100}$$

$$= \frac{3 \times x}{100}$$

$$= 0.03x$$

$\therefore$ Amount received by person $= x - 0.03x$

$\therefore$ 8,10,000 = $0.967x$

$\therefore x = 8{,}35{,}051.54$

and brokerage = 8,35,051.54 − 8,10,000 = Rs. 25,051.54

Illustration 48 A broker charges 1% from the seller and 1.5% from the buyer. If he gets Rs. 12,500 as brokerage, find the brokerage received from buyer and seller and also find the cost price of the item.

Solution

Amount of brokerage = Rs. 12,500

Combined brokerage = 1 + 1.5 = 2.5%

$$\therefore \text{Value of item} = \frac{\text{Amount of brokerage} \times 100}{\text{Rate of combined brokerage}}$$

$$= \frac{12{,}500 \times 100}{2.5}$$

$\therefore$ Cost price of item = Rs. 5,00,000

$$\therefore \text{Amount of brokerage} = \frac{\text{Rate of brokearge} \times \text{Cost price}}{100}$$

$$\therefore \text{Brokerage from seller} = \frac{1 \times 5{,}00{,}000}{100} = \text{Rs. } 5{,}000$$

$$\text{And brokerage from buyer} = \frac{1 \times 5{,}00{,}000}{100} = \text{Rs. } 7{,}500$$

Illustration 49 On increasing brokerage from 5% to 7.5% the brokerage of a broker remains unchanged. If the brokerage be Rs. 5,000, find the percentage decrease in the deal.

Solution

$$\text{Amount of sale} = \frac{\text{Amount of brokerage} \times 100}{\text{Rate of brokerage}}$$

$$= \frac{5{,}000 \times 100}{5}$$

$$= \text{Rs. } 1{,}00{,}000$$

Revised brokerage = 7.5%

$$\therefore \text{Amount of sale} = \frac{5{,}000 \times 100}{7.5}$$

$$= \text{Rs. } 66{,}666.67$$

Decrease in sales = 1,00,000 − 66,666.67 = Rs. 33,333.33

$$\therefore \text{Percentage in decrease in sale} = \frac{33{,}333.33 \times 100}{1{,}00{,}000}$$

$$= 33.33$$

ANALYTICAL EXERCISES

1. The printed price of an item is Rs. 9,000. If a discount of 8% is allowed, find the amount payable by customer.
2. A cloth merchant allows 15% discount on the cloth purchased. Mr X purchases cloth worth Rs. 3,580. How much money will Mr X pay?
3. A parcel contains 50 items. A shopkeeper paid Rs. 496 for this parcel on which he was given 15% discount. Find the list price of the item.
4. A supplier supplies 80 units of an item at the price of Rs. 90 each. The purchaser paid Rs. 5,904 for all these units. Find the trade discount and the invoice price per unit.
5. A TV dealer sells a TV set for Rs. 29,500. If the print price of the TV is Rs. 29,845, then find the rate of trade discount.
6. Mrs Radha purchases a washing machine priced 10,000 for Rs. 9,100. Calculate the rate of discount.
7. The sale price of 100 units of an item is equal to list price of 85 units of the item. Calculate the percentage discount.
8. A wholeseller gives 80 units of an item for Rs. 6,144. If the print price of the item is Rs. 96 each then find the rate of discount.
9. A store allows 15% discount on the price quoted in the price list and also gives 30% cash discount. Mr Q made a purchase worth Rs. 2,100. Calculate the amount that Mr Q has to pay.
10. A poultry farmer supplies 14 eggs for every dozen of eggs ordered and also allows 10% trade discount. Calculate the equivalent rate of discount.

11. A book store gives 20% discount on the printed price, counts 25 books as 24 and further gives 1.5% discount on reduced price of book. What amount is received per book? The print price of the book is Rs. 100.

12. A publisher supplies 147 copies of a book on the order of 140 copies of the book and also allows 12.5% trade discount. If the print price of the book be Rs. 42 then calculate the trade discount and also the net price of the book.

13. A manufacturer allows three successive discounts d_1, d_2 and d_3 percentages on sale of an item. Then prove that the selling price of item after three successive discounts $= p.\left(1-\frac{d_1}{100}\right)\left(1-\frac{d_2}{100}\right)\left(1-\frac{d_3}{100}\right)$ where p is the selling price of the item.

14. A manufacturer gives 20%, 10% and 10% successive discounts. Find the rate of equivalent discount.

15. Find the amount due for the following discounts given to an article priced Rs. 200.
(i) 10% + 20% + 5%
(ii) 20% + 5% + 10%
(iii) 20% + 10% + 5%

16. By selling 150 items of an article a retailer gains the selling price of 30 items. Find the gain percent.

17. The producer wholesaler and retailer make 10%, 15% and 25% gain, respectively. What is the production cost of item which a customer buys from the retailer for Rs. 1,265?

18. By selling an item at Rs. 75 a loss of 20% is made on the cost price. At what price it should be sold in order to gain 20% on the selling price?

19. A business man gives 20% trade discount and 20% cash discount on printed price of an item. He reduced the printed price in the ratio 6 : 5 and trade discount in the ratio 10 : 9. How much cash discount will have to be given so as to keep the selling price of the item as before?

20. A manufacturer allows 20% discount on printed price and makes 15% profit on the cost. What is the printed price of the item it he earns a profit of Rs. 480?

21. A person has purchased a house worth Rs. 20,00,000 through a broker. The broker charged 2% brokerage. Find how much money the person has to pay in purchasing the house and the amount the broker got as brokerage. If the person wants to sell the house at the cost price through the same broker at the same brokerage then find the amount received by the person.

22. A seller got Rs. 10,97,250 by selling property through a broker at 0.25% brokerage. Find selling price of the property and the total amount of brokerage that the broker received from both buyer and seller at the same brokerage.

23. The profit of supper mall in the year 2008 was Rs. 84,00,000. (i) If the manager is allowed 5% commission on profit then find out the commission payable to the manager and net profit; (ii) If the manager is allowed 5% commission after deducting his commission then find out the commission payable to the manager and net profit.

24. A manager of an MNC is paid monthly salary of Rs. 40,000 and 5% commission on profit before his salary and commission. If profit before his salary and commission for the year 2008 was Rs. 2,88,000 then find total earning of the manager and the net profit of MNC.

25. The captain of a ship company is paid monthly salary of Rs. 50,000. If in addition to this he is allowed 5% commission on net profit arrived after deducting his salary and commission. If profit before allowing such salary and commission for a voyage of 3 months is Rs. 7,80,000 then find out the total amount received by the captain and net profit of the voyage.

26. The manager of a firm gets monthly salary of Rs. 20,000 and is allowed 5% commission on net profit. He gets Rs. 3,00,000 for salary and commission for the year 2005. Find out the profit of the firm for the year 2005.

27. A traveling agent gets 1% commission on the sales and 2.5% bonus on the sales exceeding Rs. 25,000. He earned Rs. 6,900 as commission and bonus. Find the amount of sale.

28. A broker charges 0.5% from seller and 1% from the buyer. If he gets Rs. 6,075 brokerage then calculate the amount of brokerage received from buyer and from seller and also the total value of trade.

29. An agent sells 100 bags of wheat on the advice of principal at 10% commission. The principal asks the agent to purchase sugar after deducting his commission. He purchases 70 bags of sugar after deducting 5% purchase commission. If the total remuneration of the agent be Rs. 84,400, find the selling price of wheat and the purchase price of sugar.

30. A del credere agent is entitled to get 7.5% general commission and 2.5% del credere commission. He sells goods worth Rs. 87,500. Find the amount of his commission.

31. How many items priced at Rs. 24 each should a person sell to earn commission of Rs. 1,944, when the commission rate is 15%?

32. Mr Parikh wants to sell a purchased car at 10% gain. A broker can settle the sale at 15% gain but he charges 2% brokerage. Mr parikh gets Rs. 6,615 more by selling through the broker. Find the cost price of the car.

33. A plot was purchased through a booker for Rs. 7,50,000. If the broker charged Rs. 3,000 as brokerage, find the rate of brokerage.

34. An agent gets 5% commission on cash sale and 8% on credit sale. He sold goods worth Rs. 1,50,000 and received Rs. 8,850 as commission. Find cash and credit sale.

35. Mr Q gets Rs. 1,20,050 net on selling property through a broker allowing 2% brokerage. Find the amount paid to the broker and the value of property.

ANSWERS

(1) Rs. 8,280
(2) Rs. 3,043
(3) Rs. 11.60
(4) 22.5%, Rs. 73.80
(5) 1.29%
(6) 9%
(7) 15%
(8) 20%
(9) Rs. 1,695.75
(10) 22.86%
(11) Rs. 75.65
(12) 16.67% and Rs. 35
(14) 35.20%
(15) All are same Rs. 136.80
(16) 25%
(17) Rs. 800
(18) Rs. 117.79
(19) $6\frac{14}{41}\%$
(20) Rs. 4,600
(21) Rs. 40,000, Rs. 19,99,200
(22) Rs. 11,00,000, Total brokerage Rs. 5,500)
(23) (i) Commission = Rs. 4,20,000 and net profit = Rs. 79,80,000

(ii) Commission = $\frac{84,00,000 \times 5}{100 + 5}$ =

Rs. 4,00,000 and net profit = Rs. 80,00,000

(24) Rs. 19,20,000 and net profit Rs. 2,68,80,000
(25) Rs. 1,80,000 and net profit = Rs. 6,00,000
(26) Rs. 15,00,000
(27) Rs. 2,15,000
(28) 4,05,000; from seller 2,025, from buyer 4,050
(29) Rs. 490 and Rs. 466.67
(30) Rs. 8,750
(31) 540
(32) 2,20,500
(33) 0.75%
(34) Rs. 10,500, Rs. 45,000
(35) Rs. 2,450, Rs. 1,22,500

7

Simple Interest, Average Due Dates and Rebate on Bills Discounted

LEARNING OBJECTIVES

After studying this chapter, you will be able to understand:

- The basic concept of simple interest and its terminology
- How to calculate simple interest by using daily product method
- Meaning of average due date and its application to obtain rebate or to calculate simple interest

INTRODUCTION

At various stages of life, a need of additional money may arise. Money may be borrowed from a money lender or from some financial institution and will be returned after a fixed period of time. Due to variety of reasons the money lender charges some additional amount to the borrower for the use of his money. Some of the reasons are:

1. Money value changes from time to time
2. Opportunity cost due to alternative use of money
3. To neutralize the effect of inflation
4. The risk of recovering the principal

The money lender is called *creditor* and the money borrower is called *debtor*. The borrowed money is called *principal*. The time period for which money is borrowed is called *period* or *term*. The charge paid for the use of borrowed money is called *interest*. The interest paid for the use of Rs. 100 is called *interest rate*. The sum of the principal and interest is called the *amount*.

SIMPLE INTEREST

When interest is calculated uniformly on the original principal throughout the period under consideration then it is called simple interest (S.I).

Let P be the principal and R be the rate of interest and N be the number of years.
Now interest on Rs. 100 for 1 year = R

$\therefore$ Interest on Re. 1 for 1 year = Rs. $\dfrac{R}{100}$

$\therefore$ Interest on principal P for 1 year = Rs. $\dfrac{PR}{100}$

$\therefore$ Interest on principal P for N years = Rs. $\dfrac{PRN}{100}$

$\therefore$ Simple interest (S.I.) on principal P at the rate of interest R for N years is given as

$$\text{S.I.} = \frac{PRN}{100}$$

Also, amount $A = P + \text{S.I.}$

$$A = P + \frac{PRN}{100}$$

$$A = P\left(1 + \frac{RN}{100}\right)$$

ILLUSTRATIONS

Illustration 1 Calculate simple interest and amount on Rs. 50,000 at the rate of 8% for 5 years.

Solution

Here $P = 50{,}000$, $R = 8$, $N = 5$, S.I. = ?

$$\text{S.I.} = \frac{PRN}{100}$$

$$= \frac{50{,}000 \times 8 \times 5}{100}$$

$$= 20{,}000$$

and amount $A = P + I$

$$= 50{,}000 + 20{,}000$$

$$= \text{Rs. } 70{,}000$$

Illustration 2 A certain sum is borrowed from a money lender at 12% of simple interest. If at the end of 5 years Rs. 6,000 is paid as interest to the lender then find the sum.

Solution

Here $R = 12$, $N = 5$, S.I. = 6,000 and P = ?

Now S.I. = $\dfrac{PRN}{100}$

$$\therefore 6{,}000 = \frac{P \times 12 \times 5}{100}$$

$$\therefore P = \frac{6{,}000 \times 100}{12 \times 5}$$

Principal = Rs. 10,000

Illustration 3 If Rs. 40,000 is paid after 5 years at 12% rate of simple interest. Find the principal.

Solution

Here $A = 40{,}000$, $R = 12$, $N = 5$, $P = ?$

$$\text{Now } A = P\left(1 + \frac{RN}{100}\right)$$

$$40{,}000 = P\left(1 + \frac{12 \times 5}{100}\right)$$

$$40{,}000 = P(1 + 0.60)$$

$$40{,}000 = P(1.60)$$

$$P = \frac{40{,}000}{1.60}$$

Principal = Rs. 25,000

Illustration 4 A person deposited Rs. 50,000 with a money lender at 6% rate of simple interest. After some time he receives interest Rs. 15,000. Find out the period for which he deposited the money.

Solution

Here $P = 50{,}000$, $R = 6$, S.I. = 15,000 and $N = ?$

$$\text{Now S.I.} = \frac{PRN}{100}$$

$$\therefore 15{,}000 = \frac{50{,}000 \times 6 \times N}{100}$$

$$\therefore N = \frac{15{,}000 \times 100}{50{,}000 \times 6} = 5 \text{ years}$$

Illustration 5 Rs. 30,000 is invested in a company at simple interest rate of 10% per annum. After how many years will the money amount to Rs. 48,000?

Solution

Here $A = 48{,}000$, $P = 30{,}000$, $R = 10$, $N = ?$

$$\text{Now } A = P\left(1+\frac{RN}{100}\right)$$

$$\therefore 48{,}000 = 30{,}000\left(1+\frac{10\times N}{100}\right)$$

$$\therefore \frac{48{,}000}{30{,}000} = 1+\frac{N}{10}$$

$$\therefore 1.6 = 1+\frac{N}{10}$$

$$\therefore 1.6 - 1 = \frac{N}{10}$$

$$\therefore N = 10\times 0.6$$

$$\therefore N = 6 \text{ years}$$

Illustration 6 Simple interest on sum equals to 1/4 of itself in 4 years. Find the rate of interest.

Solution

Let the amount of principal be P

$$\text{Now S.I.} = \frac{1}{4}P\text{ , } N = 4$$

$$\text{Now S.I.} = \frac{PRN}{100}$$

$$\therefore \frac{1}{4}P = \frac{P\times R\times 4}{100}$$

$$\therefore \frac{P\times 100}{4\times P\times 4} = R$$

$$R = \frac{25}{4} = 6\frac{1}{4}\%$$

The required rate of simple interest = $6\frac{1}{4}\%$

Illustration 7 A person borrowed a certain sum at 6% interest per annum and immediately lent the same to another person at 10% interest per annum. He gained Rs. 1,200 in 6 months. What is the sum the person borrowed?

Solution

Let the sum borrowed = P

Now the interest on Rs. P for 6 months at 6% per annum

$$= \frac{P\times(1/2)\times 6}{100} \quad \left(\frac{1}{2} = 6 \text{ months}\right)$$

$$= \frac{3P}{100}$$

Also at 10% interest per annum, the interest on Rs. P for 6 months

$$= \frac{P \times (1/2) \times 10}{100}$$

$$= \frac{5P}{100}$$

Now the person has a profit of Rs. 1,200,

hence $\frac{5P}{100} - \frac{3P}{100} = 1,200$

$$\therefore 2P = 1,200 \times 100$$

$$\therefore P = 60,000$$

The principal that the person borrowed = Rs. 60,000

Illustration 8 A lender lent Rs. 1,00,000 to two persons in two parts. The first man borrowed at 6% and second person at 6.5% interest per annum. If the money lender receives Rs. 31,000 as total interest after 5 years then determine what sum was borrowed by each of the two persons?

Solution

Let the sum borrowed by one person = Rs. x

Then the sum borrowed by the other person = 1,00,000 – x

Now the simple interest on Rs. x for 5 years at 6% per annum

$$= \frac{x \times 5 \times 6}{100} = \frac{3x}{10} = 0.3x$$

and interest on Rs. $(1,00,000 - x)$ for 5 years at 6.5% per annum

$$= \frac{(1,00,000 - x) \times 5 \times 6.5}{100}$$

$$= \frac{3,25,000 - 32.5x}{100}$$

$$= 32,500 - 0.325x$$

Now the total interest that the lender received = Rs. 31,000

$$0.3x + 32,500 - 0.325x = 31,000$$

$$32,500 - 0.025x = 31,000$$

$$32,500 - 31,000 = 0.025x$$

$$0.025x = 1,500$$

$$\therefore x = \frac{1,500}{0.025} = 60,000$$

The sum borrowed by first person = Rs. 60,000 and
the sum borrowed by second person = Rs. 1,00,000 − 60,000 = Rs. 40,000

Illustration 9 A man lent Rs. 80,000 for 3 years and Rs. 60,000 for 5 years at the same rate of interest. He receives total interest of Rs. 29,700. What is the rate of interest?

Solution

Suppose R be the rate of interest.
Now simple interest received on Rs. 80,000 for 3 years at rate

$$R\% \text{ per annum } = \frac{80{,}000 \times R \times 3}{100} = 2400\ R$$

Also simple interest received on Rs. 60,000 for 5 years at rate

$$R\% \text{ per annum } = \frac{60{,}000 \times R \times 5}{100} = 3000\ R$$

Now total interest received = Rs. 29,700

$$2{,}400R + 3{,}000R = \text{Rs. } 29{,}700$$

$$5{,}400R = 29{,}700$$

$$R = \frac{29{,}700}{5{,}400} = 5.5\%$$

Rate of interest per annum = 5.5%

Illustration 10 A person takes a loan of Rs. 1,00,000 at the rate of 8% simple interest with the agreement that he will repay it with interest in 20 equal annual installments, one at the end of every year. How much is his yearly installment?

Solution

Suppose the sum of each installment is Rs. P.

Rate of interest per annum for Re. 1 = Re. $\frac{8}{100}$ = Re. 0.08

$\therefore$ The amount for 20 years of Re. 1 = Re. $(1 + 20 \times 0.08)$
$\therefore$ The amount for 20 years of Rs. 1,00,000
= Rs. $1{,}00{,}000(1 + 20 \times 0.08)$
But the amount of 1st installment for 19 years = Rs. $P(1 + 19 \times 0.08)$
and the amount of 2nd installment for 18 years = Rs. $P(1 + 18 \times 0.08)$
and so on. The last installment is P
Now

$$1{,}00{,}000(1 + 20 \times 0.08) = P(1 + 19 \times 0.08) + P(1 + 18 \times 0.06) + \ldots$$
$$P(1 + 1 \times 0.08) + P$$

$$\therefore\ 2{,}20{,}000 = 20P + P\ 0.08\left(19 + 18 + 17 + \ldots + 1\right)$$

$$\therefore\ 2{,}20{,}000 = 20P + P \times 0.08 \times \frac{19 \times 20}{2} \qquad \left[\because 1 + 2 + \ldots + n = \frac{n(n+1)}{2}\right]$$

$$\therefore\ 2{,}20{,}000 = 20P + 15.2P$$

$$\therefore\ 2{,}20{,}000 = 35.2P$$

$$\therefore\ P = \frac{2{,}20{,}000}{35.2}$$

$$\therefore\ P = 6250$$

Illustration 11 A man borrows Rs. 50,000 from his friend and promises that he will pay Rs. 2,500 at the end of every three months on account of principal and in addition to that the simple interest of 8% on the outstanding principal. Find the total interest that the man has to pay.

Solution

Since Rs. 50,000 is to be paid at the rate of Rs. 2,500 per 3 months so there must be total $\frac{50{,}000}{2{,}500}$ = 20 installments of principal

$\therefore$ Interest paid at the first payment = $50{,}000 \times 0.08 \times \frac{1}{4}$ = Rs. 1,000

Interest paid at the second payment = $47{,}500 \times 0.08 \times \frac{1}{4}$ = Rs. 950

Interest paid at the third payment = $45{,}000 \times 0.08 \times \frac{1}{4}$ = Rs. 900

and so on.

Lastly the interest paid at the 20th payment = $2{,}500 \times 0.08 \times \frac{1}{4}$ = Rs. 50

Hence the total interest paid in 20 installments
= 1,000 + 950 + 900 + . . . + 50 (arithmetic series)

$= \frac{n}{2}(a + l)$ where $n = 20$, $a = 1000$, $l = 50$

$= \frac{20}{2}(1000 + 50)$

= Rs. 10,500

Simple Interest by Using Daily Product Method

Usually bank accounts and postal accounts calculate interest in a particular month on the basis of minimum balance between the 10th and the last date of the calendar month. But in the case of all bank advances like overdraft, cash credit, etc., the interest is calculated on the basis of daily product method. According to this method, the changes on due (balance of receivable and liabilities) at a given date are considered in such cases. The figure thus obtained is multiplied by the number of days in consideration and finally total of this multiplication is treated as principal and interest on one day is calculated. The principal thus obtained is called the *daily product*. This method is known as daily product method in the calculation of simple interest.

Following are some important points to be considered in the use of daily product method:

1. When interest is calculated then one year is taken to be equal to 365 days.
2. In the leap year, February is of 29 days but for calculation of interest 365 days are taken for a year.
3. While calculating the number of days usually last day is taken into account and the first day is ignored. However if the year or the month ends on that day and account shows debit balance then both the first and the last days are taken into account.
4. When bills are purchased or discounted, days are calculated from the date of purchase or discounted to the date of maturity. For calculating the date of maturity 3 days of grace are taken into consideration in addition to the period of the bill.

Illustration 12 A bank has given overdraft facility to the ABC Company on its current account. Following information is given in its bank statement. Calculate simple interest at the rate of 12%.

Date	Particulars	Debit (Rs.)	Credit (Rs.)	Balance (Rs.)
1st January 2009	Balance (B.O.D.)*	20,000	–	20,000
5th January 2009	Cash A/C	–	4,000	16,000
11th January 2009	Cash A/C	–	8,000	8,000
18th January 2009	Self	3,000	–	11,000
28th January 2009	XYZ-A/C	4,000	–	15,000
31st January 2009	Balance carrried forward			

*Bank Over Draft

Solution

In order to calculate simple interest by using daily product method we prepare the following table.

Date	Debit	Credit	Balance	Days	Product (Rs.) Balance × Days
1st January 2009	20,000	–	20,000	4	80,000
5th January 2009	–	4,000	16,000	6	96,000
11th January 2009	–	8,000	8,000	7	56,000
18th January 2009	3,000	–	11,000	10	1,10,000
28th January 2009	4,000	–	15,000	4	60,000
31st January 2009	–	–	15,000	–	–
					4,02,000

Here, $P = 4{,}02{,}000,\ R = 12,\ N = \dfrac{1}{365}$

$$\text{S.I.} = \frac{PRN}{100}$$

$$= \frac{4{,}02{,}000 \times 12 \times 1}{100 \times 365} = 132.16$$

$\therefore$ Simple Interest is Rs. 132.16

Illustration 13 Chandubhai entered the following transaction with a money lender firm “Jagruti”. The firm calculates interest at 18% by daily product method. Calculate the simple interest.

Date	Money Borrowed (Rs.)	Date	Money Deposited (Rs.)
1st April 2005 (Op. Bal.)	55,000	7th April 2005	35,000
		13th April 2005	18,000
21st April 2005	25,000	26th April 2005	22,000
28th April 2005	14,000	30th April 2005	19,000
	94,000	Closing Balance	94,000

Solution

First we prepare the following table to calculate interest by daily product method.

Date	Borrowed	Deposited	Balance	Days	Product (Rs.) Balance × Days
1st April 2005	55,000	–	55,000	6	3,30,000
7th April 2005	–	35,000	20,000	6	1,20,000
13th April 2005	–	18,000	2,000	8	16,000
21st April 2005	25,000	–	27,000	5	1,35,000
26th April 2005	–	22,000	5,000	2	10,000
28th April 2005	14,000	–	19,000	3	57,000
30th April 2005	–	19,000	19,000	–	–
					6,68,000

$\therefore$ Simple Interest by using daily product method is

$$P = 6{,}68{,}000,\ R = 18,\ N = \frac{1}{365}$$

$$\text{S.I.} = \frac{PRN}{100}$$

$$= \frac{6{,}68{,}000 \times 18 \times 1}{100 \times 365}$$

$$= \text{Rs. } 329.42$$

Illustration 14 From the books of Ramesh Corporation following information is available about the account of Mr C.F. Patel. Ramesh Corporation calculates interest at 18% on the balance amount. Find the simple interest paid by Mr Patel.

Mr C.F. Patel's Account

Date	Particulars	Amount	Date	Particulars	Amount
1st January 2006	Balance b/d	2,80,000	12th January 2006	Cash A/C	1,20,000
26th January 2006	Sales A/C	3,60,000	3rd March 2006	Bank A/C	2,50,000
9th March 2006	Sales A/C	1,20,000	21st March 2006	Goods returned	40,000
6th May 2006	Sales A/C	2,00,000	12th April 2006	Cash A/C	1,20,000
			21st June 2006	Bank A/C	2,30,000
			30th June 2006	Balance c/d	2,00,000
		9,60,000			9,60,000

Solution

First we calculate daily products from the given information.

Date	Debit	Credit	Balance	Days	Product (Rs.) Balance × Days
1st January 2006	2,80,000	–	2,80,000	11	30,80,000
12th January 2006	–	1,20,000	1,60,000	14	22,40,000
26th January 2006	3,60,000	–	5,20,000	37	1,92,40,000
3rd March 2006	–	2,50,000	2,70,000	6	16,20,000
9th March 2006	1,20,000	–	3,90,000	12	46,80,000
21st March 2006	–	40,000	3,50,000	22	77,00,000
12th April 2006	–	1,20,000	2,30,000	24	55,20,000
6th May 2006	2,00,000	–	4,30,000	45	1,93,50,000
21st June 2006	–	2,30,000	2,00,000	10	20,00,000
30th June 2006	–	–	2,00,000	–	–
					6,54,30,000

Now $P = 6{,}54{,}30{,}000$, $R = 16\%$, $N = \dfrac{1}{365}$

$$\text{S.I.} = \frac{PRN}{100}$$

$$= \frac{6{,}54{,}30{,}000 \times 16 \times 1}{100 \times 365}$$

$$= \text{Rs. } 28{,}681.64$$

Illustration 15 Suresh has taken a loan from a bank to purchase a car. The loan ledger shows the following details from 1st April 2009 to 30th June 2009:

Date	Loan Taken	Amount	Date	Repayment	Amount
1st April 2009	Raising Loan	3,50,000	20th April 2009	Installment paid	15,000
			10th May 2009	Installment paid	15,000
			13th June 2009	Installment paid	15,000
			30th June 2009	Loan Balance	3,05,000
		3,50,000			3,50,000

Interest is calculated at 10% after every three months by daily product. Find the interest.

Solution

First we calculate daily products

Date	Loan Given	Amount Repaid	Balance Amount	Days	Daily Product
1st April 2009	3,50,000	–	3,50,000	19	66,50,000
20th April 2009	–	15,000	3,35,000	20	67,00,000
10th May 2009	–	15,000	3,20,000	34	1,08,80,000
13th June 2009	–	15,000	3,05,000	18	54,90,000
30th June 2009	–	3,05,000	–	–	–
					2,97,20,000

Hence $P = 2{,}97{,}20{,}000$, $R = 10$, $N = \dfrac{1}{365}$

$$\text{S.I.} = \frac{PRN}{100}$$

$$= \frac{2{,}97{,}20{,}000 \times 10 \times 1}{100 \times 365}$$

$$= \text{Rs. } 8{,}142.46$$

Average Due Date

When a person has to pay more than once to the same party on different dates then the person pays total amount to the party on one particular date without causing any loss of interest to either side. The date on which such payment is to be made is called *average due date*. To find average due date the following steps are to be followed.

1. Find due date of each transaction by adding 3 days of grace period.
2. Assume the earliest due date as zero date.
3. Calculate the number of days from the zero date up to each due date.
4. Multiply the number of days by the corresponding amount of transaction and obtain total of it.
5. Obtain the total of debit amounts or bill of exchange.
6. Now divide the amount derived in the above step by the amount obtained in the above step. This amount gives the average due period and hence find the due date.

Illustration 16 Shilpa drew the following bills on Swati which are accepted by her. It was agreed that Shilpa would pay total amount on 16th September 2009 instead of three bills. If the interest is to be paid at 18% for additional period after average due date, how much has Shilpa to pay with interest?

Bill Date	Period	Amount of Bill
4th June 2009	1 Month	10,000
6th June 2008	2 Months	18,000
8th June 2008	3 Months	14,000

Solution

First we determine average due date by considering the following table.

Bill Date	Period of Bill	Due Date of Bill	No. of Days from Zero Date	Amount	Product
4th June 2009	1 Month	7th July 2009	0	10,000	0
6th June 2009	2 Months	9th August 2009	33	18,000	5,94,000
8th June 2009	3 Months	11th September 2009	66	14,000	9,24,000
				32,000	15,18,000

Suppose zero date is 7th July 2009

$$\text{Average Period} = \frac{\text{Total of Product}}{\text{Total of Amount}}$$

$$= \frac{15,18,000}{32,000}$$

= 47.44 days = 47 days

Average due date = Zero date + Average period
= 7th July 2009 + 47 days
= 23rd August 2009

Now Shilpa pays all amount on 16th September 2009 which is the late payment from average due date 23rd August 2009. So she should pay interest for 25 days.

Now $P = 32,000$, $R = 18$, $N = \frac{25}{365}$

$$\text{S.I.} = \frac{PRN}{100}$$

$$= \frac{32,000 \times 18 \times 25}{100 \times 365}$$

= Rs. 394.52

Shilpa has to pay = Debt + Interest
= 32,000 + 394.52
= Rs. 32,394.52

Illustration 17 A wholesaler sold goods to the retailer to the following schedule.

Date of Goods Sold	Date of Payment	Bill Amount of Goods
10th April 2008	22nd May 2008	5,000
16th may 2008	13th June 2008	8,000
24th June 2008	2nd July 2008	7,000
3rd July 2008	28th July 2008	5,000

The retailer pays for all goods on 8th June 2008. What amount will he get through interest? What total amount he required to pay if the rate of interest is 16%?

Solution

First we determine the average due date for the payment.

Date of Goods Sold	Date of Payment	No. of Days from Zero Date	Bill Amount (Rs.)	Product (Rs.) Days × Amount
10th April 2008	22nd May 2008	0	5,000	0
16th May 2008	13th June 2008	22	8,000	1,76,000
10th April 2008	22nd May 2008	41	7,000	2,87,000
10th April 2008	22nd May 2008	67	5,000	3,35,000
			25,000	7,98,000

$$\text{Average Period} = \frac{\text{Total of Product}}{\text{Total of Amount}}$$

$$= \frac{7{,}98{,}000}{25{,}000}$$

$$= 31.92 \text{ days} = 32 \text{ days}$$

Average Due Date = Zero Date + Average Period
= 22nd May 2008 + 32 days
= 23rd June 2008

Now the retailer has paid entire amount on 8th June 2009 hence he gets rebate on interest for 15 days

Now $P = 25{,}000$, $R = 16$, $N = \frac{15}{365}$

$$\text{S.I.} = \frac{PRN}{100}$$

$$= \frac{25{,}000 \times 16 \times 15}{100 \times 365}$$

= Rs. 164.38

Payable amount for retailer = Debt – Rebate
= 25,000 – 164.38
= Rs. 24,835.62

Illustration 18 Mayur Corporation sold items to Meghna retailer on the following terms of payment. Meghna retailer paid Rs. 20,000 on 4th February 2008. On which day the customer pays remaining amount so that it will not cause any loss either to customer or the merchant. Now if Meghna retailer pays the remaining amount on 1st June 2008 then how much amount they have to pay together with interest, if the rate of interest is 18%?

Price of goods	5,000	15,000	6,000	4,000
Date of payments	6th January 2008	3rd February 2008	4th March 2008	23rd May 2008

Solution

First we find the average due date.

Date of Payment	No. of Days from Zero Date	Due Amount (Rs.)	Product = Days × Amount
6th January 2008	0	5,000	0
3rd February 2008	28	15,000	4,20,000
4th March 2008	57	6,000	3,42,000
23rd May 2008	137	4,000	5,48,000
		30,000	13,10,000

Suppose the due date is 6th January 2008

$$\text{Average Period} = \frac{\text{Total of Product}}{\text{Total of Amount}}$$

$$= \frac{13{,}10{,}000}{30{,}000}$$

$$= 43.67 \text{ days} = 44 \text{ days}$$

Average Due Date = Zero Date + Average Period

= 6th January 2008 + 44 days

= 19th February 2008

Now Meghna retailer pays Rs. 20,000 on 4th February 2008, which is 15 days early than the average due date.

Rebate is calculated for 15 days on Rs. 20,000 at the rate of 18%

$$\text{S.I.} = \frac{PRN}{100}$$

$$= \frac{20{,}000 \times 18 \times 15}{100 \times 365}$$

$$= \text{Rs. } 147.95$$

Now let us find out as to for how many days will the interest of Rs. 147.95 be on the remaining amount of Rs. 10,000, using the formula

$$\text{S.I.} = \frac{PRN}{100}$$

$$147.95 = \frac{10{,}000 \times 18 \times N}{100 \times 365}$$

$$N = \frac{147.95 \times 100 \times 365}{10{,}000 \times 18}$$

$$N = 30 \text{ days}$$

A sum of Rs. 20,000 is paid 15 days earlier than the due date. So if a sum of Rs. 10,000 is paid 30 days after the average due date then it will not result in any loss in interest.

New average due date = 19th February 2008 + 30 days

= 21st March 2008

Now Meghna retailer pays remaining amount on 1st June 2008, i.e., late by 72 days. So it has to pay interest on it.

Now $P = 10{,}000$, $R = 18$, $N = 72$

$$\text{S.I.} = \frac{PRN}{100}$$

$$= \frac{10{,}000 \times 18 \times 72}{100 \times 365}$$

$$= \text{Rs. } 355.07$$

Hence on 1st June 2008 Meghna retailer has to pay
Rs. 10,000 + Rs. 355.07 = Rs. 10,355.07 to Mayur Corporation

Illustration 19 In a business transaction a buyer has given a promise to the merchant that he will pay the amount of purchase in two parts: first part of Rs. 8,000 after 4 months and second part of Rs. 6,000 after 6 months of the purchase. Now it is known that the buyer has paid Rs. 4,000 after 5 months. Determine when should the remaining amount be paid so that there is no loss in interest to either parties.

Solution

Due amount = Payable amount – Paid amount
= (8,000 + 6,000) – (4,000)
= Rs. 10,000

We shall calculate the period of payment of Rs. 10,000 as under:
Rs. 8,000 × 4 months = 32,000 for one month
Rs. 6,000 × 8 months = 48,000 for one month
i.e. Rs. 80,000 for one month
Less Rs. 4,000 × 5 months = 20,000 for one month
i.e. Rs. 60,000 for one month

$$\text{Rs. } 10{,}000 \times \text{period of payment} = \frac{60{,}000}{10{,}000} = 6 \text{ months}$$

Thus if the balance of Rs. 10,000 is paid after 6 months then both the parties will not have to suffer any loss in interest.

Illustration 20 Anilbhai has taken a loan of Rs. 5,00,000 from Shalilbhai on 1st April 2005. The amount is paid in five equal installments from 1st April 2006. Rate of interest is 16% p.a. Find the due date and interest.

Solution

Average Due date

$$= \text{Date of Money Lending} + \frac{\text{Total No. of Years from Lending Money to the Date of Payment of Each Installment}}{\text{Number of Installments}}$$

$$= \text{1st April 2005} + \frac{(1+2+3+4+5)}{5}$$

= 1st April 2005 + 3 years
= 1st April 2008

Here P = 5,00,000, R = 16, N = 3 years

$$\text{S.I.} = \frac{PRN}{100}$$

$$= \frac{5,00,000 \times 16 \times 3}{100}$$

= Rs. 2,40,000

Anilbhai pays Rs. 2,40,000 as interest to shalilbhai.

Illustration 21 Ankit has taken a loan of Rs. 5,00,000 from ABC Finance Company on 1st January 2006 which is paid in five equal 6 monthly installments from 1st July 2007. Find average due date and calculate simple interest at 12% p.a.

Solution

First the number of months from the date of lending to payment date of each installment is calculated as under:

First installment is paid after 18 months
Second installment is paid after 24 months
Third installment is paid after 30 months
Fourth installment is paid after 36 months
Fifth installment is paid after 42 months
Total 150 months

Average Due Date

$$= \text{Date of Money Lending} + \frac{\text{Total No. of Years from Lending Money to the Date of Payment of Each Installment}}{\text{Number of Installments}}$$

$$= \text{1st January 2006} + \frac{150}{5}$$

= 1st January 2006 + 30 months

= 1st July 2008

Now $P = 5,00,000$, $R = 12$, $N = \frac{30}{12}$ years

$$\text{S.I.} = \frac{PRN}{100}$$

$$= \frac{5,00,000 \times 12 \times 30}{100 \times 12}$$

= Rs. 1,50,000

REBATE ON BILLS DISCOUNTED

Many financial institutions provide special type of advance facility by purchasing or discounting the bills of traders which are received from other customers. For the collection of amount of this bill, the creditor has to wait till due date but creditor can discount or sell this bill to the bank or any financial institution for immediate need of money. Discount received in advance is liability for the bank for current year.

Illustration 22 Following balances are shown in the books of MAS Finance Co. for the year ending as on 31st March 2009.

Bills discounted	Rs. 8,00,000
Discount received	Rs. 25,200
Discount received in advance	Rs. 3,500

Particulars about bills discounted

Bill No.	Date of Bill Drawn	Period	Amount	Discount Rate
1	12th January 2009	3 months	1,46,000	10%
2	7th March 2009	2 months	3,65,000	11%
3	11th March 2009	3 months	2,89,000	12%

What is the income of MAS Finance Company? What is the amount of rebate on bills discounted?

Solution

Bill No. 1

Due Date of Bill = Date of bill drawn + Period of 3 months + 3 grace days
= 12th January 2009 + 3 months + 3 days
= 15th April 2009

∴ Discount from 31st March 2009 to 15th April 2009 is discount received in advance

∴ Rebate is calculated for 15 days of April

Now $P = 1{,}46{,}000$, $R = 10$, $N = \frac{15}{365}$ years

$$\text{S.I.} = \frac{PRN}{100}$$

$$= \frac{1{,}46{,}000 \times 10 \times 15}{100 \times 365}$$

$$= \text{Rs. } 600$$

Bill No. 2

Due date of Bill = 7th March 2009 + 2 months + 3 grace days
= 10th May 2009

∴ Discount from 31st March 2009 to 10th May 2009 is discount received in advance

∴ Rebate is calculated for 30 + 10 = 40 days

Now $P = 3{,}65{,}000$, $R = 11$, $N = \frac{40}{365}$ years

$$\text{S.I.} = \frac{PRN}{100}$$

$$= \frac{3{,}65{,}000 \times 11 \times 40}{100 \times 365} = \text{Rs. } 4{,}400$$

Bill No. 3

Due date of Bill = 11th March 2009 + 3 months + 3 grace days

= 14th June 2009

∴ Discount received in advance is calculated from 31st March 2009 to 14th June 2009

∴ Rebate is calculated for 30 + 31 + 14 = 75 days

Now $P = 2{,}89{,}000$, $R = 12$, $N = \dfrac{75}{365}$ years

$$\text{S.I.} = \frac{PRN}{100}$$

$$= \frac{2{,}89{,}000 \times 12 \times 75}{100 \times 365}$$

= Rs. 7,126.03

Total rebate on bills discounted = 600 + 4,400 + 7,126.03

= Rs. 12,126.03

Discount income on bill for current year:

Discount received in advance for last year = 3,500

+ Discount received on current year = 25,200

= 35,700

$(-)$ Discount received on advance for current year = $(12{,}126.03)$

Discount income on bills for current year 16,573.97

Illustration 23 Following information is available in the books of Kalupur bank. Calculate rebate on the bills discounted. The accounting year is completed on 31st March 2005. Each bill is discounted in the bank at 10%.

Bill No.	Date Bill is Discounted	Due Date of Bill	Amount
1	25th December 2004	25th February 2005	60,000
2	27th January 2005	27th May 2005	2,00,000
3	15th February 2005	23rd June 2005	1,00,000
4	31st March 2005	2nd July 2005	40,000

Solution

Bill No. 1

Due date of this bill is before 31st March 2005, so there is no rebate on bills discounted.

Bill No. 2

Due date of this bill is 30th May 2005. Therefore discount received in advance is from 31st April 2005 to 30th May 2009

∴ Rebate is calculated for 30 + 30 = 60 days

$\therefore$ $P = 2{,}00{,}000$, $R = 10$, $N = \dfrac{60}{365}$ days

$$\therefore \text{ S.I.} = \frac{PRN}{100}$$

$$= \frac{2{,}00{,}000 \times 10 \times 60}{100 \times 365}$$

$$= \text{Rs. } 3{,}287.67$$

Bill No. 3

Due date is 26th June 2005

$\therefore$ Rebate is calculated for 30 + 31 + 26 = 87 days

Now $P = 1{,}00{,}000$, $R = 10$, $N = \dfrac{87}{365}$ days

$$\therefore \text{ S.I.} = \frac{PRN}{100}$$

$$= \frac{1{,}00{,}000 \times 10 \times 87}{100 \times 365}$$

$$= \text{Rs. } 2{,}583.56$$

Bill No. 4

Due date is 5th July 2005

$\therefore$ Rebate is calculated for 30 + 31 + 30 + 05 = 96 days

Now $P = 40{,}000$, $R = 10$, $N = \dfrac{96}{365}$ days

$$\therefore \text{ S.I.} = \frac{PRN}{100}$$

$$= \frac{40{,}000 \times 10 \times 96}{100 \times 365}$$

$$= \text{Rs. } 1{,}052.05$$

$\therefore$ Total rebate on bills discounted

= 0 + 3,287.67 + 2,583.56 + 1,052.05

= Rs. 6,723.28

Note: When discount rate is equal for all the bills then discount received in advance is easily calculated on the basis of average due date.

ANALYTICAL EXERCISES

1. Find simple interest on Rs. 10,000 for 7 years and 6 months at 6% rate of interest per annum.
2. Find simple interest on Rs. 25,000 for 15th March to 1st September at the rate of 4% per annum.

3. The simple interest on a sum equals to 1/4 of itself in 5 years. Find the rate of interest.
4. After how many years Rs. 3,00,000 will amount to Rs. 5,62,800 at 14.6% rate of interest?
5. Rs. 1,80,000 is deposited for 7 years. It amounts to Rs. 4,13,100. Find the rate of interest.
6. In what time will Rs. 12,500 amount to Rs. 14,000 at 6% per annum?
7. What sum of money will yield Rs. 17.8 as interest in 5 years at 2% per annum?
8. Yogesh deposited a sum of Rs. 1,00,000 in a bank. After 2 years he withdraws Rs. 40,000 and at the end of 5 years he receives an interest of Rs. 75,200. Find the rate of interest.
9. What time will be required for a sum of money to double at 10% simple interest?
10. Calculate interest on Rs. 18,000 for 8 months at the rate of 7.75% per annum.
11. Calculate interest on Rs. 1,00,000 for 1-1/2 years at the rate of 5.5% per year.
12. Find simple interest on Rs. 50,000 for 180 days at the rate of 6% per annum.
13. What will be the amount of Rs. 25,000 in 5.5 years at rate of 6.5% per annum?
14. Dhirubhai borrowed Rs. 50,000 from Gurubhai on 16th January 2007 at the rate of 12% per annum. He returned Rs. 25,000 on 7th June 2007. Find what sum Dhirubhai has to pay on 19th August 2008 to clear the debt.
15. Find the interest on Rs. 2,500 at the rate of 5 paise per rupee per month for the period of 10 months.
16. What sum shhould be invested at the rate of 6% per annum in order to earn Rs. 1,000 per month if the rate of interest is 8% p.a?
17. What principal will gain Re. 1 per day as simple interest at 4% per year?
18. Ketan takes a loan of Rs. 1,75,000 from a bank at 16% interest per year on 29th October 2006. On 21st February 2007, the rate of interest decreases by 1%. Find the amount paid on 25th August 2007 to clear the debt by Ketan.
19. Vinay took a loan of Rs. 50,000 from Daivik. Part of the loan is at 16% p.a. and remaining at 12% p.a. After 5 years Vinay returns Rs. 86,000 to clear the debt. Calculate the loan taken at 12% interest p.a.
20. At what per cent rate of interest Rs. 50,000 will earn Rs. 1,500 in 5 years?
21. At what per cent rate of interest a sum will double itself in 6 years?
22. The interest of Rs. 25,000 in 2-1/2 years exceeds the interest of Rs. 20,000 for the same period by Rs. 875. Find the rate of interest per year.

23. A certain sum at a certain rate of interest becomes Rs. 2,640 in 2 years and Rs. 3,000 in five years. Find the principal and rate of interest.

24. A sum of money doubles itself in 8 years. How many times the amount becomes after 40 years?

25. At 6% simple interest, a sum becomes Rs. 13,000 in 5 years. Find the interest of the sum in 8 months at 4% p.a.

26. A bank lent Rs. 25,000 to X at a certain rate of interest and Rs. 20,000 to Y at 2% higher rate of interest than X. After 4 years the bank received total Rs. 16,900 as interest. Find the rate of interest of the sum lent to X and Y.

27. A bank clerk has calculated interest on a certain principal for 6 months at 5.25% p.a. instead of 5 months at 6.25% p.a. He thus made an error of Rs. 150. What was the principal?

28. If the number of years and the rate of interest are in the ratio 5 : 1 and the interest is 1/5th of the principal. Find the rate of interest and time.

29. Two equal sums are lent at the same time at 6% p.a. and 3% p.a. The former is recovered 2 years earlier than latter, but the amount in each case is Rs. 448. Find the sum and the time for which each sum is lent?

30. A money lender has lent two equal sums to Suresh and Ramesh for 4 years and 5-1/2 years at 11% and 11.5% rate of interest respectively. If the difference of amounts received from Suresh and Ramesh is Rs. 4,620, find the sum left to each of them.

31. The basic salary of an employee is Rs. 7,000. He contributes 10% of his basic salary per month in Provident Fund. His employer also contributes 8%. If the interest credited to the Provident Fund account is 8.5% p.a., find the amount of interest after one year. Also find the amount.

32. In a bank following transactions are recorded in the account of Mr Vyas. Bank has given overdraft facility. Find interest at 12% by daily product method.

Date	**Particulars**	**Dr. (Rs.)**	**Cr. (Rs.)**	**Balance**
1st January 2008	Balance (BOD)	5,000	–	5,000
8th January 2008	A.J. Patel	3,000	–	8,000
11th January 2008	Cash A/C	–	4,000	4,000
18th January 2008	J.B. Shah	6,000	–	10,000
22nd January 2008	Cash A/C	–	5,000	5,000
26th January 2008	V.J. Shah	–	2,000	3,000
31st January 2008	Balance	–	–	3,000

33. Following information is given about a debtor's A/C in the books of a money lender. The firm calculates interest every month by daily product method. Calculate simple interest at 10% p.a.

Date	Money Borrowed	Date	Money Repaid
1st October 2009	20,000	8th October 2009	40,000
3rd October 2009	50,000	26th October 2009	30,000
12th October 2009	15,000	29th October 2009	10,000
		31st October 2009	5,000
	85,000		85,000

34. A person has borrowed money from a co-operative bank. His transactions are as under. If interest rate is 18% then calculate the interest for the month.

Date	Money Lent (Rs.)	Money Paid (Rs.)
1st August 2008	80,000	–
6th August 2008	40,000	–
13th August 2008	–	90,000
26th August 2008	1,20,000	–
28th August 2008	–	70,000

35. In a businessman's books show the following information. Find interest at 13% rate of interest for the month.

Date	Particulars	Amount	Date	Particulars	Amount
1st January 2005	Balance b/d	13,500	12th January 2005	Cash A/C	13,500
8th January 2005	Sales A/C	12,000	20th January 2005	Bank A/C	8,000
24th January 2005	Sales A/C	2,500	23rd January 2005	Bank A/C	22,000
			31st January 2005	Bank A/C	7,000
		50,500			50,500

36. Calculate interest at 13% by daily product method from the following information:

Date	Money Borrowed	Date	Money Repaid
6th June 2006	3,000	10th June 2006	2,500
14th June 2006	1,500	21st June 2006	3,200
16th June 2006	2,500	25th June 2006	800
		30th June 2006	500
	7,000		7,000

37. Calculate interest at 12% by daily product method up to 31st December 2008.

Date	Withdrawals (Debit Rs.)	Deposited (Credit Rs.)	Balance
1st January 2008	–	5,000	5,000
17th January 2008	–	4,000	9,000
3rd March 2008	6,000	–	3,000
8th April 2008	–	2,000	5,000
6th June 2008	3,000	–	2,000
9th July 2008	–	5,000	7,000
18th July 2008	3,000	–	4,000
22nd August 2008	–	4,000	8,000
26th September 2008	2,000	–	6,000
16th November 2008	3,000	–	3,000
19th December 2008	–	2,000	5,000
31st December 2008	–	–	5,000

38. Pathak & Co. lend money to Smita on the following dates. Dates of payment are also given. Find out average due date to pay all the amounts in one single payment.

Date of Lending Money	Amount Lent (Rs.)	Period
7th February 2005	15,000	3 months
9th March 2005	10,000	2.5 months
22nd April 2005	15,000	3 months
16th May 2005	10,000	2.5 months

39. Anubhav drew the following bills on Anubhai on the following dates. All the amount is paid on 3rd July 2007. Interest is collected for late payment after the average due date at 14.6%. What amount is paid by him including interest?

Bill Dates	Period of Bill	Amount of Bill
7th February 2007	2 months	10,000
12th March 2007	3 months	15,000
28th April 2007	2 months	10,000
30th May 2007	1 month	15,000

40. X sold the goods to Y on the following dates for two months credit period:

Date	12th February 2006	13th March 2006	14th April 2006	15th May 2006
Price	8,000	7,000	6,000	4,000

Y paid all amounts 20 days before the average due date. What is the rebate amount at 7.3% rate of interest?

41. Krish lent the following money to Krishna with a condition to return money on the following dates:

Date of Lending Money	Amount	Date of Returning Money	Date of Money Received	Amount
10th February 2009	5,000	23rd March 2009	1st April 2009	4,000
16th March 2009	4,000	20th April 2009	9th May 2009	5,000
20th April 2009	6,000	25th June 2009	19th June 2009	1,000

If the remaining amount is paid on 29th July 2009, find interest at 15% p.a.

42. Rs. 25,000 is to be paid after 8 months, out of which Rs. 5,000 is paid after 4 months and Rs. 10,000 is paid after 6 months. When should the remaining amount be paid without any loss of interest to either parties?

43. If all bills are discounted at 8%, find out rebate on bill discounted from the following information:

Due Date of Bill	Bill Amount	Discount Date of Bill
12th April 2008	15,000	15th December 2007
15th May 2008	21,900	16th January 2008
9th May 2008	7,300	25th February 2008
25th April 2008	14,600	8th March 2008

44. Find rebate on bills discounted from the following data:

Bill Date	Amount of Bill	Period of Bill	Discount Rate
8th November 2002	14,600	6 months	15%
9th February 2003	7,300	4 months	14%
16th March 2002	21,900	3 months	13%

45. Following balances are available in trial balance of co-operative bank as on 31st March 2009:

Name of Account	Dr. Rs.	Cr. Rs.
Rebate on bills discounted	–	30,000
Interest and discount	–	1,50,000
Bills discounted	1,00,000	–

Details about bills discounted:

Amount of Bills	**Due Date of Bill**	**Discount Rate**
36,500	25th April 2009	12%
21,900	15th May 2009	14%
41,600	19th June 2009	14.6%

Find out the total rebate on bills discounted. Also find out income of bank through discount for the year ended 31st March 2009.

ANSWERS

(1) Rs. 4,500
(2) Rs. 467.75
(3) 5%
(4) 6 years
(5) 8.5%
(6) 2 years
(7) Rs. 178
(8) 4%
(9) 10 years
(10) Rs. 930
(11) Rs. 16,500
(12) Rs. 1,479.45
(13) Rs. 36,687.50
(14) Rs. 30,958.90
(15) Rs. 1,250
(16) Rs. 75,000
(17) Rs. 9,125
(18) Rs. 1,97,126.71
(19) Rs. 20,000
(20) 6%
(21) 12.5%
(22) 7%
(23) P = Rs. 450 and $R = \frac{40}{9}\%$
(24) R = 12.5% 6 times
(25) 266.67
(26) 9.5% and 11.5%
(27) Rs. 7,20,000
(28) R = 2% and N = 10 years
(29) Rs. 400, 2 years, 4 years
(30) Rs. 24,000
(31) Rs. 696.15 and Rs. 15,816.15
(32) Rs. 54.20
(33) Rs. 328.80
(34) Rs. 1,110
(35) Rs. 161.20
(36) Rs. 17.34
(37) Rs. 629.80
(38) 22-6-05
(39) Interest is Rs. 460
(40) Rs. 100
(41) Rs. 150
(42) 12 months
(43) Rs. 397
(44) Rs. 2,914
(45) Rs. = 97,292

8

Compound Interest and Depreciation

LEARNING OBJECTIVES

After studying this chapter, you will be able to understand:

- Meaning and uses of compound interest
- Meaning of interest compounded continuously
- Meaning of nominal rate and effective rate of interest and their relation
- Meaning and usage of depreciation

INTRODUCTION

In Chapter 7, we have studied that if interest is calculated for the whole period under consideration then it is called simple interest. However, if interest for the whole period is calculated more than once then it is called *compound interest,* i.e. the interest on interest is called compound interest. When interest at the and of a specified period is calculated and added to the principal and then the interest for the next period is calculated on the basis of new principal then it is called compound interest. The fixed interval of time at the end of which the interest is calculated and added to principal is called *conversion period,* i.e., it is the time at the end of which the interest is compounded.

DERIVATION OF FORMULA FOR CALCULATING COMPOUND INTEREST

Let us consider that a sum of Rs. P is invested or borrowed at the rate of compound interest R% per year for N years. Then the interest of first period for $N = 1$ year

$$= \frac{P \times R \times 1}{100}$$

$$= \frac{PR}{100}$$

$$\therefore \text{ Amount } A = P + \mathrm{I}$$

$$= P + \frac{PR}{100}$$

$$= P\left(1+\frac{R}{100}\right) = P_1(\text{say})$$

Interest for second year = $\dfrac{P_1R\times 1}{100} = \dfrac{P_1R}{100}$

$\therefore$ Amount $A = P_1 + \dfrac{P_1R}{100}$

$$= P_1\left(1+\frac{R}{100}\right)$$

$$= P\left(1+\frac{R}{100}\right)\left(1+\frac{R}{100}\right)$$

$$= P\left(1+\frac{R}{100}\right)^2$$

Similarly the amount for 3rd year = $P\left(1+\dfrac{R}{100}\right)^3$

So in general the amount for N years $A = P\left(1+\dfrac{R}{100}\right)^N$

Now compound interest = Amount − Principal

$$= P\left(1+\frac{R}{100}\right)^N - P$$

$\therefore$ Compound interest = $p\left(1+\dfrac{R}{100}\right)^N - p$

$$\therefore \text{C.I} = P\left[\left(1+\frac{R}{100}\right)^N - 1\right]$$

Points to Remember

1. If the rate of interest is for different conversion period then

$$A = P\left(1+\frac{R_1}{100}\right)\left(1+\frac{R_2}{100}\right)\ldots\left(1+\frac{R_n}{100}\right)$$

where $R_1, R_2, \ldots, R_n$ are the rate per cent per annum of different conversion periods.

2. Compound interest C.I. = $A - P$

$$\therefore \text{C.I} = P\left[\left(1+\frac{R_1}{100}\right)\left(1+\frac{R_2}{100}\right)\ldots\left(1+\frac{R_n}{100}\right) - 1\right]$$

3. If the rate of interest is R_1 for N_1 period, R_2 for N_2 period and so on then

Amount $A = P\left(1+\dfrac{R_1}{100}\right)^{N_1}\left(1+\dfrac{R_2}{100}\right)^{N_2}\ldots\left(1+\dfrac{R_n}{100}\right)^{N_n}$

4. C.I $= A - P = P\left[\left(1+\dfrac{R_1}{100}\right)^{N_1}\left(1+\dfrac{R_2}{100}\right)^{N_2}\ldots\left(1+\dfrac{R_n}{100}\right)^{N_n} - 1\right]$

5. If the interest is compounded k times a year then the amount after N years is

$$A = P\left(1+\frac{R}{100k}\right)^{Nk} \text{ and C.I} = P\left[\left(1+\frac{R_1}{100}\right)^{Nk} - 1\right]$$

ILLUSTRATIONS

Illustration 1 Find the compound interest on Rs. 10,000 at 5.5% per annum for 5 years. Also find the amount.

Solution

$P = 10{,}000$, $R = 5.5$, $N = 5$

$$\therefore A = P\left(1+\frac{R}{100}\right)^{N}$$

$$= 10{,}000\left(1+\frac{5.5}{100}\right)^{5}$$

$$= 10{,}000\left(1.055\right)^{5}$$

$$= 10{,}000\left(1.3094\right)$$

$A =$ Rs. 13,094.93

Now C.I. $= A - P$

$= 13{,}094.93 - 1{,}000$

= Rs. 3,094.93

Illustration 2 Find the compound interest on Rs. 25,000 at 5% per annum at the end of 2-1/2 years if interest is calculated half yearly.

Solution

Here $P = 25{,}000$, $R = 5$, $N = 2\frac{1}{2} = \frac{5}{2}$ years, $k = 2$ since interest is calculated more then once in a year.

$$\therefore A = P\left(1+\frac{R}{100k}\right)^{Nk}$$

$$= 10{,}000\left(1+\frac{5}{2\times 100}\right)^{2\times 5/2}$$

$$= 25{,}000\left(1 + 0.025\right)^{5}$$

$$= 25{,}000\left(1.1314\right)$$

= Rs. 28,285.20

C.I. $= A - P$

$= 28{,}285.20 - 25{,}000$

= Rs. 3,285.20

Illustration 3 If the difference between simple interest and compound interest is Rs. 200 at 4% for 2 years on certain principal then find the principal.

Solution

Suppose the principal = Rs. 100

$$\text{S.I. for 2 years} = \frac{RPN}{100}$$

$$= \frac{100 \times 4 \times 2}{100}$$

$$= \text{Rs. } 8$$

$$\text{and C.I for 2 years} = P\left[\left(1+\frac{R}{100}\right)^{N} - 1\right]$$

$$\therefore \text{C.I.} = 100\left[\left(1+\frac{4}{100}\right)^{2} - 1\right]$$

$$= 100\left[(1.04)^{2} - 1\right]$$

$$= 100\left[1.0816 - 1\right]$$

$$= 100\left[0.0816\right]$$

$$= \text{Rs. } 8.16$$

∴ Difference between two interests = 8.16 − 8 = 0.16

Hence when difference = 0.16 then principal = Rs. 100

$$\therefore \text{When difference} = 200 \text{ then principal} = \frac{200 \times 100}{0.16}$$

$$= \text{Rs. } 1,25,000$$

Illustration 4 A certain principal doubled in 6 years. What is the rate of compound interest?

Solution

Let the principal = P

Amount after 6 years = $2P$

$$\text{Now } A = P\left(1+\frac{R}{100}\right)$$

$$2P = P\left(1+\frac{R}{100}\right)^{6}$$

$$\therefore 2 = \left(1+\frac{R}{100}\right)^{6}$$

Taking log on both sides

$$\log 2 = 6\log\left(1+\frac{R}{100}\right)$$

$$\therefore\ 0.3010 = 6\log\left(1+\frac{R}{100}\right)$$

$$\therefore\ 0.0502 = \log\left(1+\frac{R}{100}\right)$$

Taking antilog on both sides

$$1.123 = 1+\frac{R}{100}$$

$$\therefore \frac{R}{100} = 0.123$$

$$\therefore R = 12.3\%$$

Illustration 5 What sum will amount to Rs. 21,490 in 5 years at 12% compound interest payable half yearly?

Solution

Let the principal = P

$A = 21{,}490,\ N = 5,\ R = 12,\ k = 12$

$$\text{Now } A = P\left(1+\frac{R}{100k}\right)^{nk}$$

$$21{,}490 = P\left(1+\frac{12}{100\times 2}\right)^{5\times 2}$$

$$21{,}490 = P\left(1.06\right)^{10}$$

Taking log on both sides

$$\therefore \log 21{,}490 = \log P + 10\log\left(1.06\right)$$

$$\therefore 4.3322 = \log P + 10\left(0.0253\right)$$

$$\therefore 4.3322 = \log P + 0.253$$

$$\therefore \log P = 4.0792$$

$$\therefore P = \text{antilog}\left(4.0792\right)$$

$$= \text{Rs. } 12{,}000$$

Illustration 6 Compound interest on a certain sum of money for two years is Rs. 1,000 and the simple interest for the same period is Rs. 950. Find the sum and rate of interest.

Solution

Compound interest = Amount − Principal

$$\therefore\ 1{,}000 = P\left[\left(1+\frac{R}{100}\right)^2 - 1\right]$$

$$\therefore 1{,}000 = P\left(1+\frac{2R}{100}+\frac{R^2}{10{,}000}-1\right)$$

$$\therefore 1{,}000 = P\left(\frac{2R}{100}+\frac{R^2}{1{,}000}\right) \qquad (1)$$

$$\text{Now S.I} = \frac{PRN}{100}$$

$$\therefore 950 = \frac{PR\times 2}{100}$$

$$\therefore \frac{950\times 50}{R} = P$$

From eq. (1)

$$1{,}000 = \frac{950\times 50}{R}\times\frac{R}{100}\left(2+\frac{R}{100}\right)$$

$$\therefore \frac{1{,}000\times 100}{950\times 50} = 2+\frac{R}{100}$$

$$\therefore\ 2.1053 = 2+\frac{R}{100}$$

$\therefore R = 0.1053 \times 100$

$\therefore R = 10.53\%$

$$\text{Now S.I.} = \frac{PRN}{100}$$

$$950 = \frac{P\times 10.53\times 2}{100}$$

$$\therefore P = \frac{950\times 100}{10.53\times 2}$$

$\therefore P =$ Rs. 4,510.92

Illustration 7 A certain money invested on compound interest amounts to Rs. 12,100 in 2 years and Rs. 13,310 in 3 years. Find the sum invested.

Solution

Let the sum be P.

$$\text{Now } A = P\left(1+\frac{R}{100}\right)^N$$

$$\therefore 12{,}100 = P\left(1+\frac{R}{100}\right)^2 \qquad (1)$$

$$\text{and } 13{,}310 = P\left(1+\frac{R}{100}\right)^3 \qquad (2)$$

Taking eq. (2) ÷ eq. (1) gives

$$\frac{13,310}{12,100} = \frac{P\left[1+(R/100)\right]^3}{P\left[1+(R/100)\right]^2}$$

$$1.1 = 1 + \frac{R}{100}$$

$$\therefore\ 0.1 = \frac{R}{100}$$

$$\therefore R = 10\%$$

Also from eq. (1)

$$12,100 = P\left(1+\frac{10}{100}\right)^2$$

$$\therefore P = \frac{1,21,000}{(1-1)^2}$$

$$P = \text{Rs. } 10,000$$

Illustration 8 An amount of Rs. 10,000 is deposited in a bank at the rate of 12% per annum. Find the amount after 20 years.

Solution

Here P = 10,000, R = 12, N = 20.

$$\therefore A = P\left(1+\frac{R}{100}\right)^N$$

$$= 10,000\left(1+\frac{12}{100}\right)^{20}$$

$$= 1,000\,(1.12)^{20}$$

$$\therefore \log A = \log 10,000 + 20\log\,(1.12)$$

$$= 4 + 20 \times (0.0492)$$

$$= 4 + 0.9840$$

$$= 4.9840$$

$$\therefore A = \text{antilog}\,(4.9840)$$

$$\therefore A = \text{Rs. } 96,380$$

Interest Compounded Continuously

Let a principal of Rs. P be invested for N years at R per cent rate of interest per annum and let interest be compounded k times a year. In this case we have

$$A = P\left(1+\frac{R}{k\times 100}\right)^{Nk}$$

Now let us assume that the interest is compounded continuously, *i.e.*, k becomes larger and larger then we have

$$A = \lim_{x\to\infty} P\left[1+\frac{R}{k\times 100}\right]^{Nk}$$

$$= P\left[\lim_{x\to\infty}\left(1+\frac{R}{100k}\right)^{Nk}\right]$$

$$= P \lim_{x\to\infty}\left[\left(1+\frac{R}{100}\right)^{100k/R}\right]^{NR/100}$$

$$\therefore A = pe^{NR/100} \qquad \left[\because \lim_{x\to\infty}(1+x)^{1/x} = e\right]$$

Illustration 9 A person deposits Rs. 20,000 in a bank which pays an interest of 4% per annum compounded continuously. How much amount will be accumulated in his account after 5 years?

Solution

$P = 20{,}000,\ R = 4,\ N = 5$

$\therefore A = p\,e^{NR/100}$

$= 20{,}000\times e^{5\times 4/100}$

$= 20{,}000 \times e^{0.2}$

$\log A = \log 20{,}000 + 0.2\log e$

$= 4.3010 + 0.2(0.4343)\ \ (\log e = \log 2.7183 = 0.4343)$

$\log A = 4.4879$

$\therefore A = \text{antilog}\ (4.3879)$

$= \text{Rs. } 24{,}430$

Illustration 10 If interest is compounded continuously then at what rate the principal would triple itself in 8 years?

Solution

Let P be the principal.

Now $A = P\ e^{NR/100}$

$\therefore\ 3P = P\,e^{8R/100}$

$\therefore 3 = e^{0.08R}$

$\therefore \log 3 = 0.08R\log e$

$0.4771 = 0.08R \times 0.4343 \qquad \therefore\ R = \dfrac{0.4771}{0.08\times 0.4343}$

$\therefore R = 13.73\%$

NOMINAL RATE AND EFFECTIVE RATE OF INTEREST

The annual compound interest rate is called the *nominal interest rate*. When interest is compounded more than once in a year then the actual per centage of interest rate per year is called *effective interest rate*. It should be noted that if the conversion period is one year then the nominal rate and effective rate are equal. The relation between these two rates can be obtained as under:

Let R_E denote the effective rate per unit per year and R denote the nominal rate per unit per conversion period and P denote the principal amount. Since an effective rate is actual rate compounded annually, therefore, at the effective rate R_E, the principal P amounts in one year to $P(1 + R_E)$. At the rate of R per unit per conversion period, the principal P amounts to $P\left(1+\frac{R}{100k}\right)^k$

$$\text{Now } P\left(1+R_E\right)=P\left(1+\frac{R}{100k}\right)^k$$

$$\therefore\ 1+R_E=\left(1+\frac{R}{100k}\right)^k$$

$$\therefore\ R_E=\left(1+\frac{R}{100k}\right)^k-1$$

Also if the nominal rate is compounded continuously then the effective rate R_E per unit is given by

$$R_E=\lim_{x\to\infty}\left[\left(1+\frac{R}{100k}\right)^k-1\right]$$

$$=\lim_{x\to\infty}\left(1+\frac{R}{100k}\right)^k-1$$

$$=\lim_{x\to\infty}\left[\left(1+\frac{R}{100k}\right)^{100k/R}-1\right]^{R/100}-1$$

$$R_E=e^{R/100}-1$$

Illustration 11 Find the effective rate equivalent to nominal rate of 6% compounded quarterly.

Solution

Here $R = 6\%$, $k = 4$

$$R_E=\left(1+\frac{R}{100k}\right)^k-1$$

$$=\left(1+\frac{6}{4\times 100}\right)^4-1$$

$$=(1+0.015)^4-1$$

$$=1.0613-1$$

$$=0.0613$$

$\therefore$ Effective Rate $R_E = 6.13\%$

Illustration 12 Find the force of interest corresponding to the effective rate 6%.

Solution

The nominal rate R compounded continuously and equivalent to a given effective rate R_E is called the *force of interest*.

$$\text{We have } R_E = e^{R/100} - 1$$

$$0.06 = e^{R/100} - 1$$

$$\therefore\ e^{R/100} - 1 = 1.06$$

$$\therefore\ \frac{R}{100} \log e = \log 1.06$$

$$\therefore\ \frac{R}{100} (0.4343) = 0.0253$$

$$\therefore\ R = \frac{0.0253 \times 100}{0.4343}$$

$$\therefore\ R = 5.2825\%$$

Illustration 13 A man wants to invest Rs. 50,000 for 5 years. He may invest the amount at 10% per year compounded each quarterly or he may invest it at 10.50% per year. Which investment will give better return?

Solution

In the first case, the effective rate per year R_E is

$$R_E = \left(1 + \frac{R}{100k}\right)^k - 1$$

$$= \left(1 + \frac{100}{100 \times 4}\right)^4 - 1$$

$$= (1.025)^4 - 1$$

$$= 1.1038 - 1$$

$$= 0.1038$$

$\therefore$ Effective rate per year = 10.38%

When interest is compounded annually then nominal rate and effective rates are same.

Now since 10.50% > 10.38%, so the second option of investment is better.

Illustration 14 A person deposited Rs. 10,000 in a bank at 5% interest compounded annually. After 5 years the rate of interest was 6%

and after four years the rate of interest was 7%. Find the amount after 12 Years.

Solution

Here $N_1 = 5$, $R_1 = 5$, $N_2 = 4$, $R_2 = 6$, $N_3 = 3$, $R_3 = 3$

$$\text{Hence } A = P\left(1+\frac{R_1}{100}\right)^{N_1}\left(1+\frac{R_2}{100}\right)^{N_2}\left(1+\frac{R_1}{100}\right)^{N_3}$$

$$= \left(1+\frac{5}{100}\right)^5\left(1+\frac{6}{100}\right)^4\left(1+\frac{7}{100}\right)^3$$

$$= 10{,}000\left(1.05\right)^5 \times \left(1.06\right)^4 \times \left(1.07\right)^3$$

$$= 10{,}000\left(1.2763\right) \times \left(1.2625\right) \times \left(1.2250\right)$$

= Rs. 19,738.78

Illustration 15 R deposited Rs. 1,00,000 in a bank for 3 years offering interest at the rate of 6% compounded half yearly during first year, at the rate of 12% compounded quarterly during second year and at 10% compounded continuously during third year. Find the amount.

Solution

Here $K_1 = 2$, $K_2 = 4$, $R_1 = 6$, $R_2 = 12$, $R_3 = 10$ $N = 1$.

$$\therefore A = P\left(1+\frac{R}{k_1 \times 100}\right)^{k_1}\left(1+\frac{R}{k_2 \times 100}\right)^{k_2} e^{NR_3/100}$$

$$= 1{,}00{,}000\left(1+\frac{6}{2 \times 100}\right)^2\left(1+\frac{12}{4 \times 1{,}000}\right)^4 e^{(10\times 1)/100}$$

$$= 1{,}00{,}000\left(1.03\right)^2\left(1.03\right)^4 e^{0.1}$$

$$= 1{,}00{,}000\left(1.069\right)\left(1.1255\right)\left(1.1050\right)$$

= Rs. 1,31,941.74

DEPRECIATION

The decrease in the price or value of an item with time is called *depreciation*. If V is the value of an item and D is the rate of depreciation per year then the depreciated value of the item after N years is obtained by the formula:

$$\text{Depreciated value} = V\left(1-\frac{D}{100}\right)^N$$

Also when the rate of depreciation is D_1 for the first year, D_2 for the second year, D_3 for the third year then the depreciated value of the item after third year is

$$= V\left(1-\frac{D_1}{100}\right)\left(1-\frac{D_2}{100}\right)\left(1-\frac{D_1}{100}\right)$$

Illustration 16 The cost of a bike is Rs. 42,000 and its expected life is 12 years. If the depreciated value of that bike be Rs. 12,000 calculate the rate of depreciation per year.

Solution

Initial cost of bike = Rs. 42,000

Depreciated cost of bike = Rs. 12,000

$\therefore$ Depreciation in 12 years = 42,000 − 12,000

$$\therefore \text{Annual depreciation} = \frac{30{,}000}{12} = 2{,}500$$

$$\therefore \text{Rate of annual depreciation} = \frac{2{,}500 \times 100}{42{,}000} = 5.95\%$$

Illustration 17 A machine costs Rs. 72,000. If the annual rate of compound depreciation is 12.5%. Find the depreciated cost of machine after 8 years.

Solution

$$\begin{aligned}\text{Depreciated cost} &= V\left(1-\frac{D}{100}\right)^n \\ &= 72{,}000\left(1-\frac{12.5}{100}\right)^8 \\ &= 72{,}000\,(0.875)^8 \\ &= \text{antilog}\,(\log 72000 + 8\log 0.87.5) \\ &= \text{antilog}\,[4.8573 + 8(\bar{1}.9420)] \\ &= \text{antilog}\,(4.8573 + \bar{1}.5360) \\ &= \text{antilog}\,(4.3933) \\ &= \text{Rs. } 24{,}740\end{aligned}$$

Illustration 18 The initial cost a car is 1,16,000. If the cost of the car is depreciated at the end of each year by 10% then find after how many years, its depreciated value becomes Rs. 14,100.

Solution

Here depreciated value = 14,100.

V = 1,16,000, D = 10%

$$\therefore \text{Depreciated value} = V\left(1-\frac{D}{100}\right)^n$$

$$14{,}100 = 1{,}16{,}000\left(1-\frac{10}{100}\right)^n$$

$$0.1216 = a(0.9)^n$$

$$\therefore \log 0.1216 = n\log(0.9)$$

$$\bar{1}.0849 = n\left(\bar{1}.9542\right)$$

$$\therefore n = \frac{\bar{1}.0849}{\bar{1}.9542}$$

$$\therefore n = 19.98 \approx 20 \text{ years}$$

Illustration 19 The cost of a machine is Rs. 50,000. Its value is depreciated at the rate of 8% of the initial cost at the end of each year. Find its depreciated value after 20 years.

Solution

Here $V = 80{,}000$, $D = 8\%$, $n = 20$

$$\text{Now depreciated value} = V\left(1 - \frac{D}{100}\right)^n$$

$$= 80{,}000\left(1 - \frac{8}{100}\right)^{20}$$

$$= 80{,}000\left(0.92\right)^{20}$$

$$= 80{,}000\left(0.1887\right) \ \left(\text{by using calculator}\right)$$

$$= \text{Rs. } 15{,}095$$

Illustration 20 Initial cost of appliance is Rs. 64,000. The rate of depreciation for the first two years is 5%, then it becomes 8% for the next two years and it becomes 10% for the 5th year. Find the depreciated value of the appliance after 5 years.

Solution

Here $V = 64{,}000$, $D_1 = 5\%$, $n_1 = 2$, $D_2 = 8\%$, $n_2 = 2$, $D_3 = 10\%$, $n_3 = 1$

$$\text{Depreciated value} = V\left(10 - \frac{D_1}{100}\right)^{n_1}\left(10 - \frac{D_2}{100}\right)^{n_2}\left(10 - \frac{D_3}{100}\right)^{n_3}$$

$$= 64{,}000\left(1 - \frac{5}{100}\right)^2\left(1 - \frac{8}{100}\right)^2\left(1 - \frac{10}{100}\right)^1$$

$$= 64{,}000\left(0.95\right)^2 \times \left(0.92\right)^2 \times \left(0.92\right)$$

$$= \text{Rs. } 44{,}000 \ \left(\text{approx.}\right)$$

ANALYTICAL EXERCISES

1. Find the compound interest of Rs. 20,000 at the rate of 5% per annum for 1.5 years if interest is calculated half yearly.
2. A person has deposited Rs. 50,000 in a bank at compound interest of 12% per year. Find the amount in account after 20 years.
3. The compound interest of a sum at 8% p.a. for second year is Rs. 54. Find interest for the first year.

4. X borrowed Rs. 15,000 for 8 years at 6% p.a. compounded interest for the first two years, 8% p.a. for the next three years and 10% p.a. for the remaining 3 years. Find the amount paid after 8 years.
5. Find the compound interest on Rs. 6,950 at 12% per annum for 1.9 years, when interest is compounded quarterly.
6. Calculate the compound interest of Rs. 5,000 at 8% p.a. for 2 years, if the interest is calculated half yearly. What difference will if make if interest is calculated quarterly?
7. Find the compound interest on Rs. 11,920 at 12% per year if the interest is calculated (i) yearly, (ii) quarterly, (iii) monthly.
8. What sum of money will become Rs. 30,000 in 20 years at 5% p.a. compound interest?
9. What sum will amount to Rs. 28,119 in 3 years at rate of compound interest for the first year 3% p.a. for the second year 4% p.a. and for the third year 5% p.a?
10. A person borrows certain amount at 12% p.a compound interest. He lends this money at 12% p.a. compounded quarterly. If he earns Rs. 118.20 in two years then find the sum borrowed.
11. If Rs. 8,000 earns Rs. 5,040 in 5 years and the interest is compounded half yearly, find the rate of compound interest per year.
12. At what compound rate% per year a sum of Rs. 28,000 amounts to Rs. 31,670 in 1.3 years? The interest is compounded quarterly.
13. A sum at a certain rate of compound interest amounts to Rs. 2,916 in 2 years and Rs. 3,149.30 in 3 years. Find the sum and rate of compound interest.
14. The compound interest of a sum invested at a certain rate of compound interest for the second and third year is Rs. 1,760 and Rs. 1,936, respectively. Find the sum invested and the rate of interest.
15. In what time a sum of Rs. 20,000 becomes Rs. 22,050 at compound rate of 5% p.a?
16. In how many years the compound interest of Rs. 8,000 becomes Rs. 11,280 when the rate of compound interest is 10% calculated half yearly.
17. A sum lent on compound interest at 12% p.a. calculated quarterly is doubled in a certain time. Find the time period.
18. A sum invested at certain rate of compound interest doubles in 8 years. In how many years will it become nine times the sum?
19. Find the difference of simple interest and compound interest on Rs. 1,500 for 3 years at the interest rate of 6% p.a. when interest is compounded annually.
20. The simple interest of 3 years at 4% p.a. of certain sum is Rs. 303.60. Find the compound interest of this same sum for the same rate of interest.
21. On what sum the difference of its simple interest and compound interest of 3 years at 8% p.a. is Rs. 95.60?

22. X borrows a certain sum at 3% p.a. simple interest and invests the sum at 5% p.a. compound interest. After 3 years she makes a profit of Rs. 1,082. Find the sum borrowed.

23. The compound interest on certain sum for 3 years at 10% p.a. is Rs. 1,489.50. Find the simple interest of this sum at the same rate and for the same period.

24. A sum of Rs. 71,820 is divided between two persons P and Q who are at present 22 years and 23 years old and is invested at 12% p.a. compound interest. If each of them gets the same amount when they attain the age of 28 years, find the amount they received as they attained the age of 28 years and also the initial amount of investment.

25. A person deposits Rs. 6,000 in a bank at 9% p.a. compounded continuously. Find the amount that he receives after 10 years.

26. An amount triples itself in certain time if it is invested at 9% p.a. compounded continuously. Find the approximate time period.

27. How long will it take for Rs. 4,000 to amount to Rs. 7,000 if it is invested at 7% compounded continuously?

28. A national saving certificate costs Rs. 15 and realizes Rs. 20 after 10 years. Find the rate of interest when it is compounded continuously.

29. A person deposited Rs. 1,200 in saving account. For the first three years, interest was calculated semi annually at 6% p.a. then after next four years compounded quarterly at 10%. Find the amount at the end of 7 years.

30. A person invests Rs. 20,000 in a company offering interest at the rate 6% compounded half yearly during first year, at the rate of 12% compounded quarterly during the second year and at 10% compounded continuously during third year. Find the balance after 3 years.

31. Find the effective rate that is equivalent to the nominal rate of 12% compounded monthly.

32. First investment plan is 5% p.a. compounded quarterly and the second investment plan is 5.10% per year simple interest. Which plan is better?

33. The bacteria in a culture grow by 10% in first hour, decreases by 8% in seond hour, again increases by 7% in third hour. If at the end of third hour the count of bacteria is 1,51,70,000, find the number of bacteria at the initial hour.

34. What is the nominal rate of interest corresponding of the effective rate of 8%?

35. A money lender charges interest at the rate of 5 paise per rupee per month payable in advance. What effective rate of interest does he charge per year?

36. A machine depreciated in such a way that its value at the end of the year is 90% of its value at the beginning of the year. The cost of machine is Rs. 48,000 and it is eventually sold as a waste metal for Rs. 18,000. Find the number of years the machine is used.

37. A machine depreciates in value each year at 10% and at the end of fourth year it is valued at Rs. 1,31,220. Find its original cost.

38. A machine costing Rs. 70,000 depreciates at a constant rate of 6%. What is the depreciation for the 9th year? Determine the scrap value of the machine if its life is assumed to be 10 years.

39. An asset costing Rs. 15,000 will depreciate to scrap value of Rs. 1,200 in 10 years. Find the rate of depreciation.

40. A machine purchased for Rs. 50,000 is subjected to depreciation at the rate of 10% for the first 3 years and thereafter at the rate of 15% for the next 2 years. Find the value of the machine after 5 years.

ANSWERS

(1) Rs. 21,583
(2) Rs. 4,82,315
(3) Rs. 50
(4) Rs. 28,260
(5) Rs. 1,593
(6) Rs. 484, Rs. 849; Difference Rs. 11
(7) (i) Rs. 3,032.45, (ii) Rs. 3,170, (iii) Rs. 3,190
(8) Rs. 11,300 (approx.)
(9) Rs. 25,000
(10) 9,850
(11) 10% p.a.
(12) 10% p.a.
(13) Rs. 2,500, R = 8% p.a.
(14) Rs. 16,000, R = 10% p.a.
(15) 2 years
(16) 9 years
(17) 5.88 = 6 years
(18) R = 9%, n = 25.5 years
(19) Rs. 16
(20) P = Rs. 2,530, C.I. = Rs. 315
(21) P = 4,852.80
(22) Rs. 16,000
(23) P = Rs. 4,500, S.I. = Rs. 1,350
(24) P = Rs. 37,620, Q = Rs. 41,382; Initial investment = Rs. 66,650
(25) Rs. 15,710
(26) 12.2 years
(27) 8 years (approx)
(28) 2.88%
(29) Rs. 2,197
(30) Rs. 26,393
(31) 12.68%
(32) R_E = 5.09; second plan is better
(33) 1,40,09,456
(34) 7.69%
(35) 254.5%
(36) 9.31 years (approx.)
(37) Rs. 2,00,000
(38) Rs. 2,562; Rs. 32,298
(39) 22.33%
(40) Rs. 26,336.25

9

Annuity

LEARNING OBJECTIVES

After studying this chapter, you will be able to understand:

- The meaning and different terminologies of annuity
- Different types of annuity
- Derivation of formulas for different annuities
- The concept of amortization
- The meaning and uses of sinking fund

INTRODUCTION

In our day-to-day life we observe lots of money transactions. In many transactions payment is made in single transaction or in equal installments over a certain period of time. The amounts of these installments are determined in such a way that they compensate for their waiting time. In other cases, in order to meet future planned expenses, a regular saving may be done, i.e., at regular time intervals a certain amount may be kept aside, on which the person gains interest. In such cases the concept of annuity is used.

A sequence of equal payments made/received at equal intervals of time is called *annuity*. The amount of regular payment of an annuity is called *periodic payment*. The time interval between two successive payments is called *payment interval* or *payment period*. Note that the payment period may be annual, half yearly, quarterly, monthly or any fixed duration of time. The time for which the payment of an annuity is made is called *term of annuity,* i.e., it is the time interval between the first payment and the last payment.

The sum of all payments made and interest earned on them at the end of the term of annuities is called *future value of an annuity*, i.e., it is the total worth of all the payments at the end of the term of an annuity. Whereas the *present* or *capital value of an annuity* is the sum of the present values of all the payments of the annuity at the beginning of the annuity, *i.e.* it is the amount of money that must be invested in the beginning of the annuity of purchase the payments due in future. Here we note that unless mentioned specifically, the payment means yearly payment.

TYPES OF ANNUITIES

An annuity payable for a fixed number of years is called *annuity certain*. Installments of payment for a plot of land, bank security deposits, purchase of domestic durables are examples of annuity certain. Here the buyer/person knows the specified dates on which installments are to be made.

An annuity payable at regular interval of time till the happening of a specific event or the date of which cannot be accurately foretold is called *annuity contingent*, for example, the premiums on a life insurance policy, or a fixed sum paid to an unmarred girl at regular intervals of time till her marriage takes place.

An annuity payable forever is called *perpetual annuity* or *perpetuity*. In this annuity, beginning date is known but the terminal date is unknown, *i.e.,* an annuity whose payment continues forever is called perpetuity. For example, the endowment funds of trust, where the interest earned is used for welfare activities only. The principal remains the same and activity continues forever.

1. All the above types of annuities are based on the number of their periods. An annuity can also be classified on the basis of mode of payment as under.
2. An annuity in which payments of installments are made at the end of each period is called *ordinary annuity* or *annuity immediate,* e.g. repayment of housing loan, car loan etc.
3. An annuity in which payments of installments are made in the beginning of each period is called *annuity due*. In annuity due every payment is an investment and earns interest. Next payment will earn interest for one period less and so on, the last payment will earn interest of one period, e.g. saving schemes, life insurance payments, etc.
4. An annuity which is payable after the lapse of a number of periods is called *deferred annuity*. In this annuity, the term begins after certain time period termed as deferment period, e.g., pension plan of L.I.C. Many financial organizations give loan amount immediately and regular installments may start after specified time period.

Derivation of Formulae

1. *Amount of Immediate Annuity or Ordinary Annuity*

Let a be the ordinary annuity and i per cent be the rate of interest per period.

In ordinary annuity, the first installment is paid after the end of first period. Therefore it earns interest for $(n - 1)$ periods, second installment earns interest for $(n - 2)$ periods and so on. The last installment earns for $(n - n)$ period, i.e,. earns no interest.

The amount of first annuity for $(n - 1)$ period at i per cent rate per period = $a(1 + i)^{n-1}$, second annuity = $a(1 + i)^{n-2}$, third annuity = $a(1 + i)^{n-3}$ and so on.

Thus the total annuity A for n period at i per cent rate of interest is:

$$\begin{aligned} A &= a\left(1 + i\right)^{n-1} + a\left(1 + i\right)^{n-2} + \ldots + a\left(1 + i\right) + a \\ &= a + a\left(1 + i\right) + \ldots + a\left(1 + i\right)^{n-2} + a\left(1 + i\right)^{n-1} \\ &= a\left[1 + \left(1 + i\right) + \ldots + \left(1 + i\right)^{n-2} + \left(1 + i\right)^{n-2}\right] \\ &= a\,\frac{\left(1+i\right)^{n} - 1}{\left(1+i\right) - 1} \qquad \left(\text{G.P with common ratio} > 1\right) \\ A &= \frac{a}{i}\left[\left(1+i\right)^{n} - 1\right] \end{aligned}$$

2. *Present Value of Immediate Annuity (or Ordinary Annuity)*

Let a denote the annual payment of an ordinary annuity, n is the number of years and i per cent is the interest on one rupee per year and P be the present value of the annuity. In the case of immediate annuity, payments are made periodically at the end of specified period. Since the first installment is paid at the end of first year, its present value is $\frac{a}{1+i}$, the present value of second installment is $\frac{a}{\left(1+i\right)^{2}}$ and so on. If the present value of last installment is $\frac{a}{\left(1+i\right)^{n}}$ then we have

$$\begin{aligned} P &= \frac{a}{\left(1+i\right)} + \frac{a}{\left(1+i\right)^{2}} + \frac{a}{\left(1+i\right)^{3}} + \ldots + \frac{a}{\left(1+i\right)^{n}} \\ &= \frac{a}{\left(1+i\right)^{n}}\left[\left(1+i\right)^{n-1} + \left(1+i\right)^{n-2} + \ldots + 1\right] \\ &= \frac{a}{\left(1+i\right)^{n}}\left[1 + \left(1+i\right) + \left(1+i\right)^{2} + \ldots + \left(1+i\right)^{n-1}\right] \\ &= \frac{a}{\left(1+i\right)^{n}}\left[\frac{\left(1+i\right)^{n} - 1}{1+i-1}\right] \qquad \left(\text{G. P. with common ratio} > 1\right) \\ &= \frac{a}{\left(1+i\right)^{n}}\left[\frac{\left(1+i\right)^{n} - 1}{1}\right] \\ \therefore P &= \frac{a}{c}\left[1 - \frac{1}{\left(1-i\right)^{n}}\right] \end{aligned}$$

3. *Amount of Annuity Due at the End of n Period*

As defined earlier, annuity due is an annuity in which the payments are made at the beginning of each payment period. The first installment will earn interest for n periods at the rate of i per cent per period. Similarly second installment will earn interest for (n − 1) periods, and so on the last interest for on period.

Hence the amount of annuity due

$$\begin{aligned} A &= a\left(1+i\right)^{n} + a\left(1+i\right)^{n-1} + a\left(1+i\right)^{n-2} + \ldots + a\left(1+i\right)^{1} \\ &= a\left(1+i\right)\left[\left(1+i\right)^{n-1} + \left(1+i\right)^{n-2} + \ldots + 1\right] \end{aligned}$$

$$= (1+i)a\left[1+(1+i)+(1+i)^2+\ldots+(1+i)^{n-1}\right]$$

$$= (1+i)a\left[\frac{(1+i)^n-1}{1+i-1}\right]$$

$$A = (1+i)a\left[\frac{(1+i)^n-1}{1}\right]$$

$$A=(1+i)\frac{a}{i}\left[(1+i)^n-1\right]$$

4. *Present Value of Annuity Due*

Since the first installment is paid at the beginning of the first period (year), its present value will be the same as a where a is the annual payment of annuity due. The second installment is paid in the beginning of the second year, hence its present value is given as $\frac{a}{1+i}$ and so on. The last installment is paid in the beginning of nth year period, hence its present value is given as $\frac{a}{(1+i)^{n-1}}$

Thus if P denotes the present value of annuity due then

$$P = a + \frac{a}{1+i}+\frac{a}{(1+i)^2}+\ldots+\frac{a}{(1+i)^{n-1}}$$

$$= a\left[1+\frac{a}{(1+i)}+\frac{a}{(1+i)^2}+\ldots+\frac{a}{(1+i)^{n-1}}\right]$$

$$= a\frac{\left\{1-\left[1/(1+i)\right]^n\right\}}{1-1/(1+i)} \qquad \text{(G.P. with common ratio < 1)}$$

$$= a\frac{\left\{1-\left[1/(1+i)\right]^n\right\}}{(1+i-1)/(1+i)}$$

$$= a(1+i)\frac{\left[1-1/(1+i)^n\right]}{i}$$

$$P=(1+i)\frac{a}{i}\left[1-\frac{1}{(1+i)^n}\right]$$

5. *Perpetual Annuity*

Perpetual annuity is an annuity whose payment continues forever. As such the amount of perpetuity is undefined as the amount increases without any limit as time passes on. We know that the present value P of immediate annuity is given by

$$P=\frac{a}{i}\left[1-\frac{1}{(1+i)^n}\right]$$

Now as per the definition of perpetual annuity as $n \to \infty$, we know that $\frac{1}{(1+i)n} \to 0$ since $1 + i > 1$

$$\text{Hence } p = \frac{a}{i}[1-0]$$

$$\therefore\ p = \frac{a}{i}$$

6. *Deferred Annuity*

Amount of deferred annuity for *n* periods, differed *m* periods, is the value of the annuity at the end of its term and is given as

$$A = \frac{a}{i}\left[\frac{(1+i)^n - 1}{(1+i)^m}\right]$$

The present value of deferred annuity of *n* periods, deferred *m* periods, at the rate of *i* per year is given as

$$P = \frac{a}{i}\left[\frac{(1+i)^n - 1}{(1+i)^{m+n}}\right]$$

The derivation of the above formulae is left as an exercise for the students.

Note: In all the above formulae the period is of one year. Now if the payment is made more than once in a year then *i* is replaced by $\frac{i}{k}$ and *n* is replaced by *nk*, where *k* is the number of payments in a year.

AMORTIZATION

A loan is said to be amortized if it can be removed by a sequence of equal payments made over equal periods of time, which consists of the interest on the loan outstanding at the beginning of the payment period and part payments of the loan. With each payment the principal amount outstanding decreases and hence interest part on each payment decreases while the loan repayment of principal increases.

When a loan is amortized, the principal outstanding is the present value of remaining payments. Based on this, the following formulae are obtained that describe the amortization of an interest bearing loan of Rs. *A* at a rate of *i* per units per period, by *n* equal payments of Rs. *a* each when the payment is made at the end of each period.

1. Amount of each periodic payment $= \dfrac{A \times i}{1-(1+i)^{-n}}$
2. Principal outstanding of *p*th period $= \dfrac{a}{i}\left[1-(1+i)^{-n+p-1}\right]$
3. Interest contained in the *p*th payment $= a\left[1-(1+i)^{-n+p-1}\right]$

4. Principal contained in pth payment $= a\left(1 + i\right)^{-n+p-1}$
5. Total interest paid $= na - A$

SINKING FUND

It is a fund created by a company or person to meet predetermined debts or certain liabilities out of their profit at the end of every accounting year at compound rate of interest. This fund is also known *sinking fund* or *fund*. If 'a' is the periodic deposits or payments, at the rate of i per units per year then after n years the sinking fund A is $A = \frac{a}{i}\left[1(1+i)^n - 1\right]$

ILLUSTRATIONS

Illustration 1 X pays Rs. 64,000 per annum for 12 years at the rate of 10% per year. Find annuity.

Solution

Here $a = 64{,}000$, $n = 2$ and $i = \frac{12}{100} = 0.12$

$$\begin{aligned} \text{Ordinary annuity } A &= \frac{a}{i}\left[(1+i)^n - 1\right] \\ &= \frac{64{,}000}{0.1}\left[(1+0.1)^{12} - 1\right] \\ &= 64{,}000\left[(1.1)^{12} - 1\right] \end{aligned}$$

$$\begin{aligned} \text{Let } x &= (1.1)^{12} \\ \therefore \log x &= 12\log(1 - 1) \\ &= 12(0.0414) \\ x &= \text{antilog}(0.4967) \\ &= 3.1384 \end{aligned}$$

$$\begin{aligned} \therefore \text{Ordinary Annuity } A &= 6{,}40{,}000\left[3.1384 - 1\right] \\ &= 6{,}40{,}000 \times 2.1384 \\ &= \text{Rs. } 13{,}68{,}600. \end{aligned}$$

Illustration 2 If the payment of Rs. 20,000 is made quarterly for 10 years at the rate of 8% p.a., find amount of ordinary annuity.

Solution

$a = 20{,}000$, $n = 10$, $k = 4$, $i = 0.08$

Since the payment is made more than once in a year so i is replaced by $\frac{i}{k}$ and n is replaced by nk, then we have

$$A = \frac{a}{i/k}\left[\left(1+\frac{i}{k}\right)^{nk} - 1\right]$$

$$\begin{aligned}
&= \frac{20{,}000}{0.2}\left[(1+0.02)^{40} - 1\right] \\
A &= 10{,}00{,}000\left[(1.02)^{40} - 1\right] \\
\text{Let } x &= (1.05)^{40} \\
\log x &= 40\log(1.02) \\
&= 40(0.0086) \\
\therefore x &= \text{antilog}x(0.3440) \\
\therefore A &= 10{,}00{,}000(2.208 - 1) \\
&= 10{,}00{,}000(1.208) \\
&= \text{Rs. } 12{,}08{,}000
\end{aligned}$$

Illustration 3 A man deposits Rs. 30,000 on 31st December 1996. What amount does he receive on 31st December 2008, if the interest rate is 10% compounded annually?

Solution

$$a = 30{,}000,\ i = \frac{10}{100} = 0.10,\ n = 12 \text{ years}$$

$$\begin{aligned}
\therefore A &= \frac{a}{i}\left[(1+i)^{n} - 1\right] \\
&= \frac{30{,}000}{0.1}\left[(1.1)^{12} - 1\right] \\
&= 3{,}00{,}000\left[(1.1)^{12} - 1\right] \\
\text{Let } x &= (1.1)^{12} \\
\therefore \log x &= 12\log(1.1) \\
x &= \text{antilog}x(3.1391 - 1) \\
\text{Now } A &= 3{,}00{,}000(3.1391 - 1) \\
&= 3{,}00{,}000(2.1391) \\
&= \text{Rs. } 6{,}41{,}718.80
\end{aligned}$$

Illustration 4 What amount should be deposited annually so that after 16 years a person receives Rs. 1,67,160, if the interest rate is 15%?

Solution

Here $A = 1{,}67{,}160$, $n = 16$, $i = 0.15$, $a = ?$

$$\begin{aligned}
\therefore A &= \frac{a}{i}\left[(1+i)^{n} - 1\right] \\
1{,}67{,}160 &= \frac{a}{0.15}\left[(1.1)^{16} - 1\right] \\
\text{Let } x &= (1.15)^{16} \\
\log x &= 16\log(1.15) \\
&= 16(0.0607) \\
x &= \text{antilog}(0.9712) \\
&= 9.358
\end{aligned}$$

$$\text{Hence } 1{,}67{,}160 = \frac{a}{0.15}(9.358 - 1)$$

$$\therefore a = \frac{1{,}67{,}160 \times 0.15}{8.358}$$

$$\therefore a = \text{Rs. } 3{,}000$$

Illustration 5 The age of the daughter is 2 years. His father wants to get Rs. 2,00,000 when his daughter is 22 years old. He opens an account with a bank at 10% rate of compound interest. What amount should he deposit at the end of every month in this recurring account?

Solution

$A = 20{,}00{,}000$, $i = 0.10$, $n = 20$, and $k = 12$

$$\therefore A = \frac{a}{i/k}\left[\left(1 + \frac{i}{k}\right)^{nk} - 1\right]$$

$$= 20{,}00{,}000 = \frac{a}{0.10/12}\left[\left(1 + \frac{0.10}{12}\right)^{20 \times 12} - 1\right]$$

$$= \frac{129}{0.10}\left[\left(\frac{12.00}{12}\right)^{250} - 1\right]$$

$$= 120a\left[(1.0083)^{240} - 1\right]$$

$$\text{Let } x = (1.0083)^{240}$$

$$\therefore \log x = 240\log(1.0083)$$

$$= 240(0.0033)$$

$$x = \text{antilog}(0.783)$$

$$= 6.194$$

$$\text{Hence } 20{,}00{,}000 = 120a(6.164 - 1)$$

$$\therefore a = \frac{20{,}00{,}000}{120 \times 5.194}$$

$$= 3{,}208.83$$

$$a \approx 3{,}209$$

i.e. he should deposit Rs. 3,209 every month.

Illustration 6 A person deposits Rs. 4,000 in the beginning of every year. If the rate of compound interest is 14% then find the amount after 10 years.

Solution

Here $a = 4{,}000$, $i = 0.14$, $n = 10$

$$\therefore A = (1+i)\frac{a}{i}\left[(1+i)^n - 1\right]$$

$$= \left(1+0.14\right)\frac{4{,}000}{0.14}\left[\left(1+0.14\right)^{10}-1\right]$$

$$= \left(1.14\right)\frac{4{,}000}{0.14}\left(3.707\ -1\right)$$

$$= \left(1.14\right)\frac{4{,}000}{0.14}\left(2.707\right)$$

$$= \text{Rs. } 88{,}170$$

Illustration 7 A person deposits Rs. 5,000 in the beginning of every month in recurring account at the rate of 11% p.a. What amount would he get after 15 years?

Solution

$$a = 5{,}000,\ i = 0.11,\ n = 15,\ k = 12$$

$$\therefore A = \left(1+\frac{i}{k}\right)\frac{9}{i/k}\left[\left(1+\frac{i}{k}\right)^{nk}-1\right]$$

$$= \left(1+\frac{0.11}{12}\right)\frac{5{,}000}{\left(0.11/12\right)}\left[\left(1+\frac{0.11}{12}\right)^{15\times 12}-1\right]$$

$$= \left(\frac{12.11}{12}\right)\frac{5{,}000\times 12}{0.11}\left[\left(\frac{12.11}{12}\right)^{180}-1\right]$$

$$= 12.11\times\frac{5{,}000\times 12}{0.11}\left[\left(1.0092\right)^{180}-1\right]$$

$$= 12.11\times\frac{5{,}00{,}000}{11}\left(4.634\ -1\right)$$

$$= \frac{60{,}55{,}000}{11}\left(3.634\right)$$

$$= \text{Rs. } 20{,}00{,}352$$

Illustration 8 What amount should be deposited in the beginning of January, April, July and October of every year at 15% rate of compound interest to receive Rs. 40,00,000 on maturity at the end of 10 years?

Solution

$$A = 40{,}00{,}000,\ i = 0.15,\ n = 1,\ k = 4,\ a = ?$$

$$\text{Now } A = \left(1+\frac{i}{k}\right)\frac{a}{i/k}\left[\left(1+\frac{i}{k}\right)^{nk}-1\right]$$

$$40{,}00{,}000 = \left(1+\frac{0.15}{4}\right)\frac{a}{\left(0.15/4\right)}\left[\left(1+\frac{0.15}{4}\right)^{10\times 4}-1\right]$$

$$\therefore 40{,}00{,}000 = \left(\frac{4.15}{4}\right)\frac{4a}{0.15}\left[\left(\frac{4.15}{4}\right)^{40} - 1\right]$$

$$40{,}00{,}000 = \frac{4.15a}{0.15}\left[(1.0375)^{40} - 1\right]$$

$$40{,}00{,}000 = \frac{415a}{15}(4.406 - 1)$$

$$\therefore a = \frac{40{,}00{,}000 \times 15}{415 \times 3.406}$$

$$a = 42{,}448.12$$

$$a = \text{Rs. } 42{,}448$$

Illustration 9 A person purchases a machine on 1st January 2009 and agrees to pay 10 installments each of Rs. 12,000 at the end of every year inclusive of compound rate of 15%. Find the present value of machine.

Solution

Here $n = 10$, $a = 12{,}000$, $i = 0.15$

$$\text{Now } P = \frac{a}{1}\left[1 - \frac{1}{(1+i)^n}\right]$$

$$= \frac{12{,}000}{0.15}\left[1 - \frac{1}{(1+0.15)^{10}}\right]$$

$$= 80{,}000\left(1 - \frac{1}{4.016}\right)$$

$$= 80{,}000\left(\frac{3.046}{4.046}\right)$$

$$= 60{,}227.38$$

$$= \text{Rs. } 60{,}227$$

Illustration 10 A person has purchased a car on 1st April 2005. He has paid Rs. 50,000 cash and an installment of Rs. 5,000 to be paid at the end of each month for 5 years. If this amount includes the interest rate of 15% then find the purchase price of car at present.

Solution

Here $a = 5000$, $i = 0.15$, $n = 5$, $k = 12$

$$\therefore P = \frac{a}{i/k}\left\{1 - \frac{1}{\left[1 + (1/k)\right]^{nk}}\right\}$$

$$= \frac{5{,}000}{(0.15/12)}\left\{1-\frac{1}{[1+(0.15/12)]^{5\times12}}\right\}$$

$$= \frac{5{,}000\times12}{0.15}\left[1-\frac{1}{(1.0125)^{60}}\right]$$

$$= 4{,}00{,}000\left(\frac{1.138}{2.138}\right)$$

= 2,12,909.26

= Rs. 2.12.909

Hence the purchase price of that car = 2,12,909 + 50,000
= Rs. 2,62,909

Illustration 11 A photographer purchases a camera on installments. He has to pay 7 annual installments each of Rs. 36,000 right from the date of purchase. If the rate of compound interest is 16% then find the cost price (present value) of camera.

Solution

Here $a = 36000$, $n = 7$, $i = 0.16$,

$$\text{Now } P = (1+i)\frac{a}{i}\left[1-\frac{1}{(1+i)^n}\right]$$

$$= (1+0.16)\frac{36{,}000}{0.16}\left[1-\frac{1}{(1+0.16)^7}\right]$$

$$= (1.16)\frac{36{,}000}{0.16}\left[1-\frac{1}{(1.16)^7}\right]$$

$$= 2{,}61{,}000\left(1-\frac{1}{2.282}\right)$$

$$= 2{,}61{,}000\left(\frac{1.828}{2.828}\right)$$

= 1,68,708,62

= Rs. 1,68,709

Illustration 12 A proprietor of a garage took a loan on 1st January 2005 at 12% rate of compound interest. In order to repay, if he agrees to pay Rs. 15,000 in the beginning of the months January, April, July and October up to 6 years, find the present value of loan.

Solution

Here $a = 15{,}000$, $n = 6$, $i = 0.12$, $k = 4$.

$$\therefore P = \left(1+\frac{i}{k}\right)\frac{a}{i/k}\left\{1-\frac{1}{\left[1+(i/k)\right]^{nk}}\right\}$$

$$= \left(1+\frac{0.12}{4}\right)\frac{15{,}000}{(0.12/4)}\left\{1-\frac{1}{\left[1+(0.12/4)\right]^{6\times 4}}\right\}$$

$$= 1.03\frac{15{,}000\times 4}{0.12}\left[1-\frac{1}{(1.03)^{24}}\right]$$

$$= 5{,}15{,}000\left(1-\frac{1}{2.029}\right)$$

$$= 5{,}15{,}000\left(\frac{1.029}{2.029}\right)$$

$$= \text{Rs. } 2{,}61{,}180$$

Illustration 13 In order to purchase a manufacturing uni,t a person has taken a loan of Rs. 15,00,000 from State Bank of India at 12% rate of interest. If he repays the amount in 10 yearly installments then find the amount of installment.

Solution

$P = 15{,}00{,}000$, $n = 10$, $i = 0.12$, $a = ?$

$$P = \frac{a}{i}\left[1-\frac{1}{(1+i)^n}\right]$$

$$15{,}00{,}000 = \frac{a}{0.12}\left(1-\frac{1}{(1.12)^{10}}\right)$$

$$15{,}00{,}000 = \frac{a}{0.12}\left(1-\frac{1}{3.105}\right)$$

$$15{,}00{,}000 = \frac{a}{0.12}\left(\frac{2.105}{3.105}\right)$$

$$\therefore a = \frac{15{,}00{,}000\times 0.12\times 3.105}{2.105}$$

$$a = 2{,}65{,}510.68$$

$$a = \text{Rs. } 2{,}65{,}511$$

i.e. the yearly installment is Rs. 2,65,511

Illustration 14 A person has taken a loan of Rs. 7,00,000 at 16% rate of interest from a finance company. If the repayment period is of 15 years then find what installment has he to pay at the beginning of each month.

Solution

$P = 7{,}00{,}000$, $n = 15$, $i = 0.16$ and $k = 12$.

$$P = \left(1+\frac{i}{k}\right)\frac{a}{i/k}\left\{1-\frac{1}{\left[1+(i/k)\right]^{nk}}\right\}$$

$$7{,}00{,}000 = \left(1+\frac{0.16}{12}\right)\frac{a}{0.116/12}\left\{1-\frac{1}{\left[1+(0.16/12)\right]^{15\times 12}}\right\}$$

$$7{,}00{,}000 = \frac{12.16}{12}\times\frac{12a}{0.16}\left[1-\frac{1}{(1.0133)^{180}}\right]$$

$$7{,}00{,}000 = \frac{12.16a}{0.16}\left(1-\frac{1}{9.772}\right)$$

$$7{,}00{,}000 = \frac{1{,}216a}{16}\left(\frac{8.772}{9.772}\right)$$

$$\therefore a = \frac{7{,}00{,}000\times 16\times 9.772}{1{,}216\times 8.772}$$

$= 10{,}260.52$

$=$ Rs 10,261

i.e. the person has to pay Rs. 10,261 as monthly installment.

Illustration 15 A manufacturer plans to purchase a machine costing Rs. 3,00,000. Its estimated life is 10 years and the machine will add net income of Rs. 60,000 every year, with the prevailing rate of compound interest at 12%. It is advisable to purchase the machine? Why?

Solution

$a = 60{,}000$, $n = 10$, $i = 0.12$

Now we find the present value of the machine

$$P = \frac{a}{i}\left[1-\frac{1}{(1+i)^n}\right]$$

$$\therefore P = \frac{60{,}000}{0.12}\left[1-\frac{1}{(1+0.12)^{10}}\right]$$

$$= \frac{60,00,000}{12}\left[1-\frac{1}{(1-12)^{10}}\right]$$

$$= 5,00,000\left(1-\frac{1}{3.105}\right)$$

$$= 5,00,000\left(\frac{2.105}{3.105}\right)$$

= 3,38,969.4

= Rs. 3,38,969

i.e. the present value of the machine is Rs. 3,38,969 which is higher than the purchase cost. So it is advisable to purchase the machine.

Illustration 16 The chairman of a society wishes to award a gold medal to a student getting highest marks in business mathematics. If this medal costs Rs. 9,000 every year and the rate of compound interest is 15%, what amount is he required to deposit now?

Solution

Here $a = 9{,}000$, $i = 0.15$

$$P = \frac{a}{i}$$

$$= \frac{5,000}{0.15}$$

= Rs. 60,000

∴ The amount to be deposited = Rs. 60,000.

Illustration 17 A limited company wants to create a fund to help their employees in critical circumstances. If the estimated expenses per month are Rs. 18,000 and the rate of compound interest is 15% then find out the required amount to be deposited by the company for this purpose.

Solution

$a = 18{,}000$, $i = 0.15$, $k = 12$

$$\therefore P = \frac{a}{i/k}$$

$$= \frac{18,000}{0.15/12}$$

= 14,40,000

The required amount to be deposited = Rs. 14,40,000.

Illustration 18 At what rate converted semi-annually will the present value of a perpetuity of Rs. 675 payable at the end of each 6 months be Rs. 30,000?

Solution

$$P = \frac{a}{i/k}$$

$$\therefore 30{,}000 = \frac{675}{i/2}$$

$$\therefore i = \frac{675 \times 2}{30{,}000}$$

$$\therefore i = 0.045$$

∴ 4.5% is the rate of interest.

Illustration 19 Q. borrows certain sum of money at 8% p.a. compound interest and agreed to repay it in 10 equal yearly instatementd of Rs. 1,200 each. If the first installment is to be paid at the end of 5 years from the date of borrowing and the other yearly installments are paid regularly, then find the sum borrowed by Q.

Solution

$a = 1{,}200$, $i = 0.08$, $n = 10$, $m = 4$, $p = ?$

$$P = \frac{a}{i}\left[\frac{(1+i)^{n} - 1}{(1+i)^{m+n}}\right]$$

$$= \frac{12{,}00}{0.08}\left[\frac{(1+0.08)^{10-1}}{(1+0.08)^{4+10}}\right]$$

$$= \frac{1{,}200}{0.08}\left[\frac{(1.08)^{10} - 1}{(1.08)^{14}}\right]$$

$$= 15{,}000\left(\frac{2.158-1}{2.935}\right)$$

$$= 5{,}918.23$$

$$= \text{Rs. } 5{,}918$$

∴ The sum borrowed = Rs. 5,918

Illustration 20 A loan of Rs. 10,000 is to be repaid by equal annual installments of principal and interest over a period of 20 years. The rate of interest is 3% p.a. compounded annually. Find (1) Annual installment, (2) The capital contained in 8th installment and (3) The principal repaid after 12 installments have been paid.

Solution

(1) $P = 10{,}000,\ i = 0.03,\ n = 20,\ a = ?$

$$P = \frac{a}{i}\left[1-(1+i)^{-n}\right]$$

$$\therefore 1{,}000 = \frac{a}{003}\left[1-(1+0.3)^{-20}\right]$$

$$\therefore a = \frac{10{,}000}{14.88}$$

$$a = 672.04$$

$$a = 672$$

∴ The size of each annual installment is Rs. 672.

(2) Capital contained in the 8th installment

$= a\left[(1+i)^{-n+k-1}\right]$

$= 672\left[(1.03)^{-20+8-1}\right]$

$= 672(1.03)^{-13}$

$= 672 \times 0.681$

$= 457.63$

(3) The principal repaid after 12 installments

= loan amount − Principal outstanding in the beginning of 13th year

$= 10{,}000 - \dfrac{672}{0.03}\left[1-(1.03)^{-20+13-1}\right]$

$= 10{,}000 - \dfrac{675}{0.03}(1-0.7849)$

= Rs. 5,181.76

Illustration 21 A limited company issued 10% cumulative debentures of Rs. 100 each. These 5,000 cumulative debentures are to be redeemed with interest after 5 years. In order to provide this fund, the company wants to create a sinking fund by transferring some amount yearly and investing it at 12% rate of compound interest. Find out the sum to be transferred every year to the sinking fund.

Solution

Let us first find out the amount to be paid after 5 years including the amount of interest.

Here $P = 100 \times 5{,}000 = 5{,}00{,}000,\ n = 5,\ R = 100$

$$\therefore A = P\left(1+\frac{R}{100}\right)$$

$= 5{,}00{,}000\left(1+\frac{10}{100}\right)^5$

$= 5{,}00{,}000(1.10)^5$

$= 8{,}05{,}255$

So the amount to be paid after 5 years is Rs. 8,05,255.

Now

$A = 8{,}05{,}255,\ n = 5,\ i = 0.12$

$$\therefore A = \frac{a}{i}\left[(1+i)^n - 1\right]$$

$$8{,}05{,}255 = \frac{a}{0.12}\left[(1+0.12)^5 - 1\right]$$

$$8{,}05{,}255 \times 0.12 = a\left[(1.12)^5 - 1\right]$$

$$96{,}630.6 = a\left[1.7623 - 1\right]$$

$$\therefore a = \frac{96{,}630.6}{0.7623} = 1{,}26{,}761.9$$

$$a = 1{,}26{,}762$$

∴ *i.e.* Rs. 1,26,762 are to be transfused every year to sinking fund.

Illustration 22 A manufacturing unit purchased a machine for Rs. 2,50,000 the expected life of which is 10 years. It is assumed that after 10 years, a new machine would cost them 40% more. Under this assumption the manufacturing unit decided to create a sinking fund and invest it at 15% rate of compound interest. Find the amount to be transferred every year to this fund.

Solution

Purchase price of the machine after 10 years

= 2,50,000 + 40% of 2,50,000

= 2,50,000 + 1,00,000

= 3,50,000

Now $A = 3{,}50{,}000,\ i = 0.15,\ n = 10$

$$\therefore A = \frac{a}{i}\left[(a+i)^n - 1\right]$$

$$3{,}50{,}000 = \frac{a}{0.15}\left[(1+0.15)^{10} - 1\right]$$

$$3{,}50{,}000 \times 0.15 = a\left[(1.15)^{10} - 1\right]$$

$$52{,}500 = a\left[4.0456 - 1\right]$$

$$\therefore a = \frac{52{,}500}{3.0456}$$

$$\therefore a = 17{,}237.98$$

$$\therefore a = \text{Rs. } 17{,}238$$

∴ Rs. 17,238 should be transferred to sinking fund every year.

Illustration 23 A sinking fund is created by investing Rs. 3,500 half yearly. Find the amount of the fund after six years assuming that the interest rate is 4%.

Solution

Here $n = 6$, $a = 3{,}500$, $i = 0.04$, $k = 2$.

$$\begin{aligned}\therefore A &= \frac{a}{i/k}\left[(1+i/k)^{nk} - 1\right]\\ &= \frac{3,500}{0.04/2}\left[\left(1+\frac{0.04}{2}\right)^{2\times 6} - 1\right]\\ &= \frac{3,500\times 2}{0.04}\left[(1.02)^{12} - 1\right]\\ &= 1{,}75{,}000\left(1.2682 - 1\right)\\ &= \text{Rs. } 46{,}935\end{aligned}$$

Illustration 24 A company intends to create a sinking fund to replace at the end of 20th year assets costing Rs. 25,00,000. Calculate the amount to be retained out of profit every year if the interest rate is 8% per year.

Solution

Here $A = 25{,}00{,}000$, $n = 20$, $i = 0.08$

$$\begin{aligned}\therefore A &= \frac{a}{i}\left[(1+i)^{n} - 1\right]\\ 25{,}00{,}000 &= \frac{a}{0.08}\left[(1+0.08)^{20} - 1\right]\\ 25{,}00{,}000 \times 0.08 &= a\left[4.6610 - 1\right]\\ 2{,}00{,}000 &= a\left(3.6610\right)\end{aligned}$$

$$\therefore a = \frac{2,00,000}{3.3310} = 54,629.88$$

$\therefore$ Rs. 54,630 are to be invested every year.

ANALYTICAL EXERCISES

1. Find the amount of an ordinary annuity of Rs. 6,400 p.a. for 12 years at the rate of interest of 10% per year. [Given $(1.1)^{12} = 3.1384$]
2. If the payment of Rs. 2,000 is made at the end of every quarter for 10 years at the rate of 8% per year then find the amount of annuity. [Given $(1.02)^{40} = 2.2080$]

3. Find the amount of an ordinary annuity of 12 monthly payment of Rs. 1,500 that earns interest at 12% p.a. compounded monthly. [Given $(1.01)^{12} = 1.1262$]
4. A bank pays 8% p.a. interest compounded quarterly. What equal deposits have to be made at the end of each quarter for 10 years to have Rs. 30,200? [Given $(1.02)^{40} = 2.2080$]
5. Find the least number of years for which an ordinary annuity of Rs. 800. p.a. must accumulate Rs. 20,000 at 16% interest compounded annually. [Given $(1.02)^{40} = 2.2080$]
6. A firm anticipates a capital expenditure of Rs. 5,00,000 for a new machine is 5 years. How much should be deposited quarterly in sinking fund carrying 12% p.a. compounded quarterly to provide for the purchase. [$(1.03)^{20} = 1.8061$]
7. A person deposits Rs. 2,000 from his salary towards his P.F. account. The same amount is credited by his employer also. If 8% rate of compound interest is paid then find balance amount after his service of 20 years. [Given $(1.0067)^{240} = 3.3266$]
8. Find the present value of Rs. 2,000 p.a. for 14 years at 10% p.a. rate of interest [Given $(1.1)^{-14} = 0.2632$]
9. Find the present value of an annuity of Rs. 900 payable at the end of 6 months for 6 years. The money compound at 8% p.a. [Given $(1.04)^{-12} = 0.6252$]
10. A firm buys a machine on installments and pays Rs. 50,000 cash and the balance payment in 5 equal installments of Rs. 40,000 each payable at the end of each year. If the rate of interest is 10% compounded annually, find the cash price of machine. [Given $(1.1)^5 = 1.6106$]
11. A loan of Rs. 50,000 is to be repaid at 8% p.a. compound interest in 24 equal annual installments beginning at the end of first year. Find the amount of each installment. [Given $(1.06)^{24} = 4.0486$]
12. A person borrows Rs. 80,000 at 10% p.a. interest rate compounded half yearly and agrees to pay in 12 installments at the end of each six months. Find the installments. (Given log 1.05 = 0.0212 and log 5.567 = 0.7456)
13. A man retires at the age of 60 years. He gets pension of Rs. 72,000 p.a. for rest of his life. Reckoning his expected life to be upto 74 years, find single sum equivalent to his pension of interest at 16% p.a. [Given log 1.16 = 0.0645, antilog $\bar{1}.0970 = 0.1250$]
14. Find the amount at the end of 12 years of an annuity of Rs. 5,000 payable at the beginning of each year, if the money is compounded at 10% p.a.
15. A person purchased an item paying Rs. 5,000 down payment and agrees to pay Rs. 200 every three months for next 4 years. The seller charges 8% per annum interest compounded quarterly. Find the cash price of the item. Also if the person missed the first three payments, what must he pay at the time the fourth is due to bring him up to date?

16. What should be the monthly sales volume of a company if it desires to earn a 12% annual return convertible monthly on its investment of Rs. 2,00,000? Monthly costs are Rs. 3,000. The investment will have an eight year life and no scrap value.

17. What equal payments made at the beginning of each year for 10 years will pay for a piece of property priced at Rs. 6,00,000 if the money worth 5% effective? [Given $(1.05)^{-10} = 0.6139$]

18. Find the amount of annuity of Rs. 250 payable at the beginning of each month for 5 years at 15% p.a. compounded monthly. [Given $(1.0125)^{60} = 2.051$]

19. What is the present value of an annuity due of Rs. 1,500 for 16 years at 8% p.a? [Given $(1.08)^{15} = 3.1696$]

20. Find the present value of an annuity due of Rs. 45,000 payable half yearly for 13 years at 9% p.a. compounded half yearly.

21. What equal installments are made at the beginning of each month for 5 years will pay for a machine priced as Rs. 1,50,000 if the money worth 15% p.a. compounded monthly? [Given $(1.0125)^{60} = 0.4742$]

22. Find the present value of sequence of annual payment of Rs. 1,500 each, the first being made at the end of 3 years and the last at the end of 10 years. The money worth value is 8%.

23. An annuity consists of 4 annual payments of Rs. 2,500 each, the first being made at the end of 5th year. Find the amount of annuity, if the money is worth 10% effective.

24. An annuity consists of 15 semi–annual payments of Rs. 500 each first being payable after 2.5 years. Find the amount of annuity of the money is worth 8% p.a. [Given $(1.04)^{15} = 2.550$]

25. A man paid his advance taken for house constructed in equal installments of Rs. 25,000 half yearly for 10 years. If the money worth 8% p.a. compounded half yearly and payment started after initial gap of 5 years, find the sum of advance taken. [Given $(1.04)^{9} = 1.422$ and $(1.04)^{10} = 1.0880$]

26. What is the amount of perpetual annuity of Rs. 50 at 5% compound interest per year?

27. A cash prize of Rs. 1,500 is given to the student standing first in examination of business mathematics by a person every year. Find out the sum that the person has to deposit to meet this expense. Rate of interest is 12% p.a.

28. A company proposes to create a find to assist the children of their employees in education, illness, etc. The estimated expenses are Rs. 3,000 per month. If the rate of interest is 12.5% then find what amount should the company invest to meet these expenses.

29. A limited company has purchased a machine for Rs. 90,000, the expected life of machine is 7 years. After 7 years, the scrap value of machine is Rs. 2,500. A new machine would cost 25% more. Under this assumption it was decided to create a sinking fund of a sum of Rs. 11,000 every year. This amount should be invested at 12% rate of compound interest. Do you think that this is a wise decision?

30. A company establishes a sinking fund for payment of Rs. 16,00,000 debt maturing in 7 years. How much should be deposited quarterly into sinking fund carrying 12% per annum interest compounded quarterly to provide for the purpose. [Given $(1.03)^{28} = 2.2879$]

31. A company creates a sinking fund for redemption of debentures of Rs. 1,60,000 at the end of 16 years. If the company puts Rs. 6,000 at the end of each year in sinking fund and the fund accumulates 8% p.a. then find the extra money in the fund after paying off the debenture [Given: $(1.08)^{16} = 3.423$]

ANSWERS

(1) Rs. 1,36,860
(2) Rs. 1,20,800
(3) Rs. 18,930
(4) Rs. 588 (approx.)
(5) 11 years (approx.)
(6) Rs. 18,608
(7) Rs. 13,96,960
(8) Rs. 8,433
(9) Rs. 8,433
(10) Rs. 2,01,640
(11) Rs. 3,984 (approx.)
(12) Rs. 9,023 (approx.)
(13) Rs. 3,93,750 (approx.)
(14) Rs. 1,17,645
(15) Rs. 7,716 (approx.) Rs. 824 (approx.)
(16) Monthly sales volume = Rs. 6,251 (approx.)
(17) Rs. 7,40,005 (approx.)
(18) Rs. 21,280
(19) Rs. 14,334.38
(20) Rs. 81,196 (approx.)
(21) Rs. 35,220
(22) Rs. 7,355.30
(23) Rs. 11,600
(24) Rs. 3,899 (approx.)
(25) Rs. 15,040
(26) Rs. 1,000
(27) Rs. 12,500
(28) Rs. 2,88,000
(29) $A = 1,10,900$; it was a good decision
(30) Rs. 37,265
(31) Rs. 2,17,250

10 Limit and Continuity

LEARNING OBJECTIVES

After studying this chapter, you will be able to understand:

- Know the concept of limit and continuity
- Understand the theorems (statements) and formula of limit and definition of continuity
- Know how to solve the problems related to limit and continuity with the help of given illustrations

INTRODUCTION

Limit and Continuity is a branch of calculus. Intuitively we call quantity y a function of another quantity x if there is a rule by which a unique value of y is associated with a corresponding value of x.

DEFINITION OF LIMIT OF A FUNCTION

When value of x is very close to certain value a, then functional value $f(x)$ goes on becoming l. Then symbolically we can say that

$$\lim_{x\to a} f(x)=l$$

Note: Meaning of $x\to a$ $(x \text{ tends to } a)$ is $(x-a)$ must be a common factor of polynomial.

Meaning of $n\to\infty$

When the value of n increases indefinitely then we can say that $n\to\infty$ as $1/n=0$ and we write it as $\lim\limits_{n\to\infty} 1/n=0$

Working Rules of Limit

Suppose $\lim\limits_{x\to a} f(x)=l$ and $\lim\limits_{x\to a} g(x)=m$ then

1. $\lim\limits_{x\to a}\left[f(x)\pm g(x)\right]=\lim\limits_{x\to a} f(x)\pm\lim\limits_{x\to a} g(x)=l\pm m$

2. $\lim_{x \to a} \mathrm{k} f(x) = \mathrm{k}\left[\lim_{x \to a} f(x)\right] = \mathrm{k}l$ (where k = constant term)
3. $\lim_{x \to a} f(x)g(x) = \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right] = lm$
4. $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{l}{m}$ where $g(x) \neq 0$
5. $\lim_{x \to a} \log_e^{f(x)} = \log_e^{\left[\lim_{x \to a} f(x)\right]} = \log_e^l$

Some Standard Forms of Limit

1. $\lim_{x \to a} \left(\frac{x^n - a^n}{x - a}\right) = na^{n-1};\ n \in R$
2. $\lim_{h \to 0} \left(\frac{a^h - 1}{h}\right) = \log_e^a\ ;\ a \in R^+ - \{1\}$
3. $\lim_{h \to 0} \left(\frac{e^h - 1}{h}\right) = \log_e^e = 1$
4. $\lim_{n \to \infty} (1 + n)^{1/n} = e$
5. $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$
6. $\lim_{n \to \infty} |r|^n = 0$ where $|r| < 1$ $(\therefore 0 < r < 1)$

Some Standard Power Series

1. $1 + 2 + \ldots + n = \sum n = \frac{n(n+1)}{2}$
2. $1^2 + 2^2 + \ldots + n^2 = \sum n^2 = \frac{n(n+1)(2n+1)}{6}$
3. $1^3 + 2^3 + \ldots + n^3 = \sum n^3 = \left(\sum n\right)^2 = \frac{n^2(n+1)^2}{4}$
4. $1 + 1 + 1 + \ldots + 1$ n terms = $\sum 1 = n$
5. If $S_n = a + ar + ar^2 + \ldots + ar^{\infty - 1}$ then $\lim_{n \to \infty} S_n = \frac{a}{1 - r}$ where $r < 1$

DEFINITION OF CONTINUITY

If function f be defined as $x = a$ then

1. $f(x) = f(a)$ exists
2. $\lim_{x \to a^+} f(x) = f(a) = \lim_{x \to a^-} f(x)$

If above conditions are satisfied in any function then we say that $f(x)$ is continuous at $x = a$.

ILLUSTRATIONS

Illustration 1 $\lim\limits_{y\to\sqrt{2}} \dfrac{y^{5/2} - 2^{5/4}}{\sqrt{y} - 2^{1/4}}$

Solution

$$\lim_{y\to\sqrt{2}} \frac{y^{5/2} - 2^{5/4}}{\sqrt{y} - 2^{1/4}}$$

$$= \lim_{y\to\sqrt{2}} \frac{y^{5/2} - \left(\sqrt{2}\right)^{5/2}}{y^{1/2} - \left(\sqrt{2}\right)^{1/2}}$$

$$= \lim_{y\to\sqrt{2}} \left\{\frac{\left[y^{5/2} - \left(\sqrt{2}\right)^{5/2}\right] / \left(y - \sqrt{2}\right)}{\left[y^{1/2} - \left(\sqrt{2}\right)^{1/2}\right] / \left(y - \sqrt{2}\right)}\right\}$$

$$= \frac{(5/2)\left(\sqrt{2}\right)^{5/2 - 1}}{(1/2)\left(\sqrt{2}\right)^{1/2 - 1}} \qquad \left[\because \lim_{y\to a} \frac{y^n - a^n}{y - a} = na^{n-1}\right]$$

$$= \frac{5\left(\sqrt{2}\right)^{3/2}}{\left(\sqrt{2}\right)^{-1/2}}$$

$$= 5\left(\sqrt{2}\right)^2 = 10$$

Illustration 2 $\lim\limits_{x\to 0} \dfrac{a^x + b^x + c^x - 3}{x}$

Solution

$$\lim_{x\to 0} \frac{a^x + b^x + c^x - 3}{x}$$

$$= \lim_{x\to 0} \frac{a^x - 1 + b^x - 1 + c^x - 1}{x}$$

$$= \lim_{x\to 0} \left[\left(\frac{a^x - 1}{x}\right) + \left(\frac{b^x - 1}{x}\right) + \left(\frac{c^x - 1}{x}\right)\right]$$

$$= \log_e^a + \log_e^b + \log_e^c = \log_e^{abc}$$

Illustration 3 $\lim\limits_{n\to\infty} \left(\dfrac{1}{1-n^2} + \dfrac{2}{1-n^2} + \ldots + \dfrac{n}{1-n^2}\right)$

Solution

$$\lim_{n\to\infty}\left(\frac{1}{1-n^2}+\frac{2}{1-n^2}+\ldots+\frac{n}{1-n^2}\right)$$

$$=\lim_{n\to\infty}\frac{(1+2+3+\ldots+n)}{1-n^2}$$

$$=\lim_{n\to\infty}\frac{(1+2+3+\ldots+n)}{1-n^2}$$

$$=\lim_{n\to\infty}\frac{\sum n}{1-n^2}$$

$$=\lim_{n\to\infty}\frac{n[(n+1)/2]}{1-n^2}$$

$$=\frac{(1+0)}{2(0-1)}=-\frac{1}{2}\qquad n\to\infty\Rightarrow\frac{1}{n}=0$$

Illustration 4 $\lim_{x\to\infty}\left(1+\frac{9}{x}\right)^x$

Solution

$$\lim_{x\to\infty}\left(1+\frac{9}{x}\right)^x$$

$$=\lim_{x\to\infty}\left[\left(1+\frac{9}{x}\right)^{x/9}\right]^9$$

$$=e^9$$

Illustration 5 $\lim_{y\to p}\frac{\sqrt{y-q}-\sqrt{p-q}}{y^2-p^2}$

Solution

$$\lim_{y\to p}\frac{\sqrt{y-q}-\sqrt{p-q}}{y^2-p^2}$$

$$=\lim_{y\to p}\frac{\sqrt{y-q}-\sqrt{p-q}}{(y-p)(y+p)}\left(\frac{\sqrt{y-q}+\sqrt{p-q}}{\sqrt{y-q}+\sqrt{p-q}}\right)$$

$$=\lim_{y\to p}\frac{(y-p)}{(y-p)(y+p)\sqrt{y-q}+\sqrt{p-q}}$$

$$=\lim_{y\to p}\frac{1}{(y+p)\left(\sqrt{y-q}+\sqrt{p-q}\right)}$$

$$= \frac{1}{2p\left(\sqrt{p-q}+\sqrt{p-q}\right)}$$

$$= \frac{1}{4p\sqrt{p-q}}$$

Illustration 6 $\lim\limits_{y\to e} \dfrac{\log_e^y - 1}{y-e}$

Solution

$$\lim_{y\to e} \frac{\log_e^y - 1}{y-e} \qquad \text{let } y - e = x$$

$$= \lim_{y\to e} \frac{\log_e^y - \log_e^e}{y-e} \qquad \therefore y = e + x$$

$$\therefore y \to e \Rightarrow x \to 0$$

$$= \lim_{y\to e} \frac{\log_e^{(y/e)}}{y-e}$$

$$= \lim_{x\to 0} \frac{\log_e^{[(e+x)/e]}}{x}$$

$$= \lim_{x\to 0} \frac{1}{x} \log_e^{[1+(x/e)]}$$

$$= \lim_{x\to 0} \frac{1}{x} \log_e^{[1+(x/e)]^{1/x}}$$

$$= \lim_{x\to 0} \log_e^{\left\{[1+(x/e)]^{e/x}\right\}^{1/e}}$$

$$= \frac{1}{e} \log_e^{\left\{\lim\limits_{x\to 0}[1+(x/e)]^{e/x}\right\}}$$

$$= \frac{1}{e} \log_e^e = \frac{1}{e}$$

Illustration 7 $\lim\limits_{y\to 0} \dfrac{\sqrt[4]{y^4+1} - \sqrt{y^2+1}}{y^2}$

Solution

$$\lim_{y\to 0} \frac{\sqrt[4]{y^4+1} - \sqrt{y^2+1}}{y^2}$$

$$= \lim_{y\to 0} \frac{\sqrt[4]{y^4+1} - \sqrt{y^2+1}}{y^2} \left(\frac{\sqrt[4]{y^4+1} + \sqrt{y^2+1}}{\sqrt[4]{y^4+1} + \sqrt{y^2+1}}\right)$$

$$= \lim_{y\to 0} \frac{\sqrt{y^4+1} - \left(y^2+1\right)}{y^2\left(\sqrt[4]{y^4+1} + \sqrt{y^2+1}\right)}$$

$$= \lim_{y\to 0} \frac{\sqrt{y^4+1} - y^2 + 1}{x^2\left(\sqrt[4]{y^4+1} + \sqrt{y^2+1}\right)}\left(\frac{\sqrt{y^4+1} + y^2 + 1}{\sqrt{y^4+1} + y^2 + 1}\right)$$

$$= \lim_{y\to 0} \frac{\sqrt{y^4+1} - \left(y^2+1\right)}{y^2\left(\sqrt[4]{y^4+1} + \sqrt{y^2+1}\right)}$$

$$= \lim_{y\to 0} \frac{\sqrt{y^4+1} - y^2 + 1}{x^2\left(\sqrt[4]{y^4+1} + \sqrt{y^2+1}\right)}\left(\frac{\sqrt{y^4+1} + y^2 + 1}{\sqrt{y^4+1} + y^2 + 1}\right)$$

$$= \lim_{y\to 0} \frac{\left(y^4+1\right) - \left(y^2+1\right)^2}{y^2\left(\sqrt[4]{y^4+1} + \sqrt{y^2+1}\right)\left[\sqrt{y^4+1} + \left(y^2+1\right)\right]}$$

$$= \lim_{y\to 0} \frac{y^4 + 1 - y^4 - 2y^2 - 1}{y^2\left(\sqrt[4]{y^4+1} + \sqrt{y^2+1}\right)\left[\sqrt{y^4+1} + \left(y^2+1\right)\right]}$$

$$= \lim_{y\to 0} \frac{-2y^2}{y^2\left(\sqrt[4]{y^4+1} + \sqrt{y^2+1}\right)\left[\sqrt{y^4+1} + \left(y^2+1\right)\right]}$$

$$= \lim_{y\to 0} \frac{-2}{\left(\sqrt[4]{y^4+1} + \sqrt{y^2+1}\right)\left[\sqrt{y^4+1} + \left(y^2+1\right)\right]}$$

$$= \frac{-2}{(2)(2)} = -\frac{1}{2}$$

Illustration 8 $\lim\limits_{y\to 0} \frac{1}{y}\log_e^{(1+10y)}$

Solution

$$\lim_{y\to 0} \frac{1}{y}\log_e^{(1+10y)}$$

$$= \lim_{y\to 0} \log_e^{(1+10y)^{1/y}}$$

$$= \lim_{y\to 0} \log_e^{\left[(1+10y)^{1/10y}\right]^{10}}$$

$$= 10\log_e^{\left[\lim\limits_{y\to 0}(1+10y)^{1/10y}\right]}$$

$$= 10\log_e^{e} = 10$$

Illustration 9 If $f(y) = \begin{cases} \dfrac{|y|}{y}; & y \neq 0 \\ 0 & y = 0 \end{cases}$ discuss continuity at $x = 0$

Solution

Here $f\left(0\right) = 0$ (1)

$\lim\limits_{y\to 0^+} \frac{|y|}{y}$ $\lim\limits_{y\to 0^-} \frac{|y|}{y}$

let $x = 0 + h$ let $x = 0 - h$

$\therefore y \to 0^+ \Rightarrow h \to 0$ $\therefore y \to 0^- \Rightarrow h \to 0$

$\lim\limits_{h\to 0} \frac{|0+h|}{0+h}$ $\lim\limits_{h\to 0} \frac{|0-h|}{0-h}$

$\lim\limits_{h\to 0} \frac{|h|}{h} = +1$ $\lim\limits_{h\to 0} \frac{|-h|}{-h} = -1$

$\therefore \lim\limits_{y\to 0^+} f\left(y\right) \neq f\left(0\right) \neq \lim\limits_{y\to 0^-} f\left(y\right)$

$\therefore$ Function $f\left(x\right)$ is discontinuous at $y = 0$

Illustration 10 If $f\left(y\right) = \begin{cases} \frac{y^2-9}{y-3} & ; y \neq 3 \\ K & ; y = 3 \end{cases}$ the function is continuous at $y = 3$ then find K

Solution

$f\left(3\right) = k$ (1)

$\lim\limits_{y\to 3^+} \frac{y^2-9}{y-3}$ $\lim\limits_{y\to 3^-} \frac{y^2-9}{y-3}$

$= \lim\limits_{y\to 3^+} \frac{\left(y-3\right)\left(y+3\right)}{y-3}$ $= \lim\limits_{y\to 3^-} \frac{\left(y-3\right)\left(y+3\right)}{y-3}$

let $y = 3 + h$ let $y = 3 - h$

$\therefore y \to 3^+ \Rightarrow h \to 0$ $\therefore y \to 3^- \Rightarrow h \to 0$

$= \lim\limits_{h\to 0} 3 + h + 3 = 6$ $= \lim\limits_{h\to 0} 3 - h + 3 = 6$

$\therefore \lim\limits_{y\to 3^+} f\left(y\right) = f\left(3\right) = \lim\limits_{x\to 3^-} f\left(x\right)$

$6 = k = 6$

$K = 6$

Illustration 11 $\lim\limits_{y\to 0} \frac{\log_e^{\left(1+y\right)}}{y}$

Solution

$\lim\limits_{y\to 0} \frac{\log_e^{\left(1+y\right)}}{y}$

$= \lim\limits_{y\to 0} \frac{1}{y} \log_e^{\left(1+y\right)}$

$$= \lim_{y \to 0} \log_e^{(1+y)^{1/y}}$$

$$= \log_e^{\left(\lim\limits_{y \to 0}(1+y)^{1/y}\right)}$$

$$= \log_e^e \; = 1$$

Illustration 12 $\lim\limits_{y \to 0} \dfrac{3^y - 1}{\sqrt{1+y} - 1}$

Solution

$$\lim_{y \to 0} \frac{3^y - 1}{\sqrt{1+y} - 1}$$

$$= \lim_{y \to 0} \frac{3^y - 1}{\sqrt{1+y} - 1}\left(\frac{\sqrt{1+y} + 1}{\sqrt{1+y} + 1}\right)$$

$$= \lim_{y \to 0} \left(\frac{3^y - 1}{y}\right)\left(\sqrt{1+y} + 1\right)$$

$$= \left(\log_e^3\right)(2)$$

$$= 2\log_e^3 = \log_e^9$$

Illustration 13 $\lim\limits_{y \to 0} \dfrac{4^y - 2^y}{y}$

Solution

$$\lim_{y \to 0} \frac{4^y - 2^y}{y}$$

$$= \lim_{y \to 0} 2^y\left(\frac{2^y - 1}{y}\right)$$

$$= 2^0 \log_e^2 = \log_e^2$$

Illustration 14 $\lim\limits_{y \to \infty} \left(1 + \dfrac{4}{y}\right)^{y+4}$

Solution

$$\lim_{y \to \infty} \left(1 + \frac{4}{y}\right)^{y+4}$$

$$= \lim_{y \to \infty} \left(1 + \frac{4}{y}\right)^{y} \left(1 + \frac{4}{y}\right)^{4}$$

$$= \lim_{y \to \infty} \left[\left(1 + \frac{4}{y}\right)^{y/4}\right]^4 \left[\lim_{y \to \infty}\left(1 + \frac{4}{y}\right)^4\right]$$

$$= e^4 \left(1+0\right)^4 \qquad n \to \infty \; as \; \frac{1}{n} = 0$$

$$= e^4$$

Illustration 15 $\lim\limits_{y\to\infty} \dfrac{\sqrt{4y^4 - 5y^2 + 7y - 9}}{4y^2}$

Solution

$$\lim_{y\to\infty} \frac{\sqrt{4y^4 - 5y^2 + 7y - 9}}{4y^2}$$

$$= \lim_{y\to\infty} \frac{y^2\sqrt{4 - \left(5/y^2\right) + \left(7/y^3\right) - \left(9/y^4\right)}}{4y^2}$$

$$= \frac{\sqrt{4-0+0-0}}{4} \qquad \lim_{y\to\infty} as \; \frac{1}{y} = 0$$

$$= \frac{2}{4} = \frac{1}{2}$$

Illustration 16 $\lim\limits_{y\to 0} \dfrac{10^y - 5^y - 2^y + 1}{y^2}$

Solution

$$\lim_{y\to 0} \frac{10^y - 5^y - 2^y + 1}{y^2}$$

$$= \lim_{y\to 0} \frac{2^y \cdot 5^y - 5^y - 2^y + 1}{y^2}$$

$$= \lim_{y\to 0} \frac{5^y\left(2^y - 1\right) - 1\left(2^y - 1\right)}{y^2}$$

$$= \lim_{y\to 0} \left(\frac{2^y - 1}{y}\right)\left(\frac{5^y - 1}{y}\right)$$

$$= \log_e^2 \log_e^5$$

Illustration 17 $\lim\limits_{y\to 0} \left(1+3y\right)^{1/y}$

Solution

$$= \lim_{y\to 0} \left(1+3y\right)^{1/y}$$

$$= \lim_{y\to 0} \left[\left(1+3y\right)^{1/3y}\right]^3$$

$$= e^3$$

Illustration 18 $\lim_{y\to 0} \frac{e^y - e^{-y}}{y}$

Solution

$$\lim_{y\to 0} \frac{e^y - e^{-y}}{y}$$

$$= \lim_{y\to 0} \frac{e^y - \left(1/e^y\right)}{y}$$

$$= \lim_{y\to 0} \frac{e^{2y} - 1}{ye^y}$$

$$= 2\left(\lim_{y\to 0} \frac{e^{2y} - 1}{2y}\right)\left(\lim_{y\to 0} \frac{1}{e^y}\right)$$

$$= 2(1)(1)$$

$$= 2$$

Illustration 19 $\lim_{n\to\infty} \frac{1}{6} + \frac{1}{6^2} + \frac{1}{6^3} + \ldots \frac{1}{6^n}$

Solution

$$\lim_{n\to\infty} a\left(\frac{1-r^n}{1-r}\right)$$

Here, above progression is a finite G.P

$$\therefore a = \frac{1}{6}, r = \frac{1}{6}, < 1$$

$$\left[\because \lim_{n\to\infty}\left(\frac{1}{6}\right)^n = 0 \; as \; r = \frac{1}{6} < 1\right]$$

$$= \frac{(1/6)(1-0)}{(5/6)} = \frac{1}{5}$$

Illustration 20 $\lim_{x\to 0} \frac{\sqrt{1+x} - 1}{\log_e^{(1+x)}}$

Solution

$$\lim_{x\to 0} \frac{\sqrt{1+x} - 1}{\log_e^{(1+x)}}$$

$$= \lim_{x\to 0} \frac{\sqrt{1+x} - 1}{\log_e^{(1+x)}}\left(\frac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1}\right)$$

$$= \lim_{x\to 0}\left(\frac{x}{\log_e^{(1+x)}}\right)\left(\frac{1}{\sqrt{1+x} + 1}\right)$$

$$= \lim_{x\to 0} \frac{1}{(1/x)\log_e^{(1+x)}} \left(\frac{1}{\sqrt{1+x}+1}\right)$$

$$= \lim_{x\to 0} \left(\frac{1}{\log_e^{(1+x)^{1/x}}}\right)\left(\frac{1}{\sqrt{1+x}+1}\right)$$

$$= \frac{1}{\log_e^{\left[\lim\limits_{x\to 0}(1+x)^{1/x}\right]}}\left(\lim_{x\to 0}\frac{1}{\sqrt{1+x}+1}\right)$$

$$= \frac{1}{\log_e^{e}} \frac{1}{(1+1)}$$

$$= \frac{1}{2}$$

Illustration 21 $\lim\limits_{y\to 3} \dfrac{(x^2-9)}{(x-3)}$

Solution

$$\lim_{y\to 3} \frac{(y^2-9)}{(y-3)}$$

$$= \lim_{y\to 3} \frac{(y-3)(y+3)}{(y-3)}$$

$$= \lim_{y\to 3} (y+3)$$

$$= 3 + 3 = 6$$

Illustration 22 $\lim\limits_{y\to 0} \dfrac{2^y + 3^y - 2}{y}$

Solution

$$\lim_{y\to 0} \frac{2^y + 3^y - 2}{y}$$

$$= \lim_{y\to 0} \frac{2^y - 1 + 3^y - 1}{y}$$

$$= \left(\lim_{y\to 0} \frac{2^y - 1}{y}\right) + \left(\lim_{y\to 0} \frac{3^y - 1}{y}\right)$$

$$= \log_e^2 + \log_e^3$$

$$= \log_e^6$$

Illustration 23 $\lim\limits_{y\to 0} \dfrac{(1+y)^n - 1}{y}$

(a) 1 (b) n (c) $n - 1$ (d) n^{n-1}

Solution

$$\lim_{y\to 0}\frac{\left(1+y\right)^n-1}{y}$$

$$=\lim_{y\to 0}\frac{\left(1+y\right)^n-1^n}{\left(1+y\right)-1}$$

$$=n\left(1\right)^{n-1}$$

$$=n$$

Illustration 24 $\lim_{y\to 0}\frac{\sqrt{1+2y^2}-\sqrt{1-2y^2}}{y^2}$

Solution

$$\lim_{y\to 0}\frac{\sqrt{1+2y^2}-\sqrt{1-2y^2}}{y^2}$$

$$=\lim_{y\to 0}\frac{\sqrt{1+2y^2}-\sqrt{1-2y^2}}{y^2}\left(\frac{\sqrt{1+2y^2}+\sqrt{1-2y^2}}{\sqrt{1+2y^2}+\sqrt{1-2y^2}}\right)$$

$$=\lim_{y\to 0}\frac{\left(1+2y^2\right)-\left(1-2y^2\right)}{x^2\sqrt{1+2y^2}+\sqrt{1-2y^2}}$$

$$=\lim_{y\to 0}\frac{4y^2}{y^2\sqrt{1+2y^2}+\sqrt{1-2y^2}}$$

$$=\lim_{y\to 0}\frac{4}{y^2\sqrt{1+2y^2}+\sqrt{1-2y^2}}$$

$$=\frac{4}{2}=2$$

Illustration 25 $\lim_{y\to 1}\frac{1-y^{-1/7}}{1-y^{-1/3}}$

Solution

$$\lim_{y\to 1}\frac{1-y^{-1/7}}{1-y^{-1/3}}$$

$$=\lim_{y\to 1}\frac{y^{-1/7}-1}{y^{-1/3}-1}$$

$$=\lim_{y\to 1}\left[\frac{\left(y^{-1/7}-1^{-1/7}\right)/\left(y-1\right)}{\left(y^{-1/3}-1^{-1/3}\right)/\left(y-1\right)}\right]$$

$$=\frac{\left(-1/7\right)\left(1\right)^{-1/7-1}}{\left(-1/3\right)\left(1\right)^{-1/3-1}}$$

$$= \frac{(1/7)}{(1/3)} = \frac{3}{7}$$

Illustration 26 $\lim_{n\to\infty} \frac{1^3 + 2^3 + 3^3 + \ldots + n^3}{n^4}$

Solution

$$\lim_{n\to\infty} \frac{1^3 + 2^3 + 3^3 + \ldots . n^3}{n^4}$$

$$= \lim_{n\to\infty} \frac{\sum n^3}{n^4}$$

$$= \lim_{n\to\infty} \frac{\left[n^2 (n+1)^2\right]/4}{n^4}$$

$$= \lim_{n\to\infty} \frac{n^4 \left[1 + (1/n)\right]^2}{4n^4}$$

$$= \lim_{n\to\infty} \frac{\left[1 + (1/n)\right]^2}{4} \qquad \text{as } n \to \infty \Rightarrow \frac{1}{n} = 0$$

$$= \frac{1}{4}$$

Illustration 27 $\lim_{x\to\infty} (2^x - 2)(2^x + 1)^{-1}$

Solution

$$\lim_{x\to\infty} (2^x - 2)(2^x + 1)^{-1}$$

$$= \lim_{x\to\infty} \frac{(2^x - 2)}{(2^x + 1)}$$

$$= \lim_{x\to\infty} \frac{2^x \left[1 - (2/2^x)\right]}{2^x \left[1 + (1/2^x)\right]}$$

$$= \frac{1-0}{1+0} = 1 \qquad \text{as } x \to \infty \Rightarrow \frac{1}{2^x} = 0$$

Illustration 28 $\lim_{x\to\infty} 2^x (10 + x)(9 + x)^{-1} \times 2^{-x}$

Solution

$$\lim_{x\to\infty} 2^x (10+x)(9+x)^{-1} \times 2^{-x}$$

$$= \lim_{x\to\infty} \frac{2^x (10+x)}{2^x (9+x)}$$

$$= \lim_{x\to\infty} \frac{x2^x \left[(10/x)+1\right]}{x2^x \left[(9/x)+1\right]}$$

$$= 1 \text{ as } x \to \infty \Rightarrow \frac{1}{x} = 0$$

Illustration 29 $\lim\limits_{y\to\infty} y\left(\sqrt[y]{2}-1\right)$

Solution

$$\lim_{y\to\infty} y\left(\sqrt[y]{2}-1\right) \qquad \text{let } \frac{1}{y} = h$$

$$= \lim_{y\to\infty} \left(\frac{2^{1/y}-1}{1/y}\right) \qquad \therefore y \to \infty \Rightarrow h \to 0$$

$$= \lim_{h\to 0} \left(\frac{2^h - 1}{h}\right)$$

$$= \log_e^2$$

Illustration 30 $\lim\limits_{x\to 2} \dfrac{x^n - 2^n}{x-2} = 80;\ n \in N$ then find n

Solution

$$\lim_{x\to 2} \frac{x^n - 2^n}{x-2} = 80$$

$$\therefore n2^{n-1} = 80$$

$$\therefore n2^{n-1} = 5 \times 16$$

$$\therefore n2^{n-1} = 5 \times 2^4 = 5 \times 2^{5-1}$$

$$\therefore n = 5$$

ANALYTICAL EXERCISES

1. $\lim\limits_{y\to 3} 7$
2. $\lim\limits_{y\to -2} 3y + 8$

3. $\lim_{y \to 4} \sqrt{3y + 4}$

4. $\lim_{y \to 3} \dfrac{y^2 + 2y - 15}{y^2 - 9}$

5. $\lim_{y \to 2} \left(\dfrac{1}{y - 2} - \dfrac{1}{y^2 - 3y + 2} \right)$

6. $\lim_{y \to 0} \dfrac{\sqrt{a + y^2} - \sqrt{a - y^2}}{y^2}$

7. $\lim_{y \to 0} \dfrac{e^y - e^{-y}}{y}$

8. $\lim_{y \to 1} \left(\dfrac{\sqrt{3 + y} - \sqrt{5 - y}}{y^2 - 1} \right)$

9. $\lim_{y \to 0} \left(\dfrac{\sqrt{1 + y^2} - \sqrt{1 - y^2}}{y} \right)$

10. $\lim_{y \to 2} \left(\dfrac{\sqrt{3 - y} - 1}{2 - y} \right)$

11. $\lim_{y \to 1} \left(\dfrac{e^{-y} - e^{-1}}{y - 1} \right)$

12. $\lim_{y \to 0} \left(\dfrac{2^y - 1}{\sqrt{1 + y} - 1} \right)$

13. $\lim_{y \to 0} \left(\dfrac{9^y - 3^y}{4^y - 2^y} \right)$

14. $\lim_{y \to a} \dfrac{(y + 2)^{5/3} - (a + 2)^{5/3}}{y - a}$

15. $\lim_{y \to 9} \left(\dfrac{\sqrt{y} - 3}{y - 9} \right)$

16. $\lim_{y \to 0} \left(\dfrac{5^y - 1}{3^y - 1} \right)$

17. $\lim_{y \to 0} \dfrac{a^{2y} - 3a^y + 2}{2a^{2y} - a^y - 1}$

18. $\lim_{h \to 0} \dfrac{(x + h)^{-5} - x^{-5}}{h}$

19. $\lim_{y \to -1} \left(\dfrac{y^{2005} + 1}{y^{2007} + 1} \right)$

20. $\lim\limits_{y \to 0} \dfrac{\sqrt{1 + y + y^2} - 1}{y}$

21. $\lim\limits_{y \to 3} \left(\dfrac{\sqrt{y + 5} - 2\sqrt{2}}{\sqrt{y + 1} - 2} \right)$

22. $\lim\limits_{y \to 2} \dfrac{\sqrt[3]{3y + 2} - 2}{\sqrt[5]{y + 30} - 2}$

23. $\lim\limits_{y \to 3} \left(\dfrac{\sqrt{2y + 1} - \sqrt{7}}{\sqrt{3y - 1} - 2\sqrt{2}} \right)$

24. $\lim\limits_{y \to 3} \dfrac{(y - 2)^n - 1}{y - 3}$

25. $\lim\limits_{y \to 0} \dfrac{a^y - a^{2y}}{y}$

26. $\lim\limits_{y \to 2} \dfrac{\log_e^y - \log_e^2}{y - 2}$

27. $\lim\limits_{y \to 0} \left(\dfrac{1 + 2y}{1 - 2y} \right)^{1/y}$

28. $\lim\limits_{y \to 0} \dfrac{a^y + b^y - 2^{y + 1}}{y}$

29. $\lim\limits_{y \to e} \left(\dfrac{y}{e} \right)^{1/(y - e)}$

30. $\lim\limits_{y \to 3} (y - 2)^{(y - 3)^{-1}}$

31. $\lim\limits_{y \to 1} \dfrac{y^2 - y\log_e^y + \log_e^y - 1}{(y - 1)}$

32. $\lim\limits_{y \to 0} \dfrac{e^{3y} - 2e^{2y} + 2^y}{y}$

33. $\lim\limits_{y \to 0} \dfrac{e^{e^y} - 1}{e - 1}$

34. $\lim\limits_{y \to a} \dfrac{y^2 + (a + b + c)y + a(b + c)}{(y + a)}$

35. $\lim\limits_{y \to (-1)} \dfrac{\sqrt{y^3 + 1} + \sqrt{y^5 + 1}}{\sqrt{y^7 + 1}}$

36. $\lim\limits_{y \to 1} \dfrac{(y + y^2 + y^3 + \ldots + y^n) - n}{(y - 1)}$

37. $\lim\limits_{x\to 0} \dfrac{x}{\log_e^{(1+x)}}$

38. $\lim\limits_{y\to 0} \dfrac{e^y + e^{-y} - 2}{y^2}$

39. $\lim\limits_{y\to 0} \dfrac{p^y + q^y + r^y - 3}{y}$

40. $\lim\limits_{y\to 0} \left(\dfrac{1+6y^2}{1+4y^2}\right)^{1/y^2}$

41. $\lim\limits_{y\to 0} (1-4y)^{(1-y)/y}$

42. $\lim\limits_{y\to 1} \dfrac{\sqrt[n]{y} + \sqrt[n]{y^2} - 2}{y-1}$

43. $\lim\limits_{y\to 1} \dfrac{1-y^{-1/7}}{1-y^{-1/2}}$

44. $\lim\limits_{y\to 0} \dfrac{e^{ay} - e^{by}}{y}$

45. $\lim\limits_{y\to 0} \dfrac{2^{y+5} - 32}{y}$

46. $\lim\limits_{y\to 0} \dfrac{a^y - a^{-y}}{y}$

47. $\lim\limits_{y\to 1} \left(\dfrac{\log y}{1-y}\right)$

48. $\lim\limits_{y\to 0} \left(\dfrac{5^{3y} - 1}{3^{5y} - 1}\right)$

49. $\lim\limits_{y\to 0} \left(\dfrac{9^y - 3^y}{y}\right)$

50. $\lim\limits_{y\to 0} \left(\dfrac{a^y + b^y - 2}{y}\right)$

51. $\lim\limits_{y\to 0} \left(\dfrac{a^y + b^y - c^y - d^y}{y}\right)$

52. $\lim\limits_{y\to 0} \left[\dfrac{(35)^y - 5^y - 7^y + 1}{y^2}\right]$

53. $\lim\limits_{y\to 0} \left(\dfrac{a^{my} - 1}{b^{ny} - 1}\right)$

54. $\lim\limits_{y\to 0} \left(\dfrac{\log_e^{(1+y)}}{3^x - 1}\right)$

55. $\lim\limits_{y \to 0}\left(\frac{12^y - 3^y - 4^y + 1}{y^2}\right)$

56. $\lim\limits_{y \to 0}\left[\frac{\log_e^{(1+3y)}}{4^x - 1}\right]$

57. $\lim\limits_{y \to 0}\left(\frac{2^y - 1}{\sqrt{1+y} - 1}\right)$

58. $\lim\limits_{m \to \infty} P\left(1 + \frac{i}{m}\right)^{mn}$

59. $\lim\limits_{y \to \infty}\left(1 + \frac{9}{y}\right)^{y}$

60. $\lim\limits_{y \to \infty}\left(\frac{y+6}{y+1}\right)^{y+4}$

61. $\lim\limits_{y \to \infty}\left(\frac{y+1}{y+2}\right)^{2y+1}$

62. $\lim\limits_{n \to \infty}\left[\frac{1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2}{n^3}\right]$

63. $\lim\limits_{n \to \infty}\frac{1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + n(n+1)}{n^3}$

64. $\lim\limits_{y \to \infty}\left(1 + \frac{3}{y}\right)^{y+3}$

65. $\lim\limits_{y \to \infty} y\sum_{i=1}^{n}\left(e^{i/y} - 1\right)$

66. $\lim\limits_{n \to \infty}\frac{1}{n\left[1 - (m)^{1/n}\right]} =$

67. $\lim\limits_{n \to \infty}\left(1 + \frac{4}{n}\right)^{n+3}$

68. $\lim\limits_{n \to \infty}\sum_{r=1}^{n}\frac{1}{4r^2 - 1}$

69. $\lim\limits_{n \to \infty}\frac{1}{n}\sum_{r=1}^{n} 2^{r/n}$

70. $\lim\limits_{n \to \infty}\frac{(n+1) + (n+2) + \ldots + (n+n)}{n^2}$

71. $\lim\limits_{n\to\infty}\sum\limits_{2}^{n+1}\frac{1}{n^2-1}$

72. $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\frac{1}{r(r+3)}$

73. $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{r=1}^{n}5^{r/n}$

74. $\lim\limits_{n\to\infty}\frac{n\sum n^2}{\sum n^3}$

75. $\lim\limits_{y\to\infty}\left(\frac{1/y}{e^{1/y}-1}\right)$

76. $\lim\limits_{n\to\infty}\frac{10^n-3^n}{7^n-5(10)^n}$

77. $\lim\limits_{y\to 1}\frac{(y+y^2+\ldots+y^m)-m}{(y-1)}=28;\ m\in N$ then find the value of m

78. $\lim\limits_{y\to 0}\left[\frac{\log(y+a)-\log a}{y}\right]$

79. $\lim\limits_{y\to 0}\frac{\log(1+y)-\log(1-y)}{y}$

80. $\lim\limits_{y\to 0}\frac{e^{y^2}-1}{y}$

81. If $f(y)=\begin{cases}2-y\ ;\ y<2\\ 2+y \quad y\geq 2\end{cases}$ then discuss continuity of $f(y)$ at $y=2$

82. If $f(y)=\begin{cases}\frac{1}{2}-y\ ;\ 0\leq y<\frac{1}{2}\\ 1 \qquad y=\frac{1}{2}\\ \frac{3}{2}-y \quad \frac{1}{2}<y<1\end{cases}$ then $f(y)$ be continuous at $y=\frac{1}{2}$?

83. If $f(y)=\begin{cases}2y-1\ ;\ y<0\\ 2y+1 \quad y\geq 0\end{cases}$ then $f(y)$ be continuous at $y=0$?

84. $f(y)=\begin{cases}5y-4 \quad ;\ 0<y\leq 1\\ 4y^3-3y \quad 1<y\leq 2\end{cases}$ then $f(y)$ be continuous at $y=1$?

$$f(y) = \begin{cases} -y^2 & ; y \le 0 \\ 5y - 4 & ;0 < y \le 1 \\ 4y^3 - 3y & ;1 < y < 2 \\ 3y + 4 & y \ge 2 \end{cases}$$

Prove that above $f(y)$ be

85. Continuous at $y = 1$

86. Discontinuous at $y = 0$

87. Discontinuous at $y = 2$

88. If $f(y) = \begin{cases} \dfrac{y^2 - 3y + 2}{y - 1} & ; y \ne 1 \\ k & y = 1 \end{cases}$ for what value of $k (k \in R)$ becomes continuous?

89. If $f(y) = \begin{cases} a - \left(\dfrac{y^3}{a^2}\right) & ; y < a \\ a^2 - y^2 & y \ge a \end{cases}$ then $f(x)$ be continuous at $y = a$?

90. If $f(y) = \begin{cases} \dfrac{ye^{1/y}}{1 + e^{1/y}} & ; y \ne 0 \\ 0 & y = 0 \end{cases}$ then $f(y)$ be continuous at $y = 0$?

91. If $f(y) = \begin{cases} \dfrac{1}{1 + 2^{1/y}} & ; y \ne 0 \\ 0 & y = 0 \end{cases}$ then $f(y)$ be continuous at $y = 0$?

92. Prove that $f(y) = |y|$ then $f(y)$ be continuous at $y = 0$

93. If $f(y) = \begin{cases} \dfrac{1}{1 - e^{1/y}} & ; y \ne 0 \\ 1 & y = 0 \end{cases}$ then $f(y)$ be continuous at $y = 0$?

94. If $f(y) = \begin{cases} 3ay + b & ; y > 1 \\ 11 & y = 1 \\ 5ay - 2b & y < 1 \end{cases}$ If $f(x)$ is continuous at $x = 1$ then find the value of a and b

95. If $f(y) = \begin{cases} cy + 1 & ;y \le 3 \\ cy^2 - 1 & y > 3 \end{cases}$ is continuous at $y = 3$ then find the value of c

96. If $f(y) = \begin{cases} y - 3 & ;y \le 0 \\ y^2 & y > 0 \end{cases}$ then $f(y)$ be continuous at $y = 0$?

97. If $f(y) = \begin{cases} 4y + 3 & ;y \ne 4 \\ 3y + 7 & y = 4 \end{cases}$ then $f(y)$ be continuous at $y = 4$?

98. $\lim\limits_{y\to\infty} \dfrac{2y^2+5y-7}{3y^2-7y+9} = \dfrac{1}{2}$

99. $\lim\limits_{y\to\infty} \dfrac{e^y-1}{7^y-1} = \log_7^e$

100. $\lim\limits_{n\to\infty} |r|^n = 0$ then

ANSWERS

(1) 7
(2) 2
(3) 4
(4) $\frac{4}{3}$
(5) 1
(6) $\frac{1}{\sqrt{a}}$
(7) 2
(8) $\frac{1}{4}$
(9) 0
(10) $\frac{1}{2}$
(11) $\frac{-1}{e}$
(12) $\log_e^4$
(13) $\log_2^3$
(14) $\frac{5}{3}(a+2)^{2/3}$
(15) $\frac{1}{6}$
(16) $\log_3^5$
(17) $\frac{-1}{3}$
(18) $\frac{-5}{x^6}$
(19) $\frac{2005}{2007}$
(20) $\frac{1}{2}$
(21) $\frac{1}{\sqrt{2}}$
(22) 20
(23) $\frac{4\sqrt{2}}{3\sqrt{7}}$
(24) n
(25) $-\log_e^a$
(26) $\frac{1}{2}$
(27) e^4
(28) $\log_e^{ab/4}$
(29) $e^{1/e}$
(30) e
(31) 2
(32) $\log_e^2 - 1$
(33) 1
(34) $a+b+c$
(35) $\frac{\sqrt{3}+\sqrt{5}}{\sqrt{7}}$
(36) $\sum n$
(37) 1
(38) $\log_e^a \log_e^b$
(39) $\log_e^{pqr}$
(40) e^2
(41) e^{-4}
(42) $\frac{3}{n}$
(43) $\frac{2}{7}$
(44) $a-b$
(45) $32\log_e^2$
(46) $2\log_e^a$
(47) -1
(48) $\log_{243}^{125}$
(49) $\log_e^3$
(50) $\log_e^{ab}$
(51) $\log_e^{ab/cd}$
(52) $\log_e^5 \log_e^7$
(53) $\frac{m}{n}\log_b^a$

(54) $\log_3^e$

(55) $\log_e^3 \log_e^4$

(56) $3\log_4^e$

(57) $\log_e^4$

(58) pe^{in}

(59) e^{a}

(60) e^5

(61) e^{-2}

(62) $\frac{4}{3}$

(63) $\frac{1}{3}$

(64) e^3

(65) $\sum n$

(66) $-\log_m^e$

(67) e^4

(68) $-\frac{1}{2}$

(69) $\log_2^e$

(70) $\frac{3}{2}$

(71) $\frac{3}{4}$

(72) $\frac{11}{18}$

(73) $4\log_5^e$

(74) $\frac{4}{3}$

(75) 1

(76) $\frac{-1}{5}$

(77) 9

(78) $\frac{1}{a}$

(79) 2

(80) 0

(81) Discontinuous

(82) Discontinuous at $y = \frac{1}{2}$

(83) Discontinuous at $y = 0$

(84) Continuous at $y = 1$

(88) $k = -1$

(89) Continuous at $y = \mathrm{a}$

(90) Continuous at $y = 0$

(91) Discontinuous at $y = 0$

(93) Discontinuous at $y = 0$

(94) $a = 3$; $b = 2$

(95) $\frac{1}{3}$

(96) Discontinuous

(97) Continuous

(98) $\frac{1}{2}$

(99) $\log_7^e$

(100) $|r| < 1$

11

Differential Calculus

(Derivative)

LEARNING OBJECTIVES

After studying this chapter, you will be able to understand:

- Basic concept of derivative
- How to find derivative of a function by definition (first principle)
- Derivative of a function by certain direct formula and its respective application
- How to find higher order differentiation by specific algebraic method

INTRODUCTION

Differentiation is one of the most important fundamental operations in calculus. Its theory and preliminary idea is basically dependent upon the concept of limit and continuity.

Derivative is to express the rate of change in any function. Derivative means small change in the dependent variable with respect to small change in independent variable. Thus, we can say that derivative is the process of finding the derivative of a continuous function.

Derivative is a branch of calculus and its fundamentals and their applications are widely used in mathematics, statistics, economics and financial mathematics.

DEFINITION

Derivative (First Principle)

If $\lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$ exists then function $f(x)$ is said to be derivative of function f at x and it is denoted by $f'(x)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Do You Know

$y = f(x)\,(given)$

Derivative of 'y' with respect to 'x' $\dfrac{d}{dx}(y) = \dfrac{d}{dx} f(x)$

$$\therefore \frac{dy}{dx} = f'(x) = y_1 = y'$$

Standard Forms

1. $\frac{d}{dx}(\mathrm{c}) = 0$ where c = Constant
2. $\frac{d}{dx}(x^n) = nx^{n-1}$
3. $\frac{d}{dx}(x') = 1$
4. $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$
5. $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$
6. $\frac{d}{dx}(e^x) = e^x$
7. $\frac{d}{dx}(a^x) = a^x \log_e^a$
8. $\frac{d}{dx}(\log_e^x) = \frac{1}{x}$

Working Rules of Derivative

1. $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$
2. $\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$
3. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
4. $\frac{d}{dx}(uvw) = vw\frac{du}{dx} + uw\frac{dv}{dx} + uv\frac{dw}{dx}$
5. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(du/dx) - u(dv/dx)}{v^2}$ where $v \neq 0$

Chain Rules

1. If y is a function of u and u is a function of x then derivative of y with respect to x can be expressed as

   $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

2. If y is a function of u; u is a function of v and v is a function of x then derivative of y with respect to x can be expressed as

   $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$$

   Above rules (1) and (2) are said to be *chain-rules*.

Some Other Standard Forms Based on Chain Rules

1. $\frac{d}{dx}\left(y^2\right) = 2y\frac{dy}{dx} = 2yy_1$
2. $\frac{d}{dx}\left(\frac{1}{y}\right) = \left(-\frac{1}{y^2}\right)\frac{dy}{dx} = \frac{-y_1}{y^2}$
3. $\frac{d}{dx}\left(\sqrt{y}\right) = \frac{1}{2\sqrt{y}}\frac{dy}{dx} = \frac{y_1}{2\sqrt{y}}$
4. $\frac{d}{dx}\left(e^y\right) = e^y\frac{dy}{dx} = e^y y_1$
5 $\frac{d}{dx}\left(e^{\sqrt{y}}\right) = e^{\sqrt{y}}\frac{1}{2\sqrt{y}}\frac{dy}{dx} = e^{\sqrt{y}}\frac{y_1}{2\sqrt{y}}$
6. $\frac{d}{dx}a^y = a^y \log_e^a \frac{dy}{dx} = a^y \log_e^a y_1$
7. $\frac{d}{dx}\left(\log_e^y\right) = \frac{1}{y}\frac{dy}{dx} = \frac{y_1}{y}$

Relative Derivative

Suppose $y = f(t)$ and $x = g(t)$ be two non-zero constant functions then

$$\frac{dy}{dt} = \frac{d}{dt}\left[f(t)\right] = f'(t) \tag{1}$$

and $$\frac{dx}{dt} = \frac{d}{dt}\left[g(t)\right] = g'(t) \tag{2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)} \rightarrow$$

If y = (variable)$^{\text{variable}}$ then for finding $\frac{dy}{dx}$ following steps are important:

(i) Take log e both sides
(ii) Simplify the example both sides
(iii) Differentiate with respect to ‘x’ both sides
(iv) Lastly make a subject $\frac{dy}{dx}$

Second Derivative

$$y = f(x) \tag{1}$$

Differentiate with respect to ‘x’

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[f(x)\right]$$

$$\therefore \frac{dy}{dx} = f'(x) = y_1 = y' \tag{A}$$

Differentiate (A) again with respect to ‘x’

$$\therefore \frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left[f'(x)\right]=\frac{d}{dx}(y_1)=\frac{d}{dx}(y')$$

$$\therefore \frac{d^2y}{dx^2}=f''(x)=y_2=y''$$

Some Important Rules Based on Higher Derivative and Chain Rules

1. $\frac{d}{dx}(yy_1)=y\frac{d}{dx}(y_1)+y_1\frac{d}{dx}(y)$

$$=yy_2+y_1y_1$$

$$\therefore \frac{d}{dx}(yy_1)=yy_2+y_1^{\,2}$$

2. $\frac{d}{dx}(y_1^{\,2})=2y_1\frac{d}{dx}(y_1)$

$$=2y_1y_2$$

3. $\frac{d}{dx}(y^2y_1)=y^2\frac{d}{dx}(y_1)+y_1\frac{d}{dx}(y^2)$

$$=y^2y_2+y_1\left(2y\frac{d}{dx}y_1\right)$$

$$=y^2y_2+y_12y\,y_1$$

$$\therefore \frac{d}{dx}(y^2y_1)=y^2y_2+2y\,y_1^{\,2}$$

ILLUSTRATIONS

Illustration 1 If $y=\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)$ then find $\frac{dy}{dx}$

Solution

$$y=\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)$$

$$y=\left(x-\frac{1}{x}\right)$$

Differentiate with respect to ‘x’

$$\therefore \frac{dy}{dx}=\frac{d}{dx}(x)-\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$=1-\left(-\frac{1}{x^2}\right)$$

$$=1+\frac{1}{x^2}$$

Illustration 2 If $y = 6a^x - 7\log_e^x + 6x^{-2/3} + 21$ then find $\frac{dy}{dx}$

Solution

$$y = 6a^x - 7\log_e^x + 6x^{-2/3} + 21$$

Differentiate ‘y’ with respect to ‘x’

$$\frac{dy}{dx} = 6\frac{d}{dx}\left(a^x\right) - 7\frac{d}{dx}\left(\log_e^x\right) + 6\frac{d}{dx}\left(x^{-2/3}\right) + \frac{d}{dx}21$$

$$= 6a^x \log_e^a - \frac{7}{x} + 6\left(-\frac{2}{3}x^{(-2/3)-1}\right) + 0$$

$$= 6a^x \log_e^a - \frac{7}{x} - 4x^{-5/3}$$

Illustration 3 If $y = e^x x^e a^x$ then find $\frac{dy}{dx}$

Solution

$$y = e^x x^e a^x$$

Differentiate with respect to ‘x’

$$y = e^x x^e a^x$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^x x^e a^x\right)$$

$$= x^e a^x \frac{d}{dx}\left(e^x\right) + e^x a^x \frac{d}{dx}\left(x^e\right) + e^x x^e \frac{d}{dx}\left(a^x\right)$$

$$= x^e a^x e^x + e^x a^x \left(ex^{e-1}\right) + e^x x^e a^x \log_e^a$$

$$= e^x a^x \left(x^e + ex^{e-1} + x^e \log_e^a\right)$$

Illustration 4 If $y = \frac{ax + b}{cx + d}$ then find $\frac{dy}{dx}$

Solution

$$y = \frac{ax + b}{cx + d}$$

Differentiate with respect to ‘x’

$$\frac{dy}{dx} = \frac{(cx + d)(d/dx)(ax + b) - (ax + b)(d/dx)(cx + d)}{(cx + d)^2}$$

$$= \frac{(cx + d)a - (ax + b)c}{(cx + d)^2}$$

$$= \frac{acx + ad - acx - bc}{(cx + d)^2}$$

$$= \frac{ad - bc}{(cx + d)^2}$$

Illustration 5 If $y = \dfrac{e^x}{1+x^2}$ then find $\dfrac{dy}{dx}$

Solution

$$y = \frac{e^x}{1+x^2}$$

Differentiate with respect to 'x'

$$\frac{dy}{dx} = \frac{(1+x^2)(d/dx)(e^x) - e^x(d/dx)(1+x^2)}{(1+x^2)^2}$$

$$= \frac{(1+x^2)e^x - e^x(2x)}{(1+x^2)^2}$$

$$= \frac{e^x(1+x^2-2x)}{(1+x^2)^2}$$

$$= \frac{e^x(x-1)^2}{(1+x^2)^2}$$

Illustration 6 If $y = (\log_e^x)^2$ then find $\dfrac{dy}{dx}$

Solution

Here, $y = (\log_e^x)^2$

$y = u^2$ Let $u = \log_e^x$

Differentiate with respect to 'u' Differentiate with respect to 'x'

$$\frac{dy}{du} = 2u \qquad \frac{du}{dx} = \frac{1}{x} \qquad (1)$$

$$= 2\log_e^x$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (2\log_e^x)\left(\frac{1}{x}\right)$$

$$= \frac{2\log_e^x}{x}$$

Illustration 7 If $y = \log_e^{(\log_e^x)}$ then find $\dfrac{dy}{dx}$

Solution

$$y = \log_e^{(\log_e^x)}$$

$y = \log_e^{(\log_e^x)}$ Suppose

$\therefore y = \log_e^u$ $u = \log_e^x$

Differentiate y with respect to 'x' Differentiate u with respect to 'x'

$$\frac{dy}{du} = \frac{1}{u} = \frac{1}{\log_e^x} \qquad (1) \qquad \frac{du}{dx} = \frac{1}{x} \qquad (2)$$

$$\therefore \frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\frac{1}{\log_e^x}\frac{1}{x}=\frac{1}{x\log_e^x}$$

Illustration 8 If $y=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$ then find $\frac{dy}{dx}$

Solution

$$y=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$$

Differentiate with respect to 'x'

$$\frac{dy}{dx}=0+1+\frac{2x}{2\times 1!}+\frac{3x^2}{3\times 2!}+\frac{4x^3}{4\times 3!}+\ldots$$

$$=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$

$$=y$$

Illustration 9 If $x=\frac{e^\theta+e^{-\theta}}{2}$; $y=\frac{e^\theta-e^{-\theta}}{2}$ then find $\frac{dy}{dx}$

Solution

$$x=\frac{e^\theta+e^{-\theta}}{2}$$

Differentiate with respect to 'θ'

$$\frac{dx}{d\theta}=\frac{1}{2}\frac{d}{d\theta}\left(e^\theta+e^{-\theta}\right)$$

$$\frac{dx}{d\theta}=\frac{e^\theta-e^{-\theta}}{2} \qquad (1)$$

$$y=\frac{e^\theta-e^{-\theta}}{2}$$

Differentiate with respect to 'θ'

$$\frac{dy}{d\theta}=\frac{1}{2}\frac{d}{d\theta}\left(e^\theta-e^{-\theta}\right)$$

$$\frac{dy}{d\theta}=\frac{1}{2}\left(e^\theta+e^{-\theta}\right)$$

$$\frac{dy}{d\theta}=\frac{e^\theta+e^{-\theta}}{2} \qquad (2)$$

$$\therefore \frac{dy}{dx}=\frac{(dy/d\theta)}{(dx/d\theta)}=\frac{\left(e^\theta+e^{-\theta}\right)/2}{\left(e^\theta-e^{-\theta}\right)/2}=\frac{e^\theta+e^{-\theta}}{e^\theta-e^{-\theta}}$$

Illustration 10 If $xy=x+y$ then find $\frac{dy}{dx}$

Solution

$$xy=x+y$$

$$\therefore xy-y=x$$

$$\therefore y(x-1)=x$$

$$y=\frac{x}{x-1}$$

Differentiate 'y' with respect to 'x'

$$\frac{dy}{dx}=\frac{(x-1)(d/dx)(x)-x(d/dx)(x-1)}{(x-1)^2}$$

$$= \frac{(x-1)(1) - x(1-0)}{(x-1)^2}$$

$$= \frac{x-1-x}{(x-1)^2}$$

$$= \frac{-1}{(x-1)^2}$$

Illustration 11 If $\sqrt{x} + \sqrt{y} = \sqrt{a}$ then find $\frac{dy}{dx}$

Solution

$\sqrt{x} + \sqrt{y} = \sqrt{a}$

Differentiate with respect to 'x'

$$\frac{d}{dx}\left(\sqrt{x}\right) + \frac{d}{dx}\left(\sqrt{y}\right) = \frac{d}{dx}\left(\sqrt{a}\right)$$

$$\therefore \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\therefore \frac{1}{2\sqrt{y}}\frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$$

$$\therefore \frac{dy}{dx} = -\sqrt{\frac{y}{x}}$$

Illustration 12 If $y = x^x$ then find $\frac{dy}{dx}$

Solution

$y = x^x$

Taking log on both sides

$$\log_e^y = \log_e^{x^x}$$

$$\therefore \log_e^y = x\log_e^x$$

Differentiate with respect to 'x'

$$\frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}\left(\log_e^x\right) + \log_e^x \frac{d}{dx}(x)$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \log_e^x(1)$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = 1 + \log_e^x$$

$$\therefore \frac{dy}{dx} = y\left(1 + \log_e^x\right)$$

$$\therefore \frac{dy}{dx} = x^x\left(1 + \log_e^x\right)$$

Illustration 13 If $y=\left(\sqrt{x}\right)^{x}$ then find $\dfrac{dy}{dx}$

Solution

$$y=\left(\sqrt{x}\right)^{x}$$

Taking log on both sides

$$\log_e^y=\log_e^{\left(\sqrt{x}\right)^x}$$

$$\therefore \log_e^y=x\log_e^{\left(\sqrt{x}\right)}$$

Differentiate with respect to 'x'

$$\therefore \frac{1}{y}\frac{dy}{dx}=x\frac{1}{\sqrt{x}}\frac{1}{2\sqrt{x}}+\log\sqrt{x}\left(1\right)$$

$$=\frac{1}{2}+\log\sqrt{x}$$

$$\therefore \frac{dy}{dx}=y\left(\frac{1}{2}+\log\sqrt{x}\right)$$

$$=\sqrt{x}^{x}\left(\frac{1}{2}+\log\sqrt{x}\right)$$

Illustration 14 If x^x with respect to $\log\left(\log x\right)$ then find $\dfrac{dy}{dx}$

Solution

Let $u=x^x$

Taking log on both sides

$$\therefore \log u=\log^{\left(x^x\right)}$$

$$\therefore \log u=x\log x$$

Differentiate with respect to 'x'

$$\frac{1}{u}\frac{du}{dx}=x\frac{d}{dx}\log x+\log x\frac{d}{dx}\left(x\right)$$

$$\therefore \frac{1}{u}\frac{du}{dx}=x\left(\frac{1}{x}\right)+\log x\left(1\right)$$

$$\therefore \frac{du}{dx}=u\left(1+\log x\right)$$

$$=x^x\left(1+\log_e^x\right)$$

$$\therefore \frac{du}{dv}=\frac{\left(du/dx\right)}{\left(dv/dx\right)}=\frac{x^x\left(1+\log x\right)}{1/\left(x\log_e^x\right)}$$

$$\frac{du}{dv}=xx^x\log_e^x\left(1+\log_e^x\right)$$

$$=x^{x+1}\log_e^x\left(1+\log_e^x\right)$$

$$v=\log\left(\log_e^x\right)$$

$$v=\log m$$

$$m=\log_e^x$$

Differentiate with respect to 'm'

$$\frac{dm}{dx}=\frac{1}{x}$$

$$\frac{dv}{dm}=\frac{1}{m}=\frac{1}{\log_e^x}$$

$$\therefore \frac{dv}{dx}=\frac{dv}{dm}\frac{dm}{dx}$$

$$=\frac{1}{x\log_e^x}$$

Illustration 15 If 3^x with respect to e^{7x} then find $\frac{dy}{dx}$

Solution

Suppose	Suppose
$u = 3^x$	$v = e^{7x}$
Differentiate u with respect to 'x'	Differentiate v with respect to 'x'
$\therefore \frac{du}{dx} = 3^x \log_e^3$	$\therefore \frac{dv}{dx} = 7e^{7x}$

$$\therefore \frac{du}{dv} = \frac{(du/dx)}{(dv/dx)} = \frac{3^x \log_e^3}{7e^{7x}}$$

Illustration 16 If $e^{xy} - 4xy = 4$ then find $\frac{dy}{dx}$

Solution

$e^{xy} - 4xy = 4$

Differentiate with respect to 'x'

$$\therefore \frac{d}{dx}e^{xy} - 4\frac{d}{dx}(xy) = \frac{d}{dx}(4)$$

$$\therefore e^{xy}\frac{d}{dx}(xy) - 4\left[x\frac{dy}{dx} + y(1)\right] = 0$$

$$\therefore e^{xy}\left[x\frac{dy}{dx} + y(1)\right] - 4\left(x\frac{dy}{dx} + y\right) = 0$$

$$\therefore xe^{xy}\frac{dy}{dx} + ye^{xy} - 4x\frac{dy}{dx} - 4y = 0$$

$$\therefore x(e^{xy} - 4)\frac{dy}{dx} = (4y - ye^{xy})$$

$$\therefore x(e^{xy} - 4)\frac{dy}{dx} = -y(e^{xy} - 4)$$

$$\therefore \frac{dy}{dx} = \frac{-y}{x}$$

Illustration 17 If $y = e^x + e^{-x}$ then $\frac{dy}{dx}$

Solution

$y = e^x + e^{-x}$

Differentiate 'y' with respect to 'x'

$$\frac{dy}{dx} = e^x - e^{-x}$$

$$\therefore \frac{dy}{dx} = \sqrt{(e^x + e^{-x})^2 - 4e^x e^{-x}} \qquad \left(\because \alpha - \beta = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}\right)$$

$$= \sqrt{y^2 - 4}$$

Illustration 18 If $y = x^{x^{\cdot}}$ then find $\dfrac{dy}{dx}$

Solution

$$y = x^{x^{\cdot}}$$

$$\therefore y = x^y$$

$$\therefore \log_e^y = \log x^y$$

$$\therefore \log_e^y = y \log_e^x$$

Differentiate with respect to ‘x’

$$\frac{1}{y}\frac{dy}{dx} = \frac{y}{x} + \log x \frac{dy}{dx}$$

$$\therefore \left(\frac{1}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x}$$

$$\left(\frac{1 - y\log x}{y}\right)\frac{dy}{dx} = \frac{y}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y^2}{x(1 - y\log x)}$$

Illustration 19 If $y = \sqrt{x + \sqrt{x + \sqrt{x + \ldots\infty}}}$ then prove that $\dfrac{dy}{dx}$

Solution

$$y = \sqrt{x + \sqrt{x + \sqrt{x + \ldots\infty}}}$$

$$\therefore y = \sqrt{x + y}$$

$$\therefore y^2 = x + y$$

Differentiate with respect to ‘x’

$$2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$(2y - 1)\frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{2y - 1}$$

Illustration 20 If $y = \log\left(\dfrac{x^2}{e^x}\right)$ then prove that $\dfrac{d^2y}{dx^2} = -\dfrac{z}{xz}$

Solution

$$y = \log\left(\frac{x^2}{e^x}\right)$$

$$= \log_e^{x^2} - \log_e^{e^x}$$

$$= 2\log_e^x - x\log_e^e$$

$y = 2\log_e^x - x$

Differentiate ‘*y*’ with respect to ‘*x*’

$$\frac{dy}{dx} = \frac{2}{x} - 1$$

Differentiate again with respect to ‘*x*’

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 2\frac{d}{dx}\left(\frac{1}{x}\right) - \frac{d}{dx}(1)$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{2}{x^2} - 0$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{2}{x^2}$$

Illustration 21 If $y = ae^{mx} + be^{-mx}$ then prove that $\frac{d^2y}{dx^2} = m^2y$

Solution

$y = ae^{mx} + be^{-mx}$ Differentiate ‘*y*’ with respect to ‘*x*’

$$\frac{dy}{dx} = ame^{mx} - bme^{-mx}$$

Differentiate again with respect to ‘*x*’

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = am\frac{d}{dx}\left(e^{mx}\right) - bm\frac{d}{dx}\left(e^{-mx}\right)$$

$$= am^2e^{mx} + bm^2e^{-mx}$$

$$\therefore \frac{d^2y}{dx^2} = m^2\left(ae^{mx} + be^{-mx}\right)$$

$$\therefore \frac{d^2y}{dx^2} = m^2y$$

Illustration 22 If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \ldots\infty}}}$ then prove that

$$\frac{dy}{dx} = \frac{1}{x(2y-1)}$$

Solution

$$y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \ldots\infty}}}$$

$$\therefore y = \sqrt{\log x + y}$$

$$\therefore y^2 = \log x + y$$

Differentiate with respect to ‘*x*’

$$2y\frac{dy}{dx} = \frac{1}{x} + \frac{dy}{dx}$$

$$\therefore (2y-1)\frac{dy}{dx} = \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{1}{x(2y-1)}$$

Illustration 23 $y = \sqrt{e^x + \sqrt{e^x + \sqrt{e^x + \ldots\infty}}}$ then prove that $\dfrac{dy}{dx} = \dfrac{e^x}{2y-1}$

Solution

$$y = \sqrt{e^x + \sqrt{e^x + \sqrt{e^x + \ldots\infty}}}$$

$$\therefore y = \sqrt{e^x + y}$$

$$\therefore y^2 = e^x + y$$

Differentiate with respect to '*x*'

$$2y\frac{dy}{dx} = e^x + \frac{dy}{dx}$$

$$\therefore (2y-1)\frac{dy}{dx} = e^x$$

$$\therefore \frac{dy}{dx} = \frac{e^x}{2y-1}$$

Illustration 24 If $y = e^{x + e^{x + \ldots\infty}}$ then prove that $\dfrac{dy}{dx} = \dfrac{y}{1-y}$

Solution

$$y = e^{x + e^{x + \ldots\infty}}$$

$$\therefore y = e^{x+y}$$

$$\therefore \log_e^y = \log_e^{e^{x+y}}$$

$$\therefore \log_e^y = x + y\log_e^e$$

$$\therefore \log_e^y = x + y$$

Differentiate with respect to '*x*'

$$\frac{1}{y}\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\therefore \left(\frac{1}{y} - 1\right)\frac{dy}{dx} = 1$$

$$\therefore \left(\frac{1-y}{y}\right)\frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{y}{1-y}$$

Illustration 25 If $y = \log\left(x + \sqrt{1+x^2}\right)$ then prove that $\left(1+x^2\right)y_2 + xy_1 = 0$

Solution

$$y = \log\left(x + \sqrt{1+x^2}\right)$$

Differentiate with respect to '*x*'

$$y_1 = \frac{1}{x + \sqrt{1+x^2}}\frac{d}{dx}\left(x + \sqrt{1+x^2}\right)$$

$$= \frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{2x}{2\sqrt{1+x^2}}\right)$$

$$y_1 = \frac{1}{x+\sqrt{1+x^2}}\left(\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\right)$$

$$\therefore y_1 = \frac{1}{\sqrt{1+x^2}}$$

$$\therefore \sqrt{1+x^2}\,y_1 = 1$$

$$\therefore \left(1+x^2\right)y_1^{\,2} = 1 \qquad \text{(Squaring both sides)}$$

Differentiate again with respect to ‘*x*’

$$\frac{d}{dx}\left(1+x^2\right)y_1^{\,2} = \frac{d}{dx}(1)$$

$$\left(1+x^2\right)\frac{d}{dx}\left(y_1^{\,2}\right) + y_1^{\,2}\frac{d}{dx}\left(1+x^2\right) = 0$$

$$\left(1+x^2\right)2y_1y_2 + y_1^{\,2}(2x) = 0$$

$$\therefore \left(1+x^2\right)y_2 + xy_1 = 0$$

Illustration 26 If $y = \left(x+\sqrt{x^2+1}\right)^m$ then prove that $\left(1+x^2\right)y_2 + xy_1 = m^2y$

Solution

$$y = \left(x+\sqrt{x^2+1}\right)^m$$

Differentiate ‘*y*’ with respect to ‘*x*’

$$\therefore y_1 = m\left(x+\sqrt{x^2+1}\right)^{m-1}\frac{d}{dx}\left(x+\sqrt{x^2+1}\right)$$

$$\therefore y_1 = m\left(x+\sqrt{x^2+1}\right)^{m-1}\left(1+\frac{2x}{2\sqrt{x^2+1}}\right)$$

$$y_1 = \frac{m\left(x+\sqrt{x^2+1}\right)^{m-1}\left(x+\sqrt{x^2+1}\right)}{\sqrt{x^2+1}}$$

$$\therefore y_1 = \frac{m\left(x+\sqrt{x^2+1}\right)^{m}}{\sqrt{x^2+1}}$$

$$\therefore \sqrt{x^2+1}\,y_1 = my$$

$$\therefore \left(x^2+1\right)y_1^{\,2} = m^2y^2 \left(\because \text{squaring}\right)$$

Differentiate again with respect to ‘*x*’

$$\left(x^2+1\right)\frac{d}{dx}\left(y_1^{\,2}\right) + y_1^{\,2}\frac{d}{dx}\left(x^2+1\right) = m^2\frac{d}{dx}\left(y^2\right)$$

$$\therefore \left(x^2+1\right)2y_1y_2+2xy_1{}^2=m^2 2yy_1$$

$$\therefore \left(x^2+1\right)y_2+xy_1=m^2y$$

Illustration 27 If $2x=y^{1/m}+y^{-1/m}$ then prove that $\left(x^2-1\right)y_2+xy_1 = m^2y$

Solution

$$2x=y^{1/m}+y^{-1/m}$$

Differentiate with respect to ‘x’

$$\therefore 2=\frac{1}{m}y^{1/m-1}y_1-\frac{1}{m}y^{-1/m-1}y_1$$

$$=\frac{1}{m}\frac{y^{1/m}}{y}y_1-\frac{1}{m}\frac{y^{-1/m}}{y}y_1$$

$$2=\left(y^{1/m}-y^{-1/m}\right)y_1$$

$$\therefore 2my=\left[\sqrt{\left(y^{1/m}+y^{-1/m}\right)-4y^{1/m}y^{-1/m}}\right]y_1$$

$$\left[\because \alpha-\beta=\sqrt{(\alpha+\beta)^2-4\alpha\beta}\right]$$

$$\therefore 2my=\sqrt{4x^2-4}y_1$$

$$\therefore 2my=2\sqrt{x^2-1}y_1$$

$$\therefore \sqrt{x^2-1}\,y_1=my$$

$$\therefore \left(x^2-1\right)y_1{}^2=m^2y^2$$

Differentiate again with respect to ‘x’

$$\therefore \left(x^2-1\right)\frac{d}{dx}\left(y_1{}^2\right)+y_1{}^2\frac{d}{dx}\left(x^2-1\right)=m^2\frac{d}{dx}\left(y^2\right)$$

$$\therefore \left(x^2-1\right)2y_1y_2+2xy_1{}^2=m^2\left(2yy_1\right)$$

$$\therefore \left(x^2-1\right)y_2+xy_1=m^2y$$

Illustration 28 If $y=\log_7^x$ then find $\frac{dy}{dx}$

Solution

$$y=\log_7^x=\frac{\log_e^x}{\log_e^7}$$

Differentiate ‘y’ with respect to ‘x’

$$\frac{dy}{dx}=\frac{1}{\log_e^7}\frac{d}{dx}\log_e^x$$

$$\frac{dy}{dx}=\frac{1}{x\log_e^7}$$

Illustration 29 If $y = \log_e^{\sqrt{(1-x^2)/(1+x^2)}}$ then find $\frac{dy}{dx}$

Solution

$y = \log_e^{\sqrt{(1-x^2)/(1+x^2)}}$

Differentiate with respect to '*x*'

$$\frac{dy}{dx} = \frac{d}{dx}\left(\log_e^{\sqrt{(1-x^2)/(1+x^2)}}\right)$$

$$= \frac{1}{\sqrt{(1-x^2)/(1+x^2)}}\frac{d}{dx}\left(\sqrt{\frac{1-x^2}{1+x^2}}\right)$$

$$= \frac{1}{\sqrt{(1-x^2)/(1+x^2)}}\frac{1}{2\sqrt{(1-x^2)/(1+x^2)}}\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

$$= \frac{1}{2\left[(1-x^2)/(1+x^2)\right]}\left[\frac{(1+x^2)(d/dx)(1-x^2)-(1-x^2)(d/dx)(1+x^2)}{(1+x^2)^2}\right]$$

$$= \frac{(1+x^2)}{2(1-x^2)}\left[\frac{(1+x^2)(-2x)-(1-x^2)(2x)}{(1+x^2)^2}\right]$$

$$= \frac{2x(-1-x^2-1+x^2)}{2(1-x^4)}$$

$$= \frac{x(-2)}{(1-x^4)}$$

$$\therefore \frac{dy}{dx} = \frac{-2x}{1-x^4} = \frac{2x}{x^4-1}$$

Illustration 30 If $y = \frac{e^x}{a^x}$ then find $\frac{dy}{dx}$

Solution

$y = \frac{e^x}{a^x}$

Differentiate '*y*' with respect to '*x*'

$$\frac{dy}{dx} = \frac{a^x(d/dx)(e^x)-e^x(d/dx)(a^x)}{(a^x)^2}$$

$$= \frac{a^x e^x - e^x a^x \log_e^a}{\left(a^x\right)^2}$$

$$= \frac{e^x\left(1 - \log_e^a\right)}{a^x}$$

ANALYTICAL EXERCISES

1. Find $\frac{d}{dx}\left(e^0\right) =$

2. Find $\frac{d}{dx}\left(\sqrt{x}\right) =$

3. Find $\frac{d}{dx}\left(e^{2x}\right) =$

4. Find $\frac{d}{dx}\left(a^{3x}\right) =$

5. If $y = ax^2 + bx + c$ then find $\frac{dy}{dx} =$

6. Find $\frac{d}{dx}\left(x^{-1/2}\right) =$

7. If $y = \frac{x-1}{x+1}$ then find $\frac{dy}{dx} =$

8. If $y = \frac{1-x^2}{1+x^2}$ then find

9. If $y = \sqrt{3}$ then prove $\frac{dy}{dx} = 0$

10. If $y = e^{3\log_e^x}$ then find $\frac{dy}{dx} = 3x^2$

11. If $y = \sqrt{x^2 + a^2}$ then prove that $\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + a^2}}$

12. If $y = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$ then prove that $\frac{dy}{dx} = 1 - \frac{1}{x^2}$

13. If $y = \frac{px+q}{rx+s}$ then prove that $\frac{dy}{dx} = \frac{ps - qr}{(rx+s)^2}$

14. If $y = e^{2x}\left(e^x + e^{-x}\right)$ then prove that $\frac{dy}{dx} = 3e^{3x} + e^x$

15. If $y = \sqrt{a}x^5 + \sqrt{b}\log x + 7\sqrt{c} - 42$ then show that $\frac{dy}{dx} = 5\sqrt{a}x^4 + \frac{\sqrt{b}}{x}$

16. Prove that $\frac{d}{dx}\left(e^{-x}\right) = \frac{-1}{e^x}$

17. If $y = \frac{e^x}{\log_e^x}$ then prove that $\frac{dy}{dx} = \frac{e^x\left(x\log x - 1\right)}{x\left(\log x\right)^2}$

18. If $y = e^x 3^x$ then prove that $\frac{dy}{dx} = e^x 3^x (\log x - 1)$

19. If $y = \log_{\sqrt{2}}^{x}$ then prove that $\frac{dy}{dx} = \frac{1}{x \log \sqrt{2}}$

20. If $y = x^5 \log_e^x$ then prove that $\frac{dy}{dx} = x^4 (1 + 5 \log x)$

21. If $y = \log_{10}^{2} \log_{2}^{x^2}$ then prove that $\frac{dy}{dx} = \frac{2}{x \log 10}$

22. If $y = e^{x^2}$ then show that $\frac{dy}{dx} = 2xe^{x^2}$

23. If $y = \log[\log(\log x)]$ then show that $\frac{dy}{dx} = \frac{1}{x \log x \; \log(\log x)}$

24. If $y = \sqrt{ax^2 + bx}$ then show that $\frac{dy}{dx} = \frac{2ax + b}{2\sqrt{ax^2 + bx}}$

25. If $y = x + \sqrt{x^2 + 1}$ then show that $\frac{dy}{dx} = 1 + \frac{x}{\sqrt{x^2 + 1}}$

26. If $y = \log(e^{mx} + e^{nx})$ then show that $\frac{dy}{dx} = \frac{me^{mx} + ne^{nx}}{e^{mx} + e^{nx}}$

27. If $y = \log(\log \sqrt{x})$ then show that $\frac{dy}{dx} = \frac{1}{2x \log \sqrt{x}}$

28. If $y = \sqrt{a + \sqrt{a + x}}$ then show that $\frac{dy}{dx} = \frac{1}{4\sqrt{a + x}\sqrt{a + \sqrt{a + x}}}$

29. If $y = \log \sqrt{x^2 + 1}$ then show that $\frac{dy}{dx} = \frac{x}{x^2 + 1}$

30. If $y = \sqrt{4 + \sqrt{4 + \sqrt{4 + x^2}}}$ then show that

$$\frac{dy}{dx} = \frac{x}{4\sqrt{4 + x^2}\sqrt{4 + \sqrt{4 + x^2}}\sqrt{4 + \sqrt{4 + \sqrt{4 + x^2}}}}$$

31. If $y = \log\left(x + \frac{1}{x}\right)$ then prove that $\frac{dy}{dx} = \frac{x^2 - 1}{x(x^2 + 1)}$

32. If $y = \sqrt{\frac{1 + e^x}{1 - e^x}}$ then show that $\frac{dy}{dx} = \frac{e^x}{\sqrt{1 + e^x}\,(1 - e^x)^{3/2}}$

33. If $y = 3^{x^2 + 3x}$ then prove that $\frac{dy}{dx} = (2x + 3) 3^{x^2 + 3x} \log_e^3$

34. If $y = e^x \log\left(1 + x^2\right)$ then find $\dfrac{dy}{dx} =$

35. If $y = \dfrac{1}{\sqrt[3]{6x^5 - 7x^3 + 9}}$ then prove that $\dfrac{dy}{dx} = \dfrac{-\left(10x^4 - 7x^2\right)}{\left(6x^5 - 7x^3 + 9\right)^{4/3}}$

36. If $x = \dfrac{3t}{1+t^2}$ and $y = \dfrac{3t^2}{1+t^2}$ then prove that $\dfrac{dy}{dx} = \dfrac{2t}{1-t^2}$

37. If $x = at^2$ and $y = 2at$ then show that $\dfrac{dy}{dx} = \dfrac{1}{t}$

38. If $x^2 + y^2 = a^2$ then show that $\dfrac{dy}{dx} = -\dfrac{x}{y}$

39. If $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ then prove that $\dfrac{dy}{dx} = \dfrac{-xb^2}{a^2 y}$

40. If $x^{2/3} + y^{2/3} = a^{2/3}$ then prove that $\dfrac{dy}{dx} = -\sqrt[3]{\dfrac{y}{x}}$

41. If $e^x + e^y = e^{x+y}$ then show that $\dfrac{dy}{dx} = \dfrac{e^x\left(e^y - 1\right)}{e^y\left(1 - e^x\right)}$

42. If $2^x + 2^y = 2^{x+y}$ then show that $\dfrac{dy}{dx} = \dfrac{2^x\left(2^y - 1\right)}{2^y\left(1 - 2^x\right)}$

43. If $x^y = y^x$ then show that $\dfrac{dy}{dx} = \dfrac{y}{x}\left(\dfrac{y - x\log y}{x - y\log x}\right)$

44. If $y = x^{\log x}$ then show that $\dfrac{dy}{dx} = y\left(\dfrac{2\log x}{x}\right)$

45. If $y = \left(\log x\right)^2$ then prove that $\dfrac{dy}{dx} = y\left[\dfrac{1}{\log x} + \log\left(\log x\right)\right]$

46. If $y = x^{1/x}$ then prove that $\dfrac{dy}{dx} = \dfrac{y}{x^2}\left(1 - \log x\right)$

47. If $y = \left(\dfrac{1}{x}\right)^x$ then prove that $\dfrac{dy}{dx} = \dfrac{y}{x^2}\left(1 - \log x\right)$

48. If $y = \left(e^x\right)^{\log x}$ then prove that $\dfrac{dy}{dx} = y\left(1 + \log x\right)$

49. If $y = \left(\sqrt{x}\right)^x$ then prove that $\dfrac{dy}{dx} = y\left(\dfrac{1}{2} + \log\sqrt{x}\right)$

50. Find $\dfrac{dy}{dx}$ of x^2 with respect to e^x

51. Find $\dfrac{dy}{dx}$ of x^x with respect to e^x

52. If $x^y = e^{x-y}$ then prove that $\dfrac{dy}{dx} = \dfrac{\log x}{\left(1 + \log xe\right)^2}$

53. If $x\sqrt{1+y} + y\sqrt{1-x} = 0$ then prove that $\frac{dy}{dx} = \frac{-1}{(1+x)^2}$

54. If $\left(\frac{x}{x-y}\right) = \log\left(\frac{a}{x-y}\right)$ then prove that $\frac{dy}{dx} = 2 - \left(\frac{x}{y}\right)$

55. If $x\sqrt{1-y^2} + y\sqrt{1-x^2} = 191$ then prove that $\frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}$

56. If $y = \sqrt{x + \sqrt{x + \sqrt{x + \ldots\infty}}}$ then prove that $\frac{dy}{dx} = \frac{1}{2y-1}$

57. If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \ldots\infty}}}$ then prove that $\frac{dy}{dx} = \frac{1}{x(2y-1)}$

58. If $y = \sqrt[3]{\log x + \sqrt[3]{\log x + \sqrt[3]{\log x + \ldots\infty}}}$ then prove that $\frac{dy}{dx} = \frac{1}{x(3y^2-1)}$

59. If $y = ae^{3x} + be^{2x}$ then prove that $y_2 - 5y_1 = -6y$

60. If $y = x\log\left(\frac{x}{a+bx}\right)$ then prove that $x^3y_2 = (y - xy_1)^2$

61. If $y = \log\left(x + \sqrt{x^2 + a^2}\right)$ then prove that $(x^2 + a^2)y_2 + xy_1 = 0$

62. If $2x = y^{1/4} + y^{-1/4}$ then prove that $(x^2 - 1)y_2 + xy_1 + -16y = 0$

63. If $y = ax^{n+1} + bx^{-n}$ then prove that $x^2y_2 = n(n+1)y$

64. If $y^2 = x^2 + x + 1$ then prove that $4y^3y_2 = 3$

65. If $y = e^{\sqrt{x}} + e^{-\sqrt{x}}$ then prove that $xy_2 + \frac{1}{2}y_1 + \frac{-y}{4} = 0$

66. If $y = 500e^{7x} + 600e^{-7x}$ then prove that $y_2 = 49y$

67. If $y = Ae^{mx} + Be^{nx}$ then prove that $y_2 - (m+n)y_1 + mny = 0$

68. If $y = x^y$ then prove that $\frac{dy}{dx} = \frac{y^2}{x(1 - y\log x)}$

69. If $x^y y^x = k$ then prove that $\frac{dy}{dx} = -\frac{y}{x}\left(\frac{y + x\log y}{x + y\log x}\right)$

70. If $2x = y^{1/m} + y^{-1/m}$ then prove that $(x^2 - 1)\left(\frac{dy}{dx}\right)^2 = m^2y^2$

71. If $x^m y^m = (x+y)^{m+n}$ then prove that the value of $\frac{dy}{dx}$ is independent of m and n

72. If $y = \frac{x^2}{e^x}$ then show that $\frac{dy}{dx} = \frac{x(2-x)}{e^x}$

73. If $x^2y^2 + 3xy + y = 0$ then show that $\frac{dy}{dx} = \frac{-(2xy^2 + 3y)}{(2x^2y + 3x + 1)}$

74. If $y = \sqrt{\frac{1-x}{1+x}}$ then prove that $(1 - x^2)y_1 + y = 0$

75. $y = \log\left(x + \sqrt{x^2 + b^2}\right)$ then prove that $\frac{dy}{dx} = \frac{-1}{\sqrt{x^2 + a^2}}$

76. $y = \log\left(\sqrt{x - m} + \sqrt{x - n}\right)$ then prove that $\frac{dy}{dx} = \frac{1}{2\sqrt{x - m}\sqrt{x - n}}$

77. If $x^m y^n = (x + y)^{m + n}$ then prove that $\frac{dy}{dx} = \frac{y}{x}$

78. If $x^3 + y^3 - 3axy = 0$ then prove that $\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$

79. If $y = \log\left[e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right]$ then prove that $\frac{dy}{dx} = \frac{x^2 - 1}{x^2 - 4}$

80. If $y = e^{ax^2 + bx + c}$ then prove that $\frac{dy}{dx} = (2ax + b)e^{ax^2 + bx + c}$

81. If $y = \left(x + \sqrt{x^2 + a^2}\right)^m$ then prove that $\frac{dy}{dx} = \frac{my}{\sqrt{x^2 + a^2}}$

82. If $y = \sqrt{x^2 + b^2}$ then prove that $\frac{dy}{dx} = \frac{x}{2\sqrt{x^2 + a^2}}$

83. If $y = e^x + e^{-x}$ then prove that $y_1 - \sqrt{y^2 - 4} = 0$

84. If $y = \log^{5x}$ then prove that $\frac{dy}{dx} = \frac{1}{x}$

85. If $y = -3x^{-7/3}$ then prove that $\frac{dy}{dx} = 21x^{-10/3}$

86. If $y = x + \frac{1}{x}$ then prove that $\frac{dy}{dx} = 1 - \frac{1}{x^2}$

87. If $y = \left(x + \frac{1}{x^2}\right)^2$ then prove that $\frac{dy}{dx} = 2\left(x - \frac{1}{x^3}\right)$

88. If $y = \log^{\sqrt{x}}$ then prove that $\frac{dy}{dx} = \frac{1}{x}$

89. If $y = (\sqrt{x} + 1)(\sqrt{x} - 1)$ then prove that $\frac{dy}{dx} = 1$

90. $y = x + 7x^{-1}$ then prove that $\frac{dy}{dx} = 1 - \frac{7}{x^2}$

91. If $y = 7x^4 + 3x^3 - 9x - 5$ then prove that $\frac{dy}{dx} = 28x^3 + 9(x^2 - 1)$

92. If $y = \frac{1}{\sqrt[3]{x}}$ then prove that $\frac{dy}{dx} = \frac{-1}{3x^{4/3}}$

93. If $y = \log^{x^n}_{a^n}$ then prove that $\frac{dy}{dx} = \frac{1}{x \log a}$

94. If $y = x(x - 2)(x^2 + 1)$ then prove that $\frac{dy}{dx} = 4x^3 - 6x^2 + 2x - 2$

95. $y = \log(\log\sqrt{x})$ then prove that $\frac{dy}{dx} = \frac{1}{2x \log\sqrt{x}}$

96. If $y = \sqrt{x}^{\sqrt{x}}$ then prove that $\frac{dy}{dx} = \frac{y}{2\sqrt{x}}(1 + \log\sqrt{x})$

97. If $y = a^{\sqrt{x}}$ then prove that $\frac{dy}{dx} = \frac{a^{\sqrt{x}} \log^a_e}{2\sqrt{x}}$

98. $y = 5\sqrt{a} + 7\sqrt{b} - c$ then prove that $\frac{dy}{dx} = 0$

ANSWERS

(1) 0

(2) $\frac{1}{2\sqrt{x}}$

(3) $2e^{2x}$

(4) $3a^{3x} \cdot \log^a_e$

(5) $2ax + b$

(6) $-\frac{1}{2x^{3/2}}$

(7) $\frac{2}{(x + 1)^2}$

(8) $\frac{-4x}{(1 + x^2)^2}$

(9) 0

(10) $3x^2$

(34) $e^x\left(\frac{2x}{1 + x^2} + \log(1 + x^2)\right)$

(50) $\frac{2x}{e^x}$

(51) $\frac{x^x(1 + \log x)}{e^x}$

12

Integral Calculus

(Indefinite and Definite Integration)

LEARNING OBJECTIVES

After studying this chapter, you will be able to understand:

- Basic concepts of indefinite and definite integration
- How to find indefinite and definite integration by using direct formulae
- Certain applied formulae of indefinite and definite integration

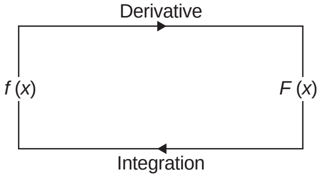
INTRODUCTION

Integration is one of the most important fundamental operations in calculus. Its theory and ideas are basically dependent upon the concept of derivative.

Applications of Integration are widely used in mathematics, statistics and mathematical economics.

CONCEPT OF INTEGRATION

Anti-derivative is called integration.



DEFINITION

$$\frac{d}{dx}\left[f(x)\right] = F(x) \qquad (1)$$

Integrate (1) with respect to '*x*'

$$\int\left[\frac{d}{dx}f(x)\right]dx = \int F(x)\,dx$$

$$\therefore\ f(x)\int F(x)dx + c$$

where c is integral constant.

SOME STANDARD FORMS

S. No.	**Integral Form**	**Integration**
(1)	$\int x^n dx =$	$\frac{x^{n+1}}{n+1} + c; n \neq -1 \in Q$
(2)	$\int 1 dx =$	$x + c$
(3)	$\int \frac{1}{x} dx =$	$\log\lvert x\rvert + c$
(4)	$\int e^x dx =$	$e^x + c$
(5)	$\int a^x dx =$	$\frac{a^x}{\log_e^a} + c$
(6)	$\int cf(x) =$	$c\int f(x)\,dx + c_1$
	Method of Substitution	
(7)	$\int f(x)\,dx =$	$\int f\left[g(t)\right]g'(t)dt$
(8)	$\int (ax+b)^n\,dx =$	$\frac{(ax+b)^{n+1}}{a(n+1)} + c$
(9)	$\int f(ax+b)dx =$	$\frac{1}{a}F(ax+b) + c$
(10)	$\int (f(x))^n\, f'(x)\,dx =$	$\frac{\left[f(x)\right]^{n+1}}{(n+1)} + c$
(11)	$\int \frac{f'(x)}{f(x)} dx =$	$\log\lvert f(x)\rvert + c$
(12)	$\int \frac{dx}{x^2 - a^2} =$	$\frac{1}{2a}\log\left\lvert\frac{x-a}{x+a}\right\rvert + c$
(13)	$\int \frac{dx}{a^2 - x^2} =$	$\frac{1}{2a}\log\left\lvert\frac{a+x}{a-x}\right\rvert + c$
(14)	$\int \frac{dx}{\sqrt{x^2 \pm a^2}} =$	$\log\left\lvert x + \sqrt{x^2 \pm a^2}\right\rvert + c$
(15)	$\int \sqrt{x^2 + a^2}dx =$	$\frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left\lvert x+\sqrt{x^2+a^2}\right\rvert + c$
(16)	$\int \sqrt{x^2 - a^2}dx =$	$\frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left\lvert x+\sqrt{x^2-a^2}\right\rvert + c$
(17)	$\int \left[f(x) + f^1(x)\right]e^x dx$	$f(x)\cdot e^x + C$

Important Applications of Integration

(1) *Integration by Parts*

$$\int uvdx = u\int vdx - \int\left(\frac{du}{dx}\int vdx\right)dx$$

where u and v functions depend upon following concepts:

L I A T E

where L = Logarithm function
I = Inverse function (Out of course)
A = Arithmetic function
T = Trigonometric function (Out of course)
E = Exponential function

Note: As per given example any function is a first latter "LIATE" that you consider as a function u and consider the remaining function as a v.

(2) *Method of Partial Fraction or Method of Linear Factors*

$$\int\frac{dx}{(x-\alpha)(x-\beta)(x-\gamma)}$$

(A) Method of calculation

$$\frac{1}{(x-\alpha)(x-\beta)(x-\gamma)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{x-\gamma} \qquad (1)$$

where α, β, γ are constants

$$\therefore \frac{1}{(x-\alpha)(x-\beta)(x-\gamma)} = \frac{A(x-\beta)(x-\gamma) + B(x-\alpha)(x-\gamma) + c(x-\alpha)(x-\beta)}{(x-\alpha)(x-\beta)(x-\gamma)}$$

$$1 = A(x-\beta)(x-\gamma) + B(x-\alpha)(x-\gamma) + C(x-\alpha)(x-\beta)$$

Now substitute the value of $(x \in R)$ so that remaining coefficient becomes constants to find A, B, C respectively then substitute value of A, B, C in (1) then Integrate (1) with respect to 'x' to both sides.

$$\int\frac{dx}{(x-\alpha)^2(x-\beta)} \qquad \left[\because \text{where } (x-\alpha) \text{ and } (x-\beta) \text{ are linear factors.}\right]$$

(B) Method of calculation

$$\frac{1}{(x-\alpha)^2(x-\beta)} = \frac{A}{(x-\alpha)} + \frac{B}{(x-\alpha)^2} + \frac{C}{(x-\beta)} \qquad (1)$$

$$\therefore \frac{1}{(x-\alpha)^2(x-\beta)} = \frac{A(x-\alpha)(x-\beta) + B(x-\beta) + C(x-\alpha)^2}{(x-\alpha)^2(x-\beta)}$$

$$\therefore 1 = A(x-\alpha)(x-\beta) + B(x-\beta) + C(x-\alpha)^2$$

Now substitute the value of x to find the constants A, B, C respectively

Now substitute the value of A, B, C in (1) and integrate both sides with respect to ‘x’

(C) Method of quadratic function

$$\int \frac{dx}{(x-\alpha)(x^2+\beta)}$$

(D) Method of calculation

$$\frac{1}{(x-\alpha)(x^2+\beta)} = \frac{A}{(x-\alpha)} + \frac{Bx+c}{x^2+\beta} \qquad (1)$$

$$\therefore \frac{1}{(x-\alpha)(x^2+\beta)} = \frac{A(x^2+\beta)+(Bx+c)(x-\alpha)}{(x-\alpha)(x^2+\beta)}$$

$$\therefore 1 = A(x^2+\beta)+(Bx+c)(x-\alpha)$$

Substitute the value of x $(\because x \in R)$ and find the constants A, B, C respectively.

Now substitute the value of A, B, C in (1) and integrate (1) with respect to ‘x’ both sides

Definite Integration

Important concepts and formulae to be remembered

Definition

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

is called definite integration, where a is said to be lower limit and b is said to be upper limit.

Standard formulae

Sr. No.	**Integral Form**	**Integration**
(1)	$\int_a^b f(x)\,dx =$	$-\int_a^b f(x)\,dx$
(2)	$\int_a^b f(x)\,dx =$	$\int_a^b f(t)\,dt$
(3)	$\int_a^b f(x)\,dx =$ $(\because a < c < b)$	$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
(4)	$\int_0^a f(x)dx =$	$\int_0^a f(a-x)dx$
(5)	$\int_a^b f(x)dx =$	$\int_a^b f(a+b-x)dx =$

Integration by Parts

(6) $$\int_a^b uv\,dx = \qquad u\int_a^b v\,dx - \int_a^b\left(\frac{du}{dx}\int v\,dx\right)dx$$

(7) $$\int_{-a}^{a} f(x)\,dx \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even function} \\ 0, \text{ if } f(x) \text{ is odd function} & \end{cases}$$

ILLUSTRATIONS

Integrate the following with respect to '*x*'

Illustration 1 $\int 7x^3 dx$

Solution

$$\int 7x^3dx = 7\int x^3dx$$

$$= 7\left(\frac{x^4}{4}\right) + c \qquad \left(\int x^n dx = \frac{x^{n+1}}{n+1} + c\right)$$

$$= \frac{7}{4}x^4 + c$$

Illustration 2 $\int\left(x + \frac{1}{x}\right)^2 dx$

Solution

$$\int\left(x + \frac{1}{x}\right)^2 dx = \int\left(x^2 + 2 + \frac{1}{x^2}\right)dx$$

$$= \int x^2dx + 2\int 1dx + \int \frac{1}{x^2}dx$$

$$= \int x^2dx + 2\int 1dx + \int x^{-2}dx$$

$$= \frac{x^3}{3} + 2x + \left(\frac{x^{-1}}{-1}\right) + c$$

$$= \frac{x^3}{3} + 2x - \frac{1}{x} + c$$

Illustration 3 $\int (5x+7)^4 dx$

Solution

$$\int (5x+7)^4 dx$$

$$= \frac{(5x+7)^5}{5(5)} + c$$

$$= \frac{(5x+7)^5}{25} + c$$

Illustration 4 $\int \sqrt{x}dx$

Solution

$$\int \sqrt{x}dx = \int x^{1/2}dx$$

$$= \frac{x^{1/2+1}}{(1/2)+1} + c$$

$$= \frac{x^{3/2}}{3/2} + c$$

$$= \frac{2}{3}x^{3/2} + c$$

Illustration 5 $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$

Solution

$$\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$$

$$= \int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}\left(\frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}\right)$$

$$= \int \frac{\sqrt{x+a}-\sqrt{x+b}}{a-b}dx \qquad \text{(Multiplication with conjunctive)}$$

$$= \frac{1}{a-b}\left[\frac{(x+a)^{3/2}}{3/2} - \frac{(x+b)^{3/2}}{3/2}\right] + c$$

$$= \frac{2}{3(a-b)}\left[(x+a)^{3/2} - (x+b)^{3/2}\right] + c$$

Illustration 6 $\int \frac{2^x + 3^x}{5^x}dx$

Solution

$$\int \frac{2^x + 3^x}{5^x}dx$$

$$= \int \left(\frac{2}{5}\right)^x dx + \int \left(\frac{3}{5}\right)^x dx$$

$$= \frac{(2/5)^x}{\log(2/5)} + \frac{(3/5)^x}{\log(3/5)} + c$$

Illustration 7 $\int \frac{e^{3x} + e^{5x}}{e^x} dx$

Solution

$$\int \frac{e^{3x} + e^{5x}}{e^x} dx$$

$$= \int e^{2x} dx + \int e^{4x} dx$$

$$= \frac{1}{2}e^{2x} + \frac{1}{4}e^{4x} + c$$

Illustration 8 $\int (4x + 2)\sqrt{x^2 + x + 1}\, dx$

Solution

$$\int (4x + 2)\sqrt{x^2 + x + 1}\, dx$$

$$= 2\int (2x + 1)(x^2 + x + 1)^{1/2} dx$$

$$= 2\left[\frac{(x^2 + x + 1)^{3/2}}{3/2}\right] + c \qquad \left\{\int f(x)^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c\right\}$$

$$= \frac{4}{3}(x^2 + x + 1)^{3/2} + c$$

Illustration 9 $\int \frac{(4 + \log x)^5}{x} dx$

Solution

$$\int \frac{(4 + \log x)^5}{x} dx$$

$$= \int (4 + \log x)^5 \left(\frac{1}{x}\right) dx$$

$$= \frac{(4 + \log x)^6}{6} + c$$

Illustration 10 $\int \frac{dx}{\sqrt{4x^2 - 9}}$

Solution

$$\int \frac{dx}{\sqrt{4x^2 - 9}}$$

$$= \frac{1}{2} \int \frac{dx}{\sqrt{x^2 - (3/2)^2}}$$

$$= \frac{1}{2} \log \left| x + \sqrt{x^2 - \frac{9}{4}} \right| + c$$

Illustration 11 $\int xe^x dx$

Solution

$$\int xe^x dx$$

$$= \int xe^x dx - \int \left[\frac{d}{dx}(x) \int e^x dx \right] dx$$

$$= xe^x - \int 1e^x dx$$

$$= xe^x - e^x + c$$

$$= e^x (x - 1) + c$$

Illustration 12 $\int xe^{-x} dx$

Solution

$$\int xe^{-x} dx$$

$$= x \int e^{-x} dx - \int \left[\frac{d}{dx}(x) \int e^{-x} dx \right] dx$$

$$= x \left(\frac{e^{-x}}{-1} \right) - \int 1 \frac{e^{-x}}{-1} dx$$

$$= -xe^{-x} + \int e^{-x} dx$$

$$= -xe^{-x} + \left(\frac{e^{-x}}{-1} \right) + c$$

$$= -xe^{-x} - e^{-x} + c$$

$$= -e^{-x} (x + 1) + c$$

Illustration 13 $\int \frac{xe^x}{(x+1)^2}dx$

Solution

$$\int \frac{xe^x}{(x+1)^2}dx$$

$$= \int \frac{x+1-1}{(x+1)^2}e^x dx$$

$$= \int \left\{ \frac{1}{x+1} + \left[\frac{-1}{(x+1)^2} \right] \right\} e^x dx$$

$$= \frac{1}{x+1}e^x + c \qquad \left\{ \because \int \left[f(x) + f'(x) \right] e^x dx = f(x)e^x \right\}$$

Illustration 14 $\int \frac{xdx}{(x^2+1)(x^2+2)}$

Solution

$$\int \frac{xdx}{(x^2+1)(x^2+2)}$$

$$= \frac{1}{2}\int \frac{dt}{(t+1)(t+2)}$$

$$= \frac{1}{2}\int \left(\frac{1}{t+1} - \frac{1}{t+2} \right) dt$$

$$= \frac{1}{2}\left\{ \log|t+1| - \log|t+2| \right\} + c$$

$$= \frac{1}{2}\log\left| \frac{t+1}{t+2} \right| + c$$

$$= \frac{1}{2}\log\left| \frac{x^2+1}{x^2+2} \right| + c$$

$$= \log\sqrt{\frac{x^2+1}{x^2+2}} + c$$

Suppose $x^2 = t$

Differentiate ‘x’ with respect to ‘t’

$$2x\frac{dx}{dt} = 1$$

$$\therefore xdx = \frac{1}{2}dt$$

Illustration 15 $\int \frac{dx}{x^2+4x+1}$

Solution

$$\int \frac{dx}{x^2+4x+1}$$

$$= \int \frac{dx}{x^2 + 4x + 4 - 3}$$

$$= \int \frac{dx}{(x+2)^2 - \left(\sqrt{3}\right)^2}$$

$$= \log\left|\frac{x + 2 - \sqrt{3}}{x + 2 + \sqrt{3}}\right| + c$$

Illustration 16 Evaluate $\int_0^2 e^{2x} dx$

Solution

$$\int_0^2 e^{2x} dx$$

$$= \left(\frac{e^{2x}}{2}\right)_0^2 = \frac{1}{2}\left(e^{2x}\right)_0^2$$

$$= \frac{1}{2}\left(e^4 - e^0\right)$$

$$= \frac{1}{2}\left(e^4 - 1\right)$$

Illustration 17 Evaluate $\int_{-3}^{-2} \frac{1}{x} dx$

Solution

$$\int_{-3}^{-2} \frac{1}{x} dx$$

$$= \left(\log|x|\right)_{-3}^{-2}$$

$$= \log|-2| - \log|-3|$$

$$= \log 2 - \log 3$$

$$= \log\frac{2}{3}$$

Illustration 18 Evaluate $\int_2^5 \frac{xdx}{x^2 + 1}$

Solution

$$\int_2^5 \frac{xdx}{x^2 + 1}$$

$$= \frac{1}{2}\int_2^5 \frac{2xdx}{x^2 + 1}$$

$$= \frac{1}{2}\left(\log\left|x^2 + 1\right|\right)_2^5 \qquad \left[\because \int \frac{f'(x)}{f(x)} dx = \log\left|f(x)\right| + c\right]$$

$$= \frac{1}{2}(\log 26 - \log 5)$$
$$= \frac{1}{2}\log\frac{26}{5}$$
$$= \log\sqrt{\frac{26}{5}}$$

Illustration 19 Evaluate $\int_0^2 xe^x dx$

Solution

$$\int_0^2 xe^x dx$$
$$= x\int_0^2 e^x dx - \int_0^2\left[\frac{d}{dx}(x)\int e^x dx\right]dx$$
$$= \left(xe^x\right)_0^2 - \int_0^2 1e^x dx$$
$$= \left(xe^x\right)_0^2 - \left(e^x\right)_0^2$$
$$= \left(2e^2 - 0\right) - \left(e^2 - e^0\right)$$
$$= 2e^2 - e^2 + 1$$
$$= e^2 + 1$$

Illustration 20 Evaluate $\int_2^3 \frac{xe^x}{(x+1)^2}dx$

Solution

$$\int_2^3 \frac{xe^x}{(x+1)^2}dx$$
$$= \int_2^3 \frac{(x+1-1)}{(x+1)^2}e^x dx$$
$$= \int_2^3\left[\frac{1}{x+1} - \frac{1}{(x+1)^2}\right]e^x dx$$
$$= \left(\frac{1}{x+1}e^x\right)_2^3$$
$$= \frac{e^3}{4} - \frac{e^2}{3}$$

Illustration 21 Evaluate $\int_{0}^{1}\left(x^{3}+2x^{2}\right)^{3}\left(3x^{2}+4x\right)dx$

Solution

$$\int_{0}^{1}\left(x^{3}+2x^{2}\right)^{3}\left(3x^{2}+4x\right)dx$$

$$=\left[\frac{\left(x^{3}+2x^{2}\right)^{4}}{4}\right]_{0}^{1}$$

$$=\frac{1}{4}\left[\left(x^{3}+2x^{2}\right)^{4}\right]_{0}^{1}$$

$$=\frac{1}{4}(3)^{4}-0$$

$$=\frac{81}{4}$$

Illustration 22 Evaluate $\int_{a}^{b}\frac{\log x}{x}dx$

Solution

$$\int_{a}^{b}\frac{\log x}{x}dx$$

$$=\frac{1}{2}\left[(\log b)^{2}-(\log a)^{2}\right]$$

$$=\frac{1}{2}(\log b+\log a)-(\log b-\log a)$$

$$=\frac{1}{2}\log ab\ \log\left(\frac{a}{b}\right)$$

Illustration 23 Evaluate $\int_{1}^{e^{2}}\frac{dx}{x(1+\log x)^{2}}$

Solution

$$\int_{1}^{e^{2}}\frac{dx}{x(1+\log x)^{2}}$$

$$=\int_{1}^{3}\frac{dt}{t^{2}}$$

$$=\int_{1}^{3}t^{-2}dt$$

$$=\left(\frac{t^{-1}}{-1}\right)_{1}^{3}$$

Suppose $1+\log x=t$

Differentiate ‘x’ with respect to ‘t’

$$\therefore 0+\frac{1}{x}\frac{dx}{dt}=1$$

$$\therefore \frac{1}{x}dx=dt \text{ and } 1+\log x=t$$

$$x=1\Rightarrow t=1+\log 1=1$$

$$x=e^{2}\Rightarrow t=1+\log e^{2}=3$$

$$= \left(-\frac{1}{t}\right)_1^3$$

$$= -\frac{1}{3} + 1$$

$$= \frac{2}{3}$$

Illustration 24 Evaluate $\int_0^2 x\sqrt{x+2}\,dx$

Solution

$$\int_0^2 x\sqrt{x+2}\,dx$$

$$= \int_0^2 (x+2-2)(x+2)^{1/2}\,dx$$

$$= \int_0^2 (x+2)^{3/2} - 2\int_0^2 (x+2)^{1/2}$$

$$= \left[\frac{(x+2)^{5/2}}{5/2}\right]_0^2 - 2\left[\frac{(x+2)^{3/2}}{3/2}\right]_0^2$$

$$= \frac{2}{5}\left(4^{5/2} - 2^{5/2}\right) - \frac{4}{3}\left(4^{3/2} - 2^{3/2}\right)$$

$$= \frac{2}{5}\left(2^5 - 4\sqrt{2}\right) - \frac{4}{3}\left(8 - 2\sqrt{2}\right)$$

$$= \frac{2^6}{5} - \frac{8\sqrt{2}}{5} - \frac{32}{3} + \frac{8\sqrt{2}}{3}$$

$$= \frac{64}{5} - \frac{32}{3} - \frac{8\sqrt{2}}{5} + \frac{8\sqrt{2}}{3}$$

$$= 32\left(\frac{2}{5} - \frac{1}{3}\right) - 8\sqrt{2}\left(\frac{1}{5} - \frac{1}{3}\right)$$

$$= 32\left(\frac{6-5}{15}\right) - 8\sqrt{2}\left(\frac{3-5}{15}\right)$$

$$= \frac{32}{15} - \frac{16\sqrt{2}}{15}$$

$$= \frac{16}{15}\left(2 + \sqrt{2}\right)$$

Illustration 25 Evaluate $\int_{-1}^{1} \frac{x^5}{a^2 - x^2} dx$

Solution

$$\int_{-1}^{1} \frac{x^5 dx}{a^2 - x^2}$$

$$\text{Here } f(x) = \frac{x^5}{a^2 - x^2}$$

$$f(-x) = \frac{(-x^5)}{a^2 - (-x^2)} = \frac{-x^5}{a^2 - x^2} = -f(x)$$

$$\therefore f(-x) = -f(x)$$

$f(x)$ is an odd function

$$\int_{-1}^{1} \frac{x^5 dx}{a^2 - x^2} = 0$$

Illustration 26 Evaluate $\int_{0}^{7} \frac{x^{5/2} dx}{x^{5/2} + (7 - x)^{5/2}}$

Solution

$$I = \int_{0}^{7} \frac{x^{5/2} dx}{x^{5/2} + (7 - x)^{5/2}} \quad (1)$$

$$= \int_{0}^{7} \frac{(7 - x)^{5/2} dx}{(7 - x)^{5/2} + [7 - (7 - x)]^{5/2}}$$

$$I = \int_{0}^{7} \frac{(7 - x)^{5/2} dx}{(7 - x)^{5/2} + x^{5/2}} \quad (2)$$

Adding eq. (1) and eq. (2) we can say that

$$2I = \int_{0}^{7} \frac{x^{5/2} + (7 - x)^{5/2}}{x^{5/2} + (7 - x)^{5/2}}$$

$$2I = \int_{0}^{7} 1dx$$

$$\therefore 2I = (x)_0^7 = 7$$

$$\therefore I = \frac{7}{2}$$

Illustration 27 Evaluate $\int_{1}^{2} \frac{(x + 3)dx}{x(x + 2)}$

Solution

$$\int_1^2 \frac{(x+3)dx}{x(x+2)}$$

$$\frac{x+3}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2} \tag{1}$$

$$\therefore \frac{x+3}{x(x+2)} = \frac{A(x+2)+B(x)}{x(x+2)}$$

$$\therefore x+3 = A(x+2)+B(x)$$

$$x = 0 \Rightarrow 3 = A(2)+0 \Rightarrow A = \frac{3}{2}$$

$$x = -2 \Rightarrow 1 = A(0)+B(-2) \Rightarrow B = -\frac{1}{2}$$

Now substitute the value of A and B in eq. (1) then integrate with respect to '*x*'

$$\int_1^2 \frac{(x+3)dx}{x(x+2)} = \int_1^2 \frac{3/2}{x}dx - \frac{1}{2}\int_1^2 \frac{dx}{x+2}$$

$$= \left(\frac{3}{2}\log|x|\right)_1^2 - \frac{1}{2}\left(\log|x+2|\right)_1^2$$

$$= \frac{3}{2}(\log 2 - \log 1) - \frac{1}{2}(\log 4 - \log 3)$$

$$= \frac{3}{2}\log 2 - \log 0 - \log 2 + \frac{1}{2}\log 3$$

$$= \frac{1}{2}\log 2 + \frac{1}{2}\log 3$$

$$= \frac{1}{2}\log 6$$

$$= \log\sqrt{6}$$

Illustration 28 Evaluate $\int_0^b \frac{\sqrt{x}dx}{\sqrt{x}+\sqrt{b-x}}$

Solution

$$I = \int_0^b \frac{\sqrt{x}dx}{\sqrt{x}+\sqrt{b-x}} \tag{1}$$

$$= \int_0^b \frac{\sqrt{b-x}dx}{\sqrt{b-x}+\sqrt{b-(b-x)}}$$

$$I = \int_0^b \frac{\sqrt{b-x}dx}{\sqrt{b-x}+\sqrt{x}} \tag{2}$$

Adding eqs. (1) and (2) we can say that

$$I + I = \int_0^b \frac{\sqrt{x} + \sqrt{b - x}}{\sqrt{x} + \sqrt{b - x}}$$

$$\therefore 2I = \int_0^b 1dx$$

$$\therefore 2I = (x)_0^b = b$$

$$\therefore I = \frac{b}{2}$$

Illustration 29 Evaluate $\int_2^4 \frac{dx}{x^2 - 4}$

Solution

$$\int_2^4 \frac{dx}{x^2 - 4}$$

$$= \int_3^4 \frac{dx}{x^2 - (2)^2}$$

$$= \frac{1}{2(2)}\left(\log\left|\frac{x-2}{x+2}\right|\right)_3^4$$

$$= \frac{1}{4}\left(\log\left|\frac{4-2}{4+2}\right| - \log\left|\frac{3-2}{3+2}\right|\right)$$

$$= \frac{1}{4}\left(\log\frac{1}{3} - \log\frac{1}{5}\right)$$

$$= \frac{1}{4}\log\frac{5}{3}$$

Illustration 30 Evaluate $\int_1^e \log x dx$

Solution

$$\int_1^e \log x dx$$

$$= \int_1^e \log x 1dx$$

$$= \log x \int_1^e 1dx - \int_1^e \left(\frac{d}{dx}\log x \int 1dx\right)dx$$

$$= (x \log x)_1^e - \int_1^e \left(\frac{1}{x}x\right)dx$$

$$= \left(x\log x\right)_1^e - \left(x\right)_1^e$$

$$= \left(e\log_e^e - 1\log 1\right) - \left(e - 1\right)$$

$$= e - 0 - e + 1$$

$$= 1$$

ANALYTICAL EXERCISES

Integrate the following

1. $\int\left(x^3 + 7x^2 - 5x\right)dx$

2. $\int\sqrt{x}dx$

3. $\int\frac{5x - 2}{\sqrt{3x - 2}}dx$

4. $\int\left(7x - 9\right)^4 dx$

5. $\int\frac{e^{2x} + e^{-2x}}{e^x}dx$

6. $\int\frac{4e^{5x} - 9e^{4x} - 3}{e^{3x}}dx$

7. $\int\frac{dx}{\sqrt{x - 1} - \sqrt{x + 1}}$

8. $\int\left(e^{x\log_e^a} - e^{a\log_e^x}\right)dx$

9. $\int\frac{\left(a^x + b^x\right)}{a^x b^x}dx$

10. $\int\left(e^{3a\log_e^x} + e^{3x\log_e^a}\right)dx$

11. $\int\left(2x + 3\right)^4 dx$

12. $\int\frac{\left(a^x - b^x\right)^2}{a^x b^x}dx$

13. $\int\sqrt{ax + b}\ dx$

14. $\int\frac{dx}{\sqrt{x + 4} + \sqrt{x + 5}}$

15. $\int\left(e^x + e^{-x}\right)^2 dx$

16. $\int\frac{e^{3x} + e^{-3x}}{e^x}dx$

17. $\int (5x + 3)^9 \, dx$

18. $\int \frac{p^x + q^x}{r^x} dx$

19. $\int \frac{e^{5\log_e^x} - e^{3\log_e^x}}{e^{4\log_e^x} - e^{2\log_e^x}} dx$

20. $\int x\sqrt{x + 2} \, dx$

21. $\int \frac{x - 1}{\sqrt{x + 4}} dx$

22. $\int \frac{dx}{(3 - 2x)^3}$

23. $\int e^{3x + 5} dx$

24. $\int \frac{2x}{(2x + 1)^2} dx$

25. $\int \frac{x\,dx}{a + bx}$

26. $\int \left(1 + \frac{1}{x}\right)(x + \log x)^2 \, dx$

27. $\int (e^x + e^{-x})^4 (e^x - e^{-x}) dx$

28. $\int x^4 (1 + x^5)^{1/3} \, dx$

29. $\int \frac{dx}{(e^x + e^{-x})^2}$

30. $\int \frac{(4 + \log x)^5}{x} dx$

31. $\int \frac{xe^{x/2}}{\sqrt{e^{x^2} + 2}} dx$

32. $\int \frac{(2x + 3)dx}{x^2 + 3x + 7}$

33. $\int \frac{10x^9 + 10^x \log_e^{10}}{x^{10} + 10^x} dx$

34. $\int \frac{dx}{x(1 + \log x)}$

35. $\int \frac{dx}{x \log x \ \log(\log x)}$

36. $\int e^x\left(a+be^x\right)^n dx$

37. $\int \frac{x^2dx}{x^2-a^2}$

38. $\int \frac{a^x \log_e^a + ax^{a-1}}{a^x+x^a}dx$

39. $\int \frac{dx}{1-9x^2}$

40. $\int \frac{x^2dx}{a^6-x^6}$

41. $\int \frac{dx}{3+2x-x^2}$

42. $\int \frac{x^2-1}{x^4+1}dx$

43. $\int \frac{dx}{\sqrt{4x^2-9}}$

44. $\int \frac{dx}{\sqrt{x^6-1}}$

45. $\int \frac{dx}{\sqrt{\left(2-x^2\right)-1}}$

46. $\int \frac{dx}{x\sqrt{\left(\log x\right)^2-5}}$

47. $\int xe^x dx$

48. $\int x^2e^x dx$

49. $\int x^3e^x dx$

50. $\int 2x^3e^{x^2} dx$

51. $\int x^3e^{ax} dx$

52. $\int e^{\sqrt{x}} dx$

53. $\int x\log x\ dx$

54. $\int x^2 \log x\ dx$

55. $\int \frac{\left(x^2+1\right)}{\left(x+1\right)^2}e^x dx$

56. $\int \frac{\left(\log x-1\right)}{\left(\log x\right)^2}dx$

57. $\int \left(\log x + \frac{1}{x^2} \right) e^x dx$

58. $\int \frac{e^x}{x} \left[x(\log x)^2 + 2\log x \right] dx$

59. $\int \frac{(1+x)}{(2+x)^2} e^x dx$

60. $\int \frac{x^2 dx}{\sqrt{x^2 - a^2}}$

61. $\int \frac{x^2 dx}{\sqrt{x^2 + 1}}$

62. $\int \frac{x^2 + 1}{\sqrt{x^2 + 4}} dx$

63. $\int \frac{dx}{(x+1)(x+2)(x+3)}$

64. $\int \frac{xdx}{(x^2+1)(x^2+2)}$

65. $\int \frac{dx}{x\left[(\log x)^2 - 5\log x + 6\right]}$

66. $\int \frac{dx}{x\left[(\log x)^2 - \log x - 2\right]}$

67. $\int \frac{2xdx}{(x^2-1)(x^2+3)}$

68. $\int_0^1 e^{2x} dx$

69. $\int_{-4}^{-1} \frac{1}{x} dx$

70. $\int_0^{1/4} \sqrt{1-4x}\; dx$

71. $\int_0^1 \frac{dx}{2x-3}$

72. $\int_2^4 \frac{xdx}{x^2+1}$

73. $\int_0^1 \frac{2xdx}{5x^2+1}$

74. $\int_0^{\log 3} \frac{e^x dx}{1+e^x}$

75. $\int_1^2 xe^x dx$

76. $\int_1^3 \frac{\log x dx}{\left(1+x^2\right)}$

77. $\int_e^{e^2} \left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right] dx$

78. $\int_0^1 \frac{x \cdot e^x}{(x+1)^2} dx$

79. $\int_0^1 xe^{x^2} dx$

80. $\int_0^1 xe^{x^2} dx$

81. $\int_0^1 xe^{-x^2} dx$

82. $\int_0^a \frac{xdx}{\sqrt{x^2+a^2}}$

83. $\int_0^{\log 2} \frac{e^x dx}{e^{2x}+5e^x+6}$

84. $\int_0^1 \frac{x}{(x+1)^2} e^x dx$

85. $\int_0^2 \frac{x^2}{x^2+(2-x)^2} e^x dx$

86. $\int_0^1 \log\left(\frac{1}{x}-1\right) dx$

87. $\int_0^1 x(1-x)^n dx$

88. $\int_a^b \frac{f(x)dx}{f(x)+f(a+b-x)}$

89. If $f(a+b-x) = f(x)$ then find the value of $\int_a^b xf(x)dx$

90. $\int_{-2}^2 \frac{x^4 dx}{a^{10}-x^{10}}$

91. $\int_0^5 \frac{x^2 dx}{x^2 + (5 - x)^2}$

92. $\int_1^3 \frac{\sqrt{x}dx}{\sqrt{x} + \sqrt{4 - x}}$

93. $\int_0^a \frac{\sqrt{x}dx}{\sqrt{x} + \sqrt{a - x}}$

94. $\int_{-1}^1 \frac{x^2 dx}{2^6 - x^6}$

95. $\int_0^4 \frac{x^4 dx}{x^4 + (4 - x)^4}$

96. $\int_{-1}^1 \log\left(\frac{2 - x}{2 + x}\right)dx$

97. $\int_0^4 \frac{\sqrt[3]{x + 5}\, dx}{\sqrt[3]{x + 5} + \sqrt[3]{9 - x}}$

98. $\int_{-1}^1 x^3 e^{x^4} dx$

99. $\int_{-1}^1 \log\left(\frac{3 - x}{3 + x}\right)dx$

100. $\int_1^2 \frac{dx}{x(1 + \log x)^2}$

ANSWERS

(1) $\frac{x^4}{4} - \frac{7}{3}x^3 - \frac{5}{2}x^2 + c$

(2) $\frac{2}{3}x^{3/2} + c$

(3) $\frac{10}{27}(3x + 2)^{3/2} - \frac{32}{9}(3x + 2)^{1/2} + c$

(4) $\frac{(7x - 9)^5}{35} + c$

(5) $e^x - \frac{e^{-3x}}{3} + c$

(6) $2e^{2x} - 9e^x + e^{-3x} + c$

(7) $-\frac{1}{3}\left[(x - 1)^{3/2} + (x + 1)^{3/2}\right] + c$

(8) $\frac{a^x}{\log_e^a} - \frac{x^{a+1}}{a + 1} + c$

(9) $-\left(\frac{1}{b^x \log_e^b} + \frac{1}{a^x \log_e^a}\right) + c$

(10) $\frac{x^{3a+1}}{3a + 1} + \frac{a^{3x}}{3\log_e^a} + c$

(11) $\frac{(2x + 3)^5}{10} + c$

(12) $\frac{(a / b)^x}{\log(a / b)} - 2x + \frac{(b / a)^x}{\log(b / a)} + c$

(13) $\frac{2}{3a}(ax+b)^{3/2}+c$

(14) $-\frac{2}{3}\left[(x+4)^{3/2}-(x+5)^{3/2}\right]+c$

(15) $\frac{1}{2}\left(e^{2x}-e^{-2x}\right)+2x+c$

(16) $\frac{1}{2}e^{2x}-\frac{1}{4e^{4x}}+c$

(17) $\frac{(5x+3)^{10}}{50}+c$

(18) $\frac{(p/r)^{x}}{\log(p/r)}+\frac{(q/r)^{x}}{\log(q/r)}+c$

(19) $\frac{x^2}{2}+c$

(20) $\frac{2}{5}(x+2)^{5/2}-\frac{4}{3}(x+2)^{3/2}+c$

(21) $\frac{2}{3}(x+4)^{3/2}-10\sqrt{x+4}+c$

(22) $\frac{1}{4(3-2x)^2}+c$

(23) $\frac{1}{3}e^{3x+5}+c$

(24) $\log\sqrt{2x+1}+\frac{1}{2(2x+1)}+c$

(25) $\frac{x}{b}-\frac{a}{b^2}\log|a+bx|+c$

(26) $\frac{(x+\log x)^3}{3}+c$

(27) $\frac{\left(e^{x}+e^{-x}\right)^5}{5}+c$

(28) $\frac{3}{20}\left(1+x^5\right)^{4/3}+c$

(29) $\frac{-1}{2\left(e^{2x}+1\right)}+c$

(30) $\frac{(4+\log x)^6}{6}+c$

(31) $\sqrt{e^{x^2}+2}+c$

(32) $\log\left|x^2+3x+7\right|+c$

(33) $\log\left|x^{10}+10^{x}\right|+c$

(34) $\log|1+\log x|+c$

(35) $\log[\log(\log x)]+c$

(36) $\frac{\left(a+be^{x}\right)^{n+1}}{b(n+1)}+c$

(37) $x+\frac{a}{2}\log\left|\frac{x-a}{x+a}\right|+c$

(38) $\log\left|a^{x}+x^{a}\right|+c$

(39) $\frac{1}{6}\log\left|\frac{1+3x}{1-3x}\right|+c$

(40) $\frac{1}{6a^3}\log\left|\frac{a^3+x^3}{a^3-x^3}\right|+c$

(41) $\frac{1}{4}\log\left|\frac{x+1}{3-x}\right|+c$

(42) $\frac{1}{2\sqrt{2}}\log\left|\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1}\right|+c$

(43) $\log\left|\sqrt{x+\sqrt{x^2-\frac{9}{4}}}\right|+c$

(44) $\frac{1}{3}\log\left|x^3+\sqrt{x^6-1}\right|+c$

(45) $\log\left|(2-x)+\sqrt{\left(2-x^2\right)-1}\right|+c$

(46) $\log\left|\log x+\sqrt{(\log x)^2-5}\right|+c$

(47) $e^{x}(x-1)+c$

(48) $\left(x^2-2x+2\right)e^{x}+c$

(49) $\left(x^3-3x^2+6x-6\right)e^{x}+c$

(50) $\left(x^2-1\right)e^{x^2}+c$

(51) $\frac{e^{ax}}{a^2}(ax-1)+c$

(52) $e^{\sqrt{x}}\left(2\sqrt{x}-1\right)+c$

(53) $\frac{x^2}{2}\log x-\frac{x^2}{4}+c$

(54) $\frac{x^3}{9}(3\log|x|-1)+c$

(55) $\left(\frac{x-1}{x+1}\right)e^{x}+c$

(56) $\dfrac{x}{\log x} + c$

(57) $\left(\log x - \dfrac{1}{x}\right)e^{x} + c$

(58) $(\log x)^{2} e^{x} + c$

(59) $\dfrac{e^{x}}{(2 + x)} + c$

(60) $\dfrac{x}{2}\sqrt{x^{2} - a^{2}} + \dfrac{a^{2}}{2}\log\left|x + \sqrt{x^{2} - a^{2}}\right| + c$

(61) $\dfrac{x\sqrt{x^{2} + 1}}{2} - \dfrac{1}{2}\log\left|x + \sqrt{x^{2} + 1}\right| + c$

(62) $\dfrac{x\sqrt{x^{2} + 4}}{2} - \log\left|x + \sqrt{x^{2} + 1}\right| + c$

(63) $\log\left|\dfrac{\sqrt{(x+1)(x+3)}}{(x+2)}\right| + c$

(64) $\log\sqrt{\dfrac{x^{2} + 1}{x^{2} + 2}} + c$

(65) $\log\left|\dfrac{\log x - 3}{\log x - 2}\right| + c$

(66) $\dfrac{1}{3}\log\left|\dfrac{\log x - 2}{\log x + 1}\right| + c$

(67) $\dfrac{1}{4}\log\left|\dfrac{x^{2} - 1}{x^{2} + 3}\right| + c$

(68) $\dfrac{1}{2}(e^{2} - 1)$

(69) $-\log 4$

(70) $\dfrac{1}{6}$

(71) $-\dfrac{1}{2}\log 3$

(72) $\dfrac{1}{2}\log\left(\dfrac{17}{5}\right)$

(73) $\dfrac{1}{5}\log 6$

(74) $\log 2$

(75) e^{2}

(76) $\dfrac{3}{4}\log 3 - \log 2$

(77) $\dfrac{e^{2}}{e} - e$

(78) $\dfrac{e}{2} - 1$

(79) $\log\dfrac{1}{8}$

(80) $\dfrac{1}{2}(e - 1)$

(81) $\dfrac{1}{2}\left(1 - \dfrac{1}{e}\right)$

(82) $a(\sqrt{2} - 1)$

(83) $\log\left(\dfrac{16}{15}\right)$

(84) $\dfrac{e - 2}{2}$

(85) 1

(86) 0

(87) $\dfrac{1}{(n + 1)(n + 2)}$

(88) $\dfrac{b - a}{2}$

(89) $\left(\dfrac{a + b}{2}\right)\int_{a}^{b} f(x)dx$

(90) $\dfrac{1}{5a^{5}}\log\left|\dfrac{a^{5} + 32}{a^{5} - 32}\right|$

(91) $\dfrac{5}{2}$

(92) 1

(93) $\dfrac{a}{2}$

(94) $\dfrac{1}{24}\log\left(\dfrac{9}{7}\right)$

(95) 2

(96) 0

(97) 2

(98) 0

(99) 0

(100) $\dfrac{\log 2}{1 + \log 2}$

13

Applications of Calculus

(Application of Derivative)

LEARNING OBJECTIVES

After studying this chapter, the student will be able to understand:

- Partial Derivative
- Application of derivatives in economics
- Application of derivatives in mathematics
- Application of integration.
- Application of partial derivatives

INTRODUCTION

Derivatives have a wide range of applications to a very large number of disciplines such as engineering, business economics, physics, social sciences, etc. and integration has a wide range of applications to a very large number of disciplines such as business and economics, biology, geometry and probability. In this chapter we shall discuss some of these applications.

PARTIAL DERIVATIVES AND THEIR APPLICATIONS

Suppose $z = f(x,y)$ is a function of two variables x and y. Keeping y as constant, if we differentiate z with respect to 'x' it is called partial derivative of z with respect to 'x' and it is denoted by $\frac{\partial z}{\partial x}$ *or* $\frac{\partial f}{\partial x}$ *or* f_x and same way keeping x as constant, if we differentiate z with respect to 'y' it is called partial derivative of z with respect to 'y' and it is denoted by $\frac{\partial z}{\partial y}$ *or* $\frac{\partial f}{\partial y}$ *or* f_y.

Thus partial derivative of $f(x,y)$ with respect to 'x' can be defined as $\frac{\partial f}{\partial x}=\lim_{h\to o}\frac{f(x+h,y)-f(x,y)}{w}$ where h is a small increment of x; similarly $f(x,y)$ with respect to 'y' can be defined as $\frac{\partial f}{\partial y}=\lim_{h\to o}\frac{f(x,y+k)-f(x,y)}{w}$.

Here $\frac{\partial f}{\partial x}$ *of* $\frac{\partial f}{\partial y}$ is first order partial derivative of f with respect to 'x' or y and $\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$ *or* $\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$ is called second order partial derivative of $f(x,y)$ with respect to 'x' or 'y' and it is also denoted by f_{xx} or f_{yy}.

In the same way we can say that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = fxy$ and $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = fyx.$

Partial derivative is used to maximize utility under certain conditions or to minimize cost under certain constraints. Before discussion of partial derivatives we should understand the meaning of utility and its related references.

Utility

Utility may be defined as the power or the property of a commodity or service to satisfy a human want. But utility is not inherent in the commodity or service. It is in the mental attitude or feeling of the consumer; therefore, utility is a subjective concept.

Total Utility

A person may not be unit of a commodity. He may require more units. The more are the units utilized, greater is the utility derived. Thus total utility is a sum total of utilities derived from the consumption of all units of a commodity. Total utility increases with the increase in the units utilized.

Marginal Utility

The marginal utility of a commodity is the utility derived from the consumption of an extra unit of a commodity. That is, marginal utility is the utility of a marginal unit.

Note: Marginal utility can be positive, negative, or zero.

Utility Function

Every consumer allocates his money income among different commodities in such a manner as to obtain maximum satisfaction from their consumption.

For example, a consumer wants to buy two commodities M and N. Suppose he purchases x units of M and y units of N. Then his total utility U will depend on x and y. That is, U is a function of x and y.

$$\therefore \; U = f(x,y)$$

It is said to be consumer's utility function and it follows following assumptions:

1. U, x and y are positive
2. For any given value of U, as x increases y decreases.
3. $f(x,y)$ is a continuous function.
4. The first and the second order partial derivatives of a function are also continuous.

Use of Partial Derivatives for Obtaining Maximum Utility

The consumer will purchase such quantitites of M and N which maximize his total utility. In doing so he has to consider the restriction of his limited income.

If the price of M is Px and N is Py per unit then since he purchases x units of M and y units of N his total expenditure will be $x\mathrm{P}x + y\mathrm{P}y$.

Suppose his budget is I, then $I = xPx + yPy$ is said to be consumer's budget equation.

Now the problem is to maximize the consumer's utility function

$U = f(x,y)$

Subject to budget equation

$I = xPx + yPy$

The problem is to find such values of x and y which maximize

$U = f(x,y)$.

Lagrange's Multiplier Method

To solve the above problem we introduce a new variable λ, multiplier.

We have

$$U = f(x,y) \tag{1}$$

$$I = xPx + yPy \tag{2}$$

$$\therefore\ I - xpx + yPy = 0 \tag{3}$$

From Eqs. (1) and (2) we can get a new function F as follows:

$$\begin{aligned} F &= (x,y,\ \lambda) = U + \lambda\,(I - xPx + yPy) \\ &= f(x,y) + \lambda\,(I - xPx + yPy) \end{aligned} \tag{4}$$

where λ is a constant, it is a Lagrange's multiplier.

To maximize F, we equate the first order partial derivatives of f with respect to x and y to zero.

$$\therefore \frac{\partial f}{\partial x} = 0 \text{ and } \therefore \frac{\partial f}{\partial y} = 0$$

We have $\dfrac{\partial f}{\partial x} = 0$; $\dfrac{\partial f}{\partial y} = 0$; $I = xPx + yPy$

Now by solving the above three equations simultaneously; we get such values of x and y which maximize F.

Use of Partial Derivatives in Minimizing Cost under Constraint

If the number of labour is denoted by x and the cost of row material denoted by y then the cost function will be $c = f(x,y)$. If z number of units are manufactured then the production can be given by $z = g(x,y)$.

The problem for the manufacturer is naturally to minimize the total cost c under the constraint of the production function $z = g(x,y)$.

From the given cost function and the constraints of the production function the function f can be written as $f = c + \lambda z$ where c and z are the functions of x and y.

Find the values of and and equating them to zero. λ can be mated and an equation in x and y can be obtained. This equation and the production function gives two simultaneous equations. Solution of these equations gives the value of x and y for the minimum cost.

ILLUSTRATIONS

Illustration 1 If $z = x^2 + 2xy + 3y^2$ then find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

Solution

Here $z = x^2 + 2xy + 3y^2$

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}x^2 + 2y\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial x}(3y^2)$$

$$= 2x + 2y(1) + 0$$

$$= 2x + 2y$$

and $$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x)^2 + 2x\frac{\partial}{\partial y}(y) + 3\frac{\partial}{\partial y}(y^2)$$

$$= 0 + 2(x) + 3(2y)$$

$$= 2x + 6y$$

Illustration 2 If $z = x^2 + y^2 + 4x - 3y + 5xy + 2$ then find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

Solution

$z = x^2 + y^2 + 4x - 3y + 5xy + 2$

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x)^2 + \frac{\partial}{\partial x}(y^2) + 4\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial x}(4y) + 5y\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial x}(2)$$

$$= 2x + 0 + 4 - 0 + 5y + 0$$

$$= 2x + 5y + 4$$

and $$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial y}(4x) - 3\frac{\partial}{\partial y}(y) + 5x\frac{\partial}{\partial y}(y) + \frac{\partial}{\partial y}(2)$$

$$= 0 + 2y + 0 - 3 + 5x + 0$$

$$= 5x + 2y - 3$$

Illustration 3 If $z = x^2 - xy + 2y^2$ then find $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 y}{\partial y^2}$

Solution

Here $z = x^2 - xy + 2y^2$

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2) - y\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial x}(2y^2)$$

$$= 2x - y + 0$$

Now $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right) = \dfrac{\partial}{\partial x}(2x - y)$

$= 2\dfrac{\partial}{\partial x}(x) - \dfrac{\partial}{\partial x}(y)$

$= 2 - 0 = 2$

Now $\dfrac{\partial z}{\partial y} = \dfrac{\partial}{\partial y}\left(x^2 - xy + 2y^2\right)$

$= \dfrac{\partial}{\partial y}\left(x^2\right) - x\dfrac{\partial}{\partial y}(y) + 2\dfrac{\partial}{\partial y}\left(y^2\right)$

$= 0 - x(1) + 2(2y)$

$= 4y - x$

Now $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial y}(4y - x)$

$= 4\dfrac{\partial}{\partial y}(y) - \dfrac{\partial}{\partial y}(x)$

$= 4(1) - 0 = 4$

Illustration 4 If $z = x^3 + 2x^2y + xy^2 - y^3$ then find $\dfrac{\partial^2 z}{\partial x^2}$; $\dfrac{\partial^2 z}{\partial y^2}$; $\dfrac{\partial^2 z}{\partial x \partial y}$; $\dfrac{\partial^2 z}{\partial y \partial z}$

Solution

Here $z = x^3 + 2x^2y + xy^2 - y^3$

$\dfrac{\partial z}{\partial x} = \dfrac{\partial}{\partial x}\left(x^3\right) + 2y\dfrac{\partial}{\partial x}\left(x^2\right) + y^2\dfrac{\partial}{\partial x}(x) - \dfrac{\partial}{\partial x}\left(y^3\right)$

$= 3x^2 + 2y(2x) + y^2(1) - 0$

$= 3x^2 + 4xy + y^2$

$\therefore \dfrac{\partial z}{\partial x} = 3x^2 + 4xy + y^2$ (1)

Now $\dfrac{\partial z}{\partial y} = \dfrac{\partial}{\partial y}\left(x^3\right) + 2x^2\dfrac{\partial}{\partial x}(y) + x\dfrac{\partial}{\partial x}\left(y^2\right) - \dfrac{\partial}{\partial y}\left(y^3\right)$

$= 0 + 2x^2(1) + x(2y) - 3y^2$

$= 2x^2 + 2xy - 3y^2$

$\therefore \dfrac{\partial z}{\partial y} = 2x^2 + 2xy - 3y^2$ (2)

Now $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(3x^2 + 4xy + y^2\right)$

$= 3\dfrac{\partial}{\partial x}\left(x^2\right) + 4y\dfrac{\partial}{\partial x}(x) + \dfrac{\partial}{\partial x}\left(y^2\right)$

$= 3(2x) + 4y(1) + 0$

$= 6x + 4y$

$$\therefore \frac{\partial^2 z}{\partial x^2} = 6x + 4y \qquad (3)$$

$$\text{Now } \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(2x^2 + 2xy - 3y^2\right) \qquad \text{[From eq. (2)]}$$

$$= 2\frac{\partial}{\partial x}\left(x^2\right) + 2y\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial x}\left(3y^2\right)$$

$$= 2(2x) + 2y(1) - 0$$

$$= 4x + 2y$$

$$\therefore \frac{\partial^2 z}{\partial x \partial y} = 4x + 2y \qquad (4)$$

$$\text{Now } \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(3x^2 + 4xy + y^2\right) \qquad \text{[From eq. (1)]}$$

$$= \frac{\partial}{\partial y}\left(3x^2\right) + 4xy\frac{\partial}{\partial y}(y) + \frac{\partial}{\partial y}\left(y^2\right)$$

$$= 0 + 4x(1) + 2y$$

$$= 4x + 2y \qquad (5)$$

$$\text{Now } \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(2x^2 + 2xy - 3y^2\right) \qquad \text{[From eq. (2)]}$$

$$= \frac{\partial}{\partial y}\left(2x^2\right) + 2x\frac{\partial}{\partial y}(y) + 3\frac{\partial}{\partial y}\left(y^2\right)$$

$$= 0 + 2x(1) - 3(2y)$$

$$= 2x - 6y$$

$$\therefore \frac{\partial^2}{\partial y^2} = 2x - 6y \qquad (6)$$

Illustration 5 If $z = x + y$ then find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

Solution

$$z = x = y$$

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(e^{x+y}\right) = e^{x+y}(1+0) = e^{x+y}$$

$$\text{and } \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(e^{x+y}\right) = e^{x+y}(1+0) = e^{x+y}$$

Illustration 6 If c then find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

Solution

Here $z = (4x + 1)(y + 3)$

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(4x+1)(y+3)$$

$$= (y+3)\frac{\partial}{\partial x}(4x+1)$$

$$= (y+3)[4(1) + 0]$$

$$= 4(y+3)$$

Now $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(4x+1)(y+3)$

$$= (4x+1)\frac{\partial}{\partial y}(4x+1)$$

$$= (4x+1)(1+0)$$

$$= (4x+1)$$

$$\therefore \frac{\partial z}{\partial y} = 4x + 1$$

Illustration 7 If $z = \frac{2x+1}{y+9}$ then evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

Solution

$$z = \frac{2x+1}{y+9}$$

Now $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{2x+1}{y+9}\right)$

$$= \frac{1}{y+9}\frac{\partial}{\partial x}(2x+1)$$

$$= \frac{2(1)+0}{y+9} = \frac{2}{y+9}$$

$$\frac{\partial z}{\partial x} = \frac{2}{y+9} \qquad (1)$$

Now $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{2x+1}{y+9}\right)$

$$= (2x+1)\left[\frac{(y+9)(\partial/\partial y)(1) - 1(\partial/\partial y)(y+9)}{(y+9)^2}\right]$$

$$= \left(2x+1\right) = \left(2x+1\right)\left(\frac{\left(y+9\right)0-1\left(1+0\right)}{\left(y+9\right)^2}\right)$$

$$= -\frac{\left(2x+1\right)}{\left(y+9\right)^2}$$

Illustration 8 If $z = x^2 + 4xy - y^2$ then find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Also obtain their values when $x = 1, y = 1$

Solution

Here $z = x^2 + 4xy - y^2$

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(x^2\right) + 4y\frac{\partial}{\partial x}\left(x\right) - \frac{\partial}{\partial x}\left(y^2\right)$$

$$= 2x + 4y\left(1\right)$$

$$\therefore \frac{\partial z}{\partial x} = 2x + 4y$$

$$\therefore \left(\frac{\partial z}{\partial x}\right) = 2\left(1\right) + 4\left(1\right) = 6$$

and $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + 4xy + y^2\right)$

$$= \frac{\partial}{\partial y}\left(x^2\right) + 4y\frac{\partial}{\partial y}\left(x\right) - \frac{\partial}{\partial y}\left(y^2\right)$$

$$= 0 + 4x\left(1\right) - 2y$$

$$\frac{\partial z}{\partial y} = 4x - 2y$$

$$\therefore \left(\frac{\partial z}{\partial y}\right) = 4\left(1\right) - 2\left(1\right) = 4 - 2 = 2$$

Illustration 9 If $z = \frac{x+y}{x-y}$ then prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$

Solution

Here $z = \frac{x+y}{x-y}$

$$\therefore \frac{\partial z}{\partial x} = \frac{\left(x-y\right)\left(\partial/\partial x\right)\left(x+y\right) - \left(x+y\right)\left(\partial/\partial x\right)\left(x-y\right)}{\left(x-y\right)^2}$$

$$= \frac{(x-y)(1+0)-(x+y)(1-0)}{(x-y)^2}$$

$$= \frac{x-y-x-y}{(x-y)^2}$$

$$\therefore \frac{\partial z}{\partial x} = \frac{-2y}{(x-y)^2} \qquad (1)$$

Similarly $\dfrac{\partial z}{\partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{x+y}{x-y}\right)$

$$= \frac{(x-y)(\partial/\partial y)(x+y)-(x+y)(\partial/\partial y)(x-y)}{(x-y)^2}$$

$$= \frac{(x-y)(0+1)-(x+y)(0-1)}{(x-y)^2}$$

$$\therefore \frac{\partial z}{\partial y} = \frac{x-y+x+y}{(x-y)^2}$$

$$\therefore \frac{\partial z}{\partial y} = \frac{2x}{(x-y)^2} \qquad (2)$$

Now L.H.S. $= x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y}$

$$= x\left(\frac{-2y}{(x-y)^2}\right) + y\left(\frac{2x}{(x-y)^2}\right)$$

$$= \frac{-2xy}{(x-y)^2} + \frac{2xy}{(x-y)^2}$$

$$= 0$$

$$= \text{R.H.S.}$$

Illustration 10 If z = $\log(x^2 + y^2)$ then prove that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$

Solution

$$z = \log(x^2 + y^2)$$

$$\therefore \frac{\partial z}{\partial x} = \frac{1}{x^2+y^2} \times \frac{\partial}{\partial x}(x^2 + y^2)$$

$$\therefore \frac{\partial z}{\partial x} = \frac{2x}{\left(x^2 + y^2\right)}$$

Now $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right) = \dfrac{\partial}{\partial x}\left[\dfrac{2x}{\left(x^2 + y^2\right)}\right]$

$$= \frac{\left(x^2 + y^2\right)2 - 2x\left(2x\right)}{\left(x^2 + y^2\right)^2}$$

$$= \frac{2x^2 + 2y^2 - 4x^2}{\left(x^2 + y^2\right)^2}$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = \frac{2y^2 - 2x^2}{\left(x^2 + y^2\right)^2} \qquad (1)$$

Now $z = \log\left(x^2 + y^2\right)$

$$\therefore \frac{\partial z}{\partial y} = \frac{1}{x^2 + y^2}\frac{\partial}{\partial y}\left(x^2 + y^2\right)$$

$$= \frac{2y}{x^2 + y^2}$$

Now $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{2y}{x^2 + y^2}\right)$

$$= \frac{\left(x^2 + y^2\right)2 - 2y\left(2y\right)}{\left(x^2 + y^2\right)^2}$$

$$= \frac{2x^2 + 2y^2 - 4y^2}{\left(x^2 + y^2\right)^2}$$

$$= \frac{2x^2 - 2y^2}{\left(x^2 + y^2\right)^2}$$

$$\therefore \frac{\partial^2 z}{\partial y^2} = \frac{2x^2 - 2y^2}{\left(x^2 + y^2\right)^2} \qquad (2)$$

L.H.S. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2}$

$$= \frac{2y^2 - 2x^2}{\left(x^2 + y^2\right)^2} + \frac{2x^2 - 2y^2}{\left(x^2 + y^2\right)^2}$$

$$= \frac{2y^2 - 2x^2 + 2x^2 - 2y^2}{\left(x^2 + y^2\right)^2}$$

$$= \frac{0}{\left(x^2 + y^2\right)^2}$$

$= 0$

= R.H.S.

Illustration 11 U = $xy + 2x$ is the utility function of a consumer. If the consumer's income is Rs. 60 and the prices of the two commodities are respectively Rs. 4 and Rs. 2, find such quantities of x and y which maximize the utility function subject to the budget restriction.

Solution

Here $U = xy + 2x$ (1)

$I = 60$ (2)

But $I = xpx + ypy$

$60 = 4x + 2y$

$60 - 4x - 2y = 0$ (3)

From (1) and (3), we get the new function F as follows:

$F = \left(x, y, \lambda + 2x + \lambda \left(60 - 4x - 2y\right)\right.$

where λ is a constant. It is, therefore, a Lagrange's multiplier.

Now taking partial derivatives of F with respect to x and y equating them to zero, we get

$$\frac{\partial F}{\partial x} = y + 2 - 4\lambda = 0$$

$\therefore 4\lambda = y + 2$ (4)

and $\frac{\partial F}{\partial x} = x - 2\lambda = 0$

$\therefore 2\lambda = x$ (5)

$\lambda = \frac{x}{2}$ (6)

Now substituting eq. (4), we get

$2x = y + 2$

$\therefore 2x - y = 2$ (7)

$$\therefore \begin{cases} 4x + 2y = 60 \\ 4x - 2y = 4 \end{cases}$$

$\therefore 4y = 56$

$\therefore y = 14$

Now substituting $y = 14$ in eq. (7)

$\therefore 2x - 14 = 2$

$\therefore\ 2x = 16$

$\therefore\ x = 8$

$\therefore$ To maximize the utility, quantities 8 and 14, respectively, of the two commodities should be purchased.

Illustration 12 The utility function of a consumer is $4 = 4x^3y^3$. His budget equation is $x + 2y = 12$. Determine quantities x and y such that the consumer gets maximum satisfaction.

Solution

Here $4 = 4x^3y^3$ (1)

and $x + 2y = 12$

$\therefore\ 12 - x - 2y = 0$ (2)

From equation (1) and (2) we get a new function F as (x, y, λ)

$F = 4x^3y^3 + \lambda(12 - x - 2y)$ (3)

Taking partial derivatives with respect to x and y respectively and equating them to zero, we get

$$\frac{\partial F}{\partial x} = 12x^3y^3 - \lambda = 0$$

$\therefore \lambda = 12x^3y^3$ (4)

and $\dfrac{\partial F}{\partial y} = 12x^3y^3 - 2\lambda = 0$ (5)

Now substituting $\lambda = 12x^3y^3$ in eq. (5), we get

$\therefore 12x^3y^2 - 2\left(12x^3y^3\right) = 0$

$\therefore 12x^3y^2\left(x - 2y\right) = 0$

$\therefore x - 2y = 0$

$\therefore x = 2y$

Now substituting $x = 2y$ in eq. (2)

$\therefore 12 - 2y - 2y = 0$

$\therefore 4y = 12$

$\therefore y = 3$

But $x = 2y$

$\therefore x = 6$

Illustration 13 The utility function U = $(x+2)^{2/3}(y+1)^{1/3}$ and the budget equation is $2x + y = 7$. Find x and y for maximum utility.

Solution

Here utility function

$U = \left(x+2\right)^{2/3}\left(y+1\right)^{1/3}$

and the budget equation is

$2x + y = 7$

$$\therefore 7 - 2x - y = 0 \tag{1}$$

$$F = f\left(x, y, \lambda\right) = \left(x+2\right)^{1/3}\left(y+2\right)^{1/3} + \lambda\left(7 - 2x - y\right)$$

Now $\dfrac{\partial F}{\partial x} = \dfrac{2}{3}\left(x+2\right)^{1/3}\left(y+1\right)^{1/3} - 2\lambda = 0$

$$\therefore \lambda = \frac{1}{3}\left(x+2\right)^{-1/3}\left(y+1\right)^{1/3} \tag{2}$$

and $\dfrac{\partial F}{\partial y} = \left(x+2\right)^{2/3}\dfrac{1}{3}\left(y+1\right)^{1/3-1} - \lambda = 0$

$$\therefore \lambda = \frac{1}{3}\left(x+2\right)^{2/3}\left(y+1\right)^{-2/3} \tag{3}$$

Now comparing the value of from Eqs. (2) and (3) we get

$$\therefore \frac{1}{3}\left(x+2\right)^{-1/3}\left(y+1\right)^{1/3} = \frac{1}{3}\left(x+2\right)^{2/3}\left(y+1\right)^{-2/3}$$

$$\therefore \frac{\left(y+1\right)^{1/3}}{\left(y+1\right)^{-2/3}} = \frac{\left(x+2\right)^{2/3}}{\left(x+2\right)^{-1/3}}$$

$$\therefore \left(y+1\right)^{1/3+2/3} = \left(x+2\right)^{2/3+1/3}$$

$$\left(y+1\right) = \left(x+2\right)^{1/3}$$

$y + 1 = x + 2$

$$x - y = -1 \tag{4}$$

Now from Eqs. (1) and (4) we get

$$\begin{cases} 2x + y = 7 \\ x - y = -1 \end{cases}$$

$3x = 6$

$\therefore x = 2$

Now $x - y = -1$

$\therefore 2 - y = -1$

$\therefore y = 3$

For maximum utility $x = 2$ and $y = 3$

Illustration 14 The total cost of a manufacturing concern is C = $x + 3y$ and its production function is $xy = 75$. Find the value of x and y such that the cost is minimum.

Solution

Here the cost function is C = $x + 3y$ and the production function is $xy = 75$

$\therefore\ F = x + 3y + (75 - xy)$

Now for minimizing the cost, partial derivatives of F with respect to x and y are equating zero.

$$\therefore \frac{\partial F}{\partial x} = 1 + 0 - \lambda y = 0$$

$$\therefore \lambda = \frac{1}{y}$$

and $\dfrac{\partial F}{\partial y} = 0 + 3 - \lambda x = 0$

$$\therefore \lambda = \frac{3}{x}$$

Now comparing both the values of λ we get $\dfrac{1}{y} = \dfrac{3}{x}$

$\therefore\ x = 3y$

Substituting $x = 3y$ in $75 = xy$

$$\therefore 75 = (3y)y$$

$$\therefore y^2 = 25$$

$$\therefore\ y = 5$$

But $x = 3y = 3(5) = 15$

$\therefore\ x = 15$ and $y = 5$ the cost will be minimum.

Illustration 15 The cost function of a commodity is C = $2x + 5y$, where x shows the labour and y shows the capital and its production function is $x\sqrt{y} = 625$. Find the value of x and y such that the cost is minimum.

Solution

Here C = $2x + 5y$ and $x\sqrt{y} = 625$

$$\therefore F = 2x + 5y + \lambda\left(625 - x\sqrt{y}\right)$$

$$\therefore \frac{\partial F}{\partial x} = 2 - \lambda\sqrt{y} = 0$$

$$\therefore \lambda = \frac{2}{\sqrt{y}} \qquad (1)$$

and $\dfrac{\partial F}{\partial y} = 5 + \lambda\left(0 - \dfrac{x}{2\sqrt{y}}\right) = 0$

$$\therefore 5 - \frac{\lambda x}{2\sqrt{y}} = 0$$

$$\therefore \lambda = \frac{10\sqrt{y}}{x} \qquad (2)$$

By comparing (1) and (2) we get

$$\frac{2}{\sqrt{y}} = \frac{10\sqrt{y}}{x}$$

$$\therefore\ 2x = 10y$$

$$\therefore\ x = 5y$$

But $x\sqrt{y} = 625$

$$\therefore 5y\sqrt{y} = 625$$

$$\therefore y\sqrt{y} = 125$$

$$\therefore y^{3/2} = (5)^3$$

$$\therefore y = (5)^2 = 25$$

But $x = 5y = 5(25) = 125$

ANALYTICAL EXERCISES

1. If $f(x,y) = x^2 + 7xy + 2y^2$ then find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

2. If $z = x^2 + 7xy + y^2 + 2x + 5y + 7$ then find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

3. If $f(x,y) = 2x^2 - 3xy + 2y^2$ then find $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$

4. If $z = x^3 + x^2y + xy^2 + y^3$ then find $\frac{\partial^2 z}{\partial x^2}$; $\frac{\partial^2 z}{\partial y^2}$; $\frac{\partial^2 z}{\partial x \cdot \partial y}$ and $\frac{\partial^2 z}{\partial y \cdot \partial x}$

5. If $f(x,y) = 2e^{3x+y}$ then obtain $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

6. If $f(x,y) = (5x+2)(3y+1)$ then obtain $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

7. If $f(x,y) = \frac{3x+1}{y+2}$ then obtain $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

8. If $z = 3x^2 + 7xy - 2y^2$ then find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Also obtain their values when $x = 2; y = 3$

9. If (1) $z = x^3y^3$ and (2) $z = ax^3 + hx^2y + by^3$ then obtain $\frac{\partial^2 z}{\partial x}$; $\frac{\partial^2 z}{\partial x \cdot \partial y}$; $\frac{\partial^2 z}{\partial y \cdot \partial x}$ and $\frac{\partial^2 z}{\partial y^2}$

10. If $z = 3xy - y^3 + \left(y^2 - 2x\right)^{3/2}$ then check that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

11. If $z_1 = x^3 - 3xy^2$ and $z_2 = 3x^2y - y^3$ then prove that $\frac{\partial^2 z_1}{\partial x^2} + \frac{\partial^2 z_1}{\partial y^2} = \frac{\partial^2 z_2}{\partial x^2} + \frac{\partial^2 z_2}{\partial y^2}$

12. If $f(x,y) = \log(x+y)$ then prove that $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2}$

13. If $z = \log\left(\frac{x^2 + y^2}{xy}\right)$ then prove that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

14. If $z = \frac{y}{x} \log x$ then prove that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

15. If $f(x,y) = \log\left(x^3 + y^3 - x^2y - xy^2\right)$ then prove that $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = \frac{2}{x+y}$

16. If $f(x,y) = \frac{x}{x^2 + y^2}$ then prove that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = -f(x,y)$

17. If $z = x^2 \log\left(\frac{y}{x}\right)$ then prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$

18. If $z = 3x^2 + 4xy + 3y^2$ then prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$

19. If $z = x^3 + y^3 + x^{3/2} y^{3/2}$ then prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$

20. If $z = \frac{xy}{x^2 + y^2}$ then prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$

21. If $z = \frac{x^2 + y^2}{2x^3 + y^3}$ then prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -z$

22. If $f(x,y) = \log(x+y)$ then prove that $fxy = fyx$

23. If $f(x,y) = x^3 + 2x^2y + xy^2 - y^3$ then prove that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

24. If $f(x,y) = x^3 + x^2y + xy^2$ then show that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

25. In manufacturing of a commodity the labour x and the capital y are utilized. If the cost function of a commodity is $c = 4x + 5y$ and the production is $40{,}000 = xy^2$, find the values of x and y such that the cost is minimum.

26. The demand function of a particular commodity is $y = 15e^{-x/3}\left(0 \le x \le 8\right)$ where y is the price per unit and x is the number of units determined. Determine the price and the quantity for which the revenue is maximum.

27. The utility function of a consumer is $U = -2x^3 + 5xy - y^2$. The price of commodity A is Rs. 1 per unit and that of commodity B is Rs. 3 per unit. If the total budget of that person is Rs. 8, find the quantities of the two commodities he should purchase to derive maximum utility. What will be the maximum utility?

28. The utility function is $U = 24x + 48y - x^2 - y^2$ and the budget equation is $x + 3y = 14$. Find the values of x and y so that the consumer gets maximum utility.

29. The utility function is $4 = 48 - (x-5)^2 - 3(y-4)^2$ and the budget equation is $x + 3y = 9$. Find the values of x and y so that the consumer gets maximum utility.

30. The utility function of a consumer is $U = 108 - (x-6)^2 - 2(y-6)^2$ and his total budget is Rs. 25. The consumer purchases commodities A and B in quantities x and y respectively and their prices per unit are Rs. 3 and 4 respectively. Find the quantities of A and B such that the consumer gets maximum utility.

31. The utility function of a consumer is $U = xy$ and his total income is Rs. 100. The price of commodity A is Rs. 2 per unit and that of commodity B is Rs. 5 per unit. x and y are the units purchased of commodity A and B respectively. Find the values of x and y so that the consumer gets maximum utility.

32. The utility function is $U = x^{1/3} y^{2/3}$. The prices of two commodities are Rs. 1 and Rs. 4 per unit respectively. The budget for these commodities is Rs. 12. Find the values of x and y so that the consumer gets maximum utility.

33. The utility function of a consumer is $U = x^{2/3} y^{2/3}$ and his total income is Rs. 100. The price of commodity A is Rs. 2 per unit and that of commodity B is Rs. 5 per unit. Find the values of x and y such that the utility is maximum.

34. The cost function of a manufacturer is $c = 2x + 3y$ and the production is $xy = 600$. Find the values of x and y such that the cost is minimum.

35. In manufacturing a commodity the labour x and the capital y are utilized. If the cost function of a commodity is $c = 4x + 4y$ and the production function is $40 = x^{2/3} y^{1/3}$, find the values of x and y so that the cost is minimum.

ANSWERS

(1) $\frac{\partial f}{\partial x} = 2x + 7y$; $\frac{\partial f}{\partial y} = 7x + 4y$

(2) $\frac{\partial z}{\partial x} = 2x + 7y + 2$; $\frac{\partial z}{\partial y} = 7x + 2y + 5$

(3) 4 ; 4

(4) $6x + 2y$; $2x + 6y$; $2x + 2y$; $2x + 2y$

(5) $6e^{3x+y}$; $2e^{3x+y}$

(6) $5(3y + 1)$; $3(5x + 2)$

(7) $\frac{3}{y+2}$; $-\frac{(3x+1)}{(y+2)^2}$

(8) $\frac{\partial z}{\partial x} = 6x + 7y$; 33; $\frac{\partial z}{\partial y} = 7x - 4y$; 2

(9) (1) $2y^3$; $6xy^2$; $6xy^2$; $6x^2y$
(2) $6ax + 2hy$; $2hx$; $2hx$; $6by$

(25) $x = 25$; $y = 40$

(26) When $x = 3$, price for maximum revenue $= \frac{15}{e}$

(27) $x = 2$; $y = 2$, maximum utility = 8

(28) $x = 5$; $y = 3$

(29) $x = 3$; $y = 2$

(30) $x = 3$; $y = 4$

(31) $x = 25$; $y = 10$

(32) $x = 4$; $y = 2$

(33) $x = 25$; $y = 10$

(34) $x = 30$; $y = 20$

(35) $x = 80$; $y = 10$

DEFINITIONS OF SOME FUNCTIONS IN ECONOMICS

TOPIC A

Demand Function

It is the relationship between demand and price of a commodity. If the price of a commodity x is p and its demand is q then this fact can be expressed mathematically as

$$q = \phi\ (\mathrm{p}),\ \mathrm{p} > 0,\ q > 0$$

Cost Function

The amount spent on the production of a commodity 'x' is called its *cost*. If 'c' denotes the production cost of x units of the commodity x then this fact can be expressed as $c(x) = \phi\ (x),\ x > 0$.

Revenue Function

Revenue obtained by selling the output produced is denoted by R. Suppose x units of output are sold at the rate of Rs. q per unit. Then the revenue due to these x units is $q \times x$.

$$\therefore R(x) = q \times x = (\text{output}) \times (\text{price}),\ q > 0,\ x > 0$$

Profit Function

The excess of total revenue over the total cost of production is called the *profit* and it is denoted by p.

$$p(x) = R(x) - c(x),\ x > 0$$

where $R(x)$ and $c(x)$ denote respectively the revenue and cost functions.

Production Function

The production of a firm depends upon factors such as investment, labor, machinery, raw materials, etc. The production function, therefore, is a function of more than one variable. Thus if PF denotes the production function then

$$PF = \mathrm{f}(\mathrm{L, k, m, r}),\ \mathrm{L, k, m, r} > 0$$

where L, k, m, r denote respectively labor, investment, machinery and raw material.

Demand Function

The demand function expresses a relationship between the unit price that a product can sell for and the number of units that can be sold at that price. It is important to note that if p represents the price per unit sold, $\frac{dp}{dx} < 0$.

Average Rate of Change of c

Let the cost function $c = f\ (q)$. It gives the total cost of producing and marketing q units of a product x. If the production level is increased by Δq units to $q + \Delta q$ then c becomes $f\ (q + \Delta q)$.

$$\therefore \ \frac{\Delta p}{\Delta q} = \frac{\text{change in total cost } c}{\text{change in output } q} = \frac{f(q+\Delta q) - f(q)}{\Delta q}$$

Now the quantity is called the average rate of change of c with respect to the output v over the interval $(q + \Delta q, q)$

Average Cost

The average concept expresses the variation of one quantity over a specified range of the values of the second quantity.

Let the total cost c of the output x be given by $c = f(x)$. Then the average cost is defined as the ratio of the total cost to the output and it is denoted by AC.

$$\therefore \text{Average cost } (\text{AC}) = \frac{\text{Total cost}}{\text{output}} = \frac{c}{x}$$

$$\therefore \text{Total cost } (c) = (\text{AC}) \times x$$

Marginal Cost

The marginal cost MC, is defined with respect to the quantity 'x' as

$$\text{Marginal cost } (\text{MC}) = \frac{dc}{dx}$$

In other words, marginal cost at a certain level of output is the change in cost that results when the output is increased by a unit amount of output from this level.

ILLUSTRATIONS

Illustration 1 The total cost function of a commodity is given by $c(x) = 0.5x^2 + 2x + 50$, where c denotes the total cost and x denotes the quantity produced. Find the average cost and the marginal cost.

Solution

$$\text{AC} = \frac{c(x)}{x} = \frac{0.5x^2 + 2x + 50}{x} = 0.5x + 2 + \frac{50}{x}$$

$$\text{MC} = \frac{d}{cx}c(x) = \frac{d}{dx}\left(0.5x^2 + 2x + 2\right) = x + 2$$

Illustration 2 If $\bar{c} = 0.05q^2 + 16 + (100/q)$ is the manufacturer's average cost function what is the marginal cost when 50 units are produced. Also interpret the result.

Solution

Total cost = Average cost × quantity produced

$$c = \bar{c} \times q = 0.05q^3 + 16q + 100$$

$$\therefore \text{MC} = \frac{dc}{dq} = \frac{d}{dq}\left(0.05q^3 + 16q + 100\right)$$

$$= 0.05\left(3q^2\right) + 16 = 0.15q^2 + 16$$

When 50 units are produced, the marginal cost is

$$\left(\frac{dc}{dq}\right)_{\text{at } q=50} = 0.15(50)^2 + 16 = 391$$

Illustration 3 The total cost $c(x)$ of a firm is $c(x) = 0.0005x^3 - 0.02x^2 - 30x + 5000$ where x is the output. Determine (i) average cost (2) slope of AC (3) Marginal cost (MC) (4) slope of MC (5) value of x for which MVC = AVC where VC denotes variable cost.

Solution

Here $c(x) = 0.0005x^3 - 0.02x^2 - 30x + 5000$

Variable cost (VC) = $0.0005x^3 - 0.02x^2 - 30x$

1. Average cost (AC) = $\dfrac{c(x)}{x} = 0.0005x^2 - 0.02x - 30 + \dfrac{5000}{x}$

2. Slope of AC = $\dfrac{d}{dx}(\text{AC}) = \dfrac{d}{dx}\left(0.0005x^2 - 0.02x - 30 + \dfrac{5000}{x}\right)$

$$= 0.001x - 0.02 - \frac{5000}{x^2}$$

3. Marginal cost = NC = $\dfrac{dc}{dx}$

$$= \frac{d}{dx}\left(0.0005x^3 - 0.02x^2 - 30x + 5000\right)$$

$$= 0.0015x^2 - 0.04x - 30$$

4. Slope of NC = $\dfrac{d}{dx}(\text{NC}) = \dfrac{d}{dx}\left(0.0015x^2 - 0.04x^2 - 30\right)$

$$= 0.003x - 0.04$$

5. Average variable cost = $\dfrac{vc}{x} = \dfrac{0.0005x^3 - 0.02x^2 - 30x}{x}$

$\therefore$ AVC = $0.0005x^2 - 0.02x - 30$

Value of x for which MVC = AVC

$\Rightarrow 0.0015x^2 - 0.04x - 30 = 0.0005x^2 - 0.02x - 30$

$\Rightarrow 0.001x^2 - 0.02x = 0$

$\Rightarrow x(0.001x - 0.02) = 0$

$\Rightarrow x = 0$ or $x = 20$

But $x \neq 0 \therefore x = 20$

Illustration 4 If the total cost function is given by $c(x) = a + bx + cx^2$ where x is the quantity of output show that $\dfrac{d}{dx}(\text{AC}) = \dfrac{1}{x}(\text{MC} - \text{AC})$ where MC and AC are marginal and average costs.

Solution

$c(x) = a + bx + cx^2$

$\therefore$ AC = $\dfrac{c(x)}{x} = \dfrac{a}{x} + b + cx$, MC $= \dfrac{dc}{dx} = b + 2cx$

$$\therefore \frac{1}{x}\left(\text{MC}-\text{AC}\right)=\frac{1}{x}\left[\left(b+2cx\right)-\left(\frac{a}{x}+b+cx\right)\right]$$

$$=\frac{1}{x}\left(cx-\frac{a}{x}\right)=c-\frac{a}{x^2} \quad (1)$$

Also $\frac{d}{dx}\left(\text{AC}\right)=\frac{d}{dx}\left(\frac{a}{x}+b+cx\right)=c-\frac{a}{x^2}$ (2)

From (1), (2) $\frac{d}{dx}\left(\text{AC}\right)=\frac{1}{x}=\frac{1}{x}\left(\text{MC}-\text{AC}\right)$

TOPIC B

Relation Between AC and MC

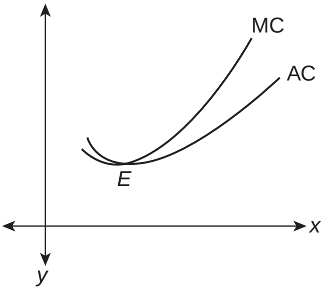
Consider a firm producing a single commodity. The total cost $c(x)$ of production is the function of its output say $c = f(x)$, where x is the quantity of output.

The average cost (AC) and the marginal cost (MC) are given by

$\text{AC}=\frac{c}{x}$, $\text{MC}=\frac{dc}{dx}$

Now $\frac{d}{dx}\left(\text{AC}\right)=\frac{d}{dx}\left(\frac{c}{x}\right)=\frac{x\left(dc/dx\right)-c\times 1}{x^2}$

$\therefore \frac{d}{dx}\left(\text{AC}\right)=\frac{1}{x}\left(\frac{dc}{dx}-\frac{c}{x}\right)=\frac{1}{x}\frac{d}{dx}\left(\text{MC}-\text{AC}\right)$



Interpretation

(1) If AC curve is rising then $\frac{d}{dx}\left(\text{AC}\right)>0$

$\Rightarrow \frac{1}{x}\left(\text{MC}-\text{AC}\right)>0 \Rightarrow \text{MC}-\text{AC}>0$
$\Rightarrow \text{MC}>\text{AC}$

(2) If AC curve is falling then $\frac{d}{dx}\left(\text{AC}\right)<0$

$\Rightarrow \frac{1}{x}\left(\text{MC}-\text{AC}\right)<0 \Rightarrow \text{MC}-\text{AC}<0 \Rightarrow \text{MC}<\text{AC}$

Thus when AC decreases, MC < AC.

(3) On the lowest point of AC curve

$\frac{d}{dx}\left(\text{AC}\right)=0 \Rightarrow \text{MC}-\text{AC}=0 \Rightarrow \text{MC}=\text{AC}$

Thus MC = AC at the lowest point of AC curve.

TOPIC C

Average Revenue and Marginal Revenue

The total revenue R from the sale of x units produced is given by $R = p \times x$ where p is price per unit.

Average Revenue

Average revenue AR is defined as

$$\text{AR} = \frac{\text{Total revenue}}{\text{Quantity}} \Rightarrow \text{AR} = \frac{R}{x}$$

$$\text{Since } R = p \times x \Rightarrow \text{AR} = \frac{R}{x} = \frac{px}{x} = p$$

Thus average revenue is the price per unit.

Marginal Revenue

Marginal revenue is defined as the rate of change of total revenue with respect to the quantity demand, MR $= \dfrac{dR}{dx}$

It indicates the rate at which revenue changes with respect to the units sold.

Marginal revenue is also interpreted as the approximate revenue received by selling additional units of output of R.

$$\text{Relative rate of change of } R = \frac{dR}{dx} \times \frac{1}{R}$$

$$\text{Per cent rate of change of revenue } R = \frac{dR}{dx}\frac{100}{R}$$

We now give below the brief marginal analysis of a firm operating under **perfect competition** and **under monopoly.**

(I) *Perfect Competition*

In perfect competition, the price p is constant. Then

1. $\text{AR} = \dfrac{R}{x} = \dfrac{px}{x} = p = \text{constant}$
2. $\text{MR} = \dfrac{dR}{dx} = \dfrac{d}{dx}(px) = \text{constant}$

Thus under perfect competition
AR = MR $= p =$ constant

(II) *Monopoly*

A monopolist is a sole supplier of goods produced. In this case, he fixes the price p of the goods according to the demand in the market. Thus under monopolist, the price p of the goods is a function of demand and it is not a constant.

In this case

(I) Average cost $R = \dfrac{px}{x} = p$ which is not a constant but p is a function of x.

(II) $\text{MR} = \dfrac{dR}{dx} = \dfrac{d}{dx}(px) = p + x\dfrac{dp}{dx}$

Thus in this case AR $\neq$ MR.

TOPIC D

Marginal Revenue Product

Let a factory have t employees, who produce a total of x units of a product per day. Then x is a function of t. Let R be the total revenue obtained by selling 'x' units. Then R can also be considered as a function of t.

Let $\frac{dR}{dt}$ be the rate of change of revenue with respect to the number of employees.

Then $\frac{dR}{dt}$ is called the *marginal revenue product*. It is interpreted as the change in revenue when the factory bears one extra employee.

Now $R = px$ where p is the price per unit.

$$\frac{d\mathrm{R}}{dt} = \frac{d}{dx}(px) = p\,\frac{dx}{dt} + x\,\frac{dp}{dx}\frac{dx}{dt}$$

$$= \frac{dx}{dt}\left(p + x\frac{dp}{dx}\right)$$

$$\therefore \text{Marginal revenue product}(\text{MRP}) = \frac{dx}{dt}\left(p + x\frac{dp}{dx}\right)$$

Illustration 5 Let p be the price per unit of a certain product. When there is a sale of q units, the relation between p and q is given by $p = \frac{100}{3q+1} - 4$

1. Find the marginal revenue function.
2. When $q = 10$ find the relative change of R and also the percentage rate of change of R at $q = 10$.

Solution

Revenue R for sale of q units at price p is given by

$$R = pq = \left(\frac{100}{3q+1} - 4\right)q = \frac{100q}{3q+1} - 4q$$

(1) Marginal revenue function

$$\frac{dR}{dq} = \frac{(3q+1)100 - 100q \times 3}{(3q+1)^2} - 4 = \frac{100}{(3q+1)^2} - 4$$

(2) Relative rate of change of R $\frac{(dR/\mathrm{dq})}{R}$

$$= \frac{\left[100/(3q+1)^2\right] - 4}{\left[100q/(3q+1)\right] - 4q} = \frac{100 - 4(3q+1)^2}{(3q+1)(96q - 12q)}$$

When $q = 10$ the relative rate of change of

$$R = \frac{100-4(31)^2}{31(966-1200)} = \frac{3744}{7440} = 0.503$$

Percentage rate of change of R at $q = 10$ is

$$\frac{dR/dq}{R} \times 100 = 0.503 \times 100 = 50.3$$

Illustration 6 The revenue R due to the sale of q units of a product is given by $R = 250q - 05q^2$. (1) How fast does R change with respect to q? (2) When $q = 10$ find the relative rate of change of R and the percentage rate of change of R.

Solution

$$R = f(q) = 25q - 0.5q^2$$

Rate of change of $R = \frac{dR}{dq} = 25 - q$

Relative rate of change of R when $q = 10$ is

$$\frac{f^1(q)}{f(q)} = \left.\frac{25-q}{25q-0.05q^2}\right\}_{q=10} = \frac{25-10}{250-50} = \frac{15}{245} = \frac{3}{40}$$

Percentage rate of change of R at $q = 10 = \frac{f^1(q)}{f(q)} \times 100$

$$= \frac{3}{40} \times 100 = \frac{300}{40}\%$$

Illustration 7 If $p = \frac{5}{q-5} - 6q$ represents the demand function for a product, where p is the price per unit of q units, find the marginal revenue.

Solution

Revenue R received for selling q units $= p \times q$

$$\therefore R = pq = \left(\frac{5}{q-5} - 6q^2\right)q = \frac{5q}{q-5} - 6q^2$$

$$\therefore \text{MR} = \frac{d\text{R}}{dq} = \frac{d}{dq}\left(\frac{5q}{q-5} - 6q^2\right)$$

$$= \frac{5(q-5) - 5q(1)}{(q-5)^2} - 12q$$

$$= -\left[\frac{25}{(q-5)^2} + 12q\right]$$

Illustration 8 The total revenues received from the sale of x units of product is given by $R(x) = 200 + \frac{x^2}{5}$ Find. (1) AR, (2) AM, (3) MR when $x = 25$, (4) the actual revenue from the sale of 26 units.

Solution

Revenue $R = p \times x = 200x + \frac{x^3}{5}$

(1) AR $= \frac{R}{x} = \frac{200+(x^2/5)}{x} = \frac{200}{x} + \frac{x}{5}$

(2) AM $= \frac{dR}{dx} = \frac{d}{dx}\left(200 + \frac{x^2}{5}\right) = \frac{2x}{5}$

(3) MR when $x = 25\left(\frac{dR}{dx}\right)_{x=25} = \frac{2}{5}25 = 100$

(4) Revenue from sale of 26th unit

$= c(26) - c(25)$

$= \left(200 + \frac{26^2}{5}\right) - \left(200 + \frac{25^2}{5}\right) = 10.2$

Illustration 9 Suppose the demand per month for a commodity is 24 if the price is Rs. 16 and 12 if the price is Rs. 22. Assuming that demand curve is linear, determine (1) demand function, (2) the total revenue function and (3) the marginal revenue function.

Solution

(1) Let the linear demand function be

$p = a + bx$

It is given that

$$\begin{cases} a + 24b = 16 \\ a + 12b = 22 \end{cases} \Rightarrow a = 28, b = -\frac{1}{2}$$

$p = 28 - \frac{1}{2}x$

(2) The total revenue function is $R = px = 28x - \frac{x^2}{2}$

(3) MR $= \frac{dR}{dx} = \frac{d}{dx}\left(28x - \frac{x^2}{2}\right) = 28 - x$

Illustration 10 Find the total revenue and marginal revenue for a firm operating under perfect competition with a current price of Rs. 5 per unit.

Solution

Revenue $R = p \times x = 5x$

Now MR $= \frac{dR}{dx} = 5$ which is constant.

Illustration 11 Find the MR function for a monopolist confronted with the demand $x = 12 - 3p$

Solution

Here $x = 12 - 3p \Rightarrow p = 4 - \frac{x}{3}$

Revenue $= R = px = 4x - \frac{x^2}{3}$

Marginal revenue MR $= \frac{dR}{dx} = 4 - \frac{2x}{3}$

Illustration 12 A manufacturer determines that t employees will produce a total of x units of a product per day when $x = 2t$. If the demand equation for a product $p = -0.5x + 20$, determine the marginal revenue product when $t = 5$. Interpret your result.

Solution

$$\begin{aligned} R &= p \times x = (-0.5x + 20)x = -0.5x^2 + 20x \\ &= -0.5(2t)^2 + 20(2t) \\ &= -2t^2 + 40t \qquad (\therefore x = 2t) \end{aligned}$$

$$\text{MR} = \frac{dR}{dx} = -4t + 40$$

$$\left(\frac{dR}{dt}\right)_{t=5} = -4(5) + 40 = 26c$$

If sixth employee is burdened, the extra revenue generated is approximately 20.

TOPIC E

Consumption Function, MPC and MPS

Consumption Function

The consumption function expresses a relationship between the total income I and the total rational consumption C. It is denoted by $c = f(I)$

Marginal Propensity to Consume (MPC)

It is the rate of change of the consumption w.r. to income.

$\therefore$ Marginal propensity to consume (MPC) $= \frac{dc}{dt}$

Marginal Propensity to Save (MPS)

Let s denote the saving; then saving S = Total income − Total consumption.

$$\therefore S = I - C$$

Marginal propensity to save (MPS) $= \frac{ds}{dt}$

Illustration 13 If the consumption function is given by $c = 8 + 9t^{3/2}$, determine the marginal consumption function.

Solution

Consumption function $c = 8 + 9t^{3/2}$

Marginal consumption function = $\frac{dc}{dt} = \frac{27}{2}t^{1/2}$

Illustration 14 The consumption function $c = f(I)$ gives relationship between total income (I) and total consumption (C). What is meant by marginal propensity to consume and marginal propensity to save? If $c = 5 \times 3\sqrt{I}$, determine the marginal propensity to save when $I = \sqrt{27}$

Solution

$c = 5 \times \mathrm{I}^{1/3}$ and $S = I - c = I - 5 \times I^{1/3}$

Marginal propensity to save $\frac{dS}{dI} = 1 - \frac{5}{3} I^{-2/3}$

When $I = \sqrt{27}$, $\frac{dS}{dI} = 1 - \frac{5}{3}\left(\sqrt{27}\right)^{-2/3} = 1 - \frac{5}{3 \times 3} = \frac{4}{9}$

Illustration 15 If the consumption is given by $c = 7I + 15\sqrt{I}$, where I is the income, when $I = 25$, determine (1) the marginal propensity to consume, (2) marginal propensity to save.

Solution

Here $c = 7I + 15\sqrt{I}$

∴ Marginal propensity to consume

$$\frac{dc}{dI} = \frac{d}{dI}\left(7I + 15\sqrt{I}\right) = 7 + 15\,\frac{1}{2\sqrt{I}}$$

When $I = 25 \Rightarrow \left(\frac{dc}{dI}\right)_{I=25} = 7 + \frac{15}{2\sqrt{25}} = 7 + \frac{15}{2.5} = 7 + \frac{3}{2} = \frac{17}{2}$

Also marginal propensity to save

$$\frac{dS}{dI} = \frac{d}{dI}\left(I - c\right) = 1 - \left(7 - \frac{15}{2\sqrt{I}}\right)$$

When $I = 25$ then $\left(\frac{dS}{dI}\right)_{I=25} = 1 - \left(7 + \frac{25}{10}\right) = -6 - \frac{5}{2}$

$$= -\frac{17}{2}$$

TOPIC F

Rate of Growth

The rate of growth g of continuous function over a time is given by the ratio of the derivative of the function to the function, i.e.

$$g_c = \frac{f^1(t)}{f(t)}$$

Where $f^1(t)$ is the change in the function for a very small increment of time while the denominator $f(t)$ is the base value of the function at time t.

Growth Rate of Discrete Function

It is denoted by g_d and is defined as

$$g_d = \left[\frac{\Delta f(t)}{\Delta(t)}\right] \div f(t)$$

Illustration 16 Determine the growth rate for the continuous function $f(t) = t^2 + 3t + 2$; interpret the result.

Solution

$$f^1(t) = \frac{d}{dt}(t^2 + 3t + 2) = 2t + 3$$

$$\text{Growth rate } g_{(c)} = g_d = \frac{f^1(t)}{f(t)} = \frac{2t+3}{t^2+3t+2} = \phi(t)$$

Interpretation

Since $\phi(t)$ is a function of t, so the growth rate depends upon the time t.

Illustration 17 If Rs. 180 is received quarterly on an investment of Rs. 18,000 find the rate of interest.

Solution

Here $\Delta f(t) = 180$, $f(t) = 18{,}000$, $\Delta t = \frac{1}{4}$

Rate of interest = Average rate of growth

$$= \frac{\Delta f(t)}{\Delta(t)} \div f(t) = \frac{\Delta f(t)}{\Delta t\ f(t)} = \frac{180}{(1/4)18000} = \frac{1}{100} = 4\%$$

ANALYTICAL EXERCISES

1. If $\bar{c} = 0.01q + 5 + \frac{500}{q}$ is the manufacturer's average cost equation, find the marginal cost function. What is the marginal cost, when 100 units are produced? Interpret the result.
2. The total cost $c(x)$ of *a* firm is $c(x) = 0.0005x^3 - 0.7x^2 - 30x + 3000$, where x is the output. Determine (1) AC, (2) Slope of AC, (3) MC, (4) Slope of MC, (5) value of x for which MVC = AVC where VC denotes the variable cost.
3. Prove that marginal cost is $\frac{1}{x}$ (MC − AC) for the total cost function $c(x) = 3x^3 + 2x^2 + 4x + 7$.
4. If $c(x)$ rupees is the total cost of manufacturing x toys and $c(x) = 500 + \frac{50}{x} + \frac{x^2}{10}$, find the average cost, the marginal cost, when $x = 20$ and the actual cost of manufacturing the first toy is Rs. 20.

5. The total cost function TC = $\sqrt{ax^2 + b} - c,$ Find MC. Show that MC increases as output increases.

6. If $p = \dfrac{100}{q+2} - 4$ represents the demand function for a product p being the price per unit of q units, find the MR function.

7. The price p per kg when x kg of a certain commodity is demanded is $p = \dfrac{1000}{5x+100} - 5,\ 0 < x < 90.$ Find

(1) Rate of change of price with respect to x
(2) The revenue function
(3) The marginal revenue
(4) The MR at $x = 10$ and $x = 40$
(5) Interpret the answer to (4)

8. The monopolist's demand curve is $p = 300 + 5x$. Find (1) MR function, (2) the relationship between the slope of MR and AR. (3) At what price is the MR '0'?

9. Show that the marginal revenue can always be expressed as $p + x\dfrac{dp}{dx}$. Deduce that the slope of the demand curve is numerically equal to $\dfrac{p}{x}$ at the output when MR = 0.

10. What is the MR function for the demand law $p = \sqrt{a - bx}$? When is MR = 0

11. A manufacturer determines t employees will produce a total of a product per day, where $x = 200t - \dfrac{t^2}{20}$. If the demand equation for the product is $p = -0.1x + 70$ determine the MR of product when $t = 40$

12. If $c = 7 + 0.6I - 0.25\sqrt{I}$ is a consumption function, find MPC and MPS when $I = 16$.

13. If the consumption function is given by $c = 6 + \dfrac{3I}{4} - \dfrac{\sqrt{I}}{3}$, determine the MPC and MPS when $I = 25$.

14. Determine the growth rate for the continuous function for $f(t) = 9t^4$.

15. If Rs. 60 is received semi-annually on an investment of Rs. 1,000, find the rate of interest.

ANSWERS

(1) $\dfrac{dc}{dq} = 0.02q + 5$, Marginal cost when $q = 100 = 7$

(2) (1) AC $= 0.005x^2 - 0.7x - 30 + \dfrac{3000}{x}$

(2) $0.001x - 0.7 - \dfrac{3000}{x^2}$

(3) MC $= 0.0015x^2 - 1.4x - 30$

(4) $0.003x - 1.4$

(5) $x = 700$.

(4) AC = 27.125, MC, = 3.875

(5) $\dfrac{d}{dx}\text{MC} = \dfrac{ab}{(ax2+b)^{3/2}} > 0$

(6) $\dfrac{d\text{R}}{dq} = \dfrac{200}{(q+2)^2} - 4$

(7) (1) $\frac{dp}{dx}=\frac{-50000}{(5x+100)^2}$

(2) $R(x)=\frac{10,000x}{5x+100}-5x$

(3) $\text{MR}=\frac{1,00,000}{(5x+100)^2}-5$

(8) (1) MR = 300 + 10x,

(2) 5101x of MR = .2,

(3) p = 150

(10) $\frac{dR}{dx}=\frac{2a-3bx}{2-bx}$, $x=\frac{2a}{3b}$

(11) 36

(12) MPC = 0.569, MPS = 0.431

(13) MPC = 0.716, MPS = 0.284.

(14) $g_c=\frac{4}{t}$

(15) 12%

MAXIMA AND MINIMA CONCEPT

A sufficient condition for maxima and minima:

If $f(x)$ possesses continuous derivatives up to the second order in a certain neighbourhood of a point c then

(1) $f(x)$ is maximum at $x = c$ if $f^{1}(c) = 0$ and $f^{11}(c) < 0$

(2) $f(x)$ is minimum at $x = c$ if $f^{1}(c) = 0$ and $f^{11}(c) > 0$

ILLUSTRATIONS

Illustration 1 Given the perimeter of a rectangle show that its area is maximum where it is a square.

Solution

Let x and y be the length and breadth of a rectangle and A be its area.

$\therefore$ The perimeter of the rectangle $= 2(x + y) = 2c$(say)

$\therefore x + y = c \Rightarrow y = c - x$

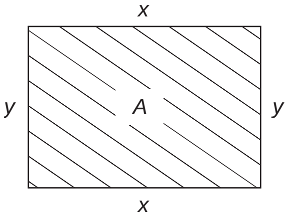
Now the area of rectangle A $= xy = x(c - x) = cx - x^2$

$\therefore \frac{dA}{dx}=c-2x$

Now $\frac{dA}{dx}=0\Rightarrow c-2x=0 \quad \Rightarrow x=\frac{c}{2}$

But $\frac{d^2A}{dx^2} = -2 < 0$

$\therefore$ A is maximum where $x=\frac{c}{2}\Rightarrow y=\frac{c}{2}$



Hence the area is maximum when the sides of the rectangle are equal, i.e. when it is a square.

Illustration 2 The cost c of manufacturing a certain article is given by $c = q^2 - 4q + 100$, where q is the number of articles manufactured. Find the maximum value of c.

Solution

$c = q^2 - 4q + 100 \Rightarrow \frac{dc}{dq}=2q-4$

Now for the maximum $\Rightarrow \frac{dc}{dq} = 0 \Rightarrow 2q - 4 = 0 \Rightarrow q = 2$

also $\frac{d^2c}{dq^2} = 2 - 0 = 2 > 0$ its given maximum costs.

$\therefore$ maximum cost $= (2)^2 - 4(2) + 100$

$= 4 - 8 + 100$

$=$ Rs. 96

Illustration 3 The profit $p(x)$ due to advertising x, in hundreds of rupees is given by $p(x) = 120 + 80x - x^2$

(1) What amount of advertising fetches maximum profit?

(2) What is the maximum profit?

Solution

(1) Here $p(x) = 120 + 80x - x^2$

$\therefore \frac{dp}{dx} = 0 + 80 - 2x$

Now $\frac{dp}{dx} = 0 \Rightarrow 80 - 2x = 0 \Rightarrow x = 40$

Now for maximum profit $\frac{d^2p}{dx^2} = 0 - 2 = -2 < 0$

$\therefore$ Profit is maximum at $x = 40$

(2) Maximum profit $= 120 + 80(40) - (40)2 = 1720$ in hundreds.

Illustration 4 The cost of manufacturing a particular type of ball is given by $c(x) = x^2 - 1200x + 360040$ where x denotes the number of balls produced. How many balls should the company manufacturer at the cost which is minimum and what would be the cost per ball at this level of production?

Solution

$c(x) = x^2 - 1200x + 360040$

$\therefore \frac{dc}{dx} = 2x - 1200$

$\therefore \frac{dc}{dx} = 0 \Rightarrow 2x - 1200 = 0 \Rightarrow x = 600$

Also $\frac{d^2c}{dx^2} = 2 - 0 = 2 > 0$

Hence the cost is minimum when $x = 600$

Also cost per ball at this level

$c(600) = (600)^2 - 1200(600) + 360040$

$=$ Rs. 40 per ball

Illustration 5 A television company charges Rs. 6,000 per unit on an order of 50 sets or less sets. The charge is reduced by Rs. 75 per set for each order in excess of 50. Find the largest size of order the company should allow so as to receive maximum revenue.

Solution

Let x units television be the size of the order

$$R(x) : p \times x = \left[6000 - 75(x - 50)\right]x$$
$$= 6000x - 75x^2 + 3750x$$
$$R(x) = 9750x - 75x^2$$

For $R(x)$ to be maximum or minimum

$$\frac{dR}{dx} = 0 \Rightarrow 9750 - 150x = 0 \Rightarrow x = 65$$

Now $\frac{d^2R}{dx^2} = 0 - 150 = -150 < 0$

$\Rightarrow R(x)$ is maximum when $x = 65$

Hence the required largest order is of 65 sets.

Illustration 6 If $p\left(\frac{100}{q+2} - 2\right)$ represents the demand function for a product, where p is the price per unit of 'q' units, find the marginal revenue. Also determine the limit of 'q' for selling which yields maximum revenue.

Solution

Revenue function $R(q) = p \times q = q\left(\frac{100}{q+2} - 2\right)q$

$$\therefore R(q) = \frac{100q}{q+2} - 2q$$

$$\therefore \text{MR} = \frac{dR}{dq} = \frac{100}{q+2} - \frac{100q}{(q+2)^2} - 2$$

For maximum revenue $\frac{dR}{dq} = 0$

$$\therefore \text{MR} = \frac{100}{q+2} - \frac{100q}{(q+2)^2} - 2 = 0$$

$$\therefore 100(q + 2) - 100q - 2(q + 2)^2 = 0$$
$$\therefore q^2 + 4q - 96 = 0$$
$$\therefore (q + 12)(q - 8) = 0$$
$$\therefore q = -12, 8$$

$q = -12$ is discarded as $q > 0$ so $q = 8$.

Now $\frac{d^2R}{dq^2} = \frac{d}{dq}\left(\frac{dR}{dq}\right) = \frac{100}{(q+2)^2} - \frac{100}{(q+2)^2} + \frac{200q}{(q+2)^3}$

$$\frac{d^2R}{dq^2} = \frac{-400}{(q+2)^3}$$

Now $\left(\frac{d^2R}{dq^2}\right)_{q=8} = \frac{-400}{(q+2)^3} < 0$

$\therefore$ The revenue is maximum when $q = 8$.

Illustration 7 Given a quadratic cost function $c(x) = ax^2 + bx + c$ minimize the average cost and show that the average cost is equal to marginal cost that value.

Solution

$$\text{AC} = \frac{c}{x} = \frac{ax^2+bx+c}{x} = ax+b+\frac{c}{x}$$

$$\therefore \frac{d(\text{AC})}{dx} = a + 0 - \frac{c}{x^2}$$

$$\frac{d(\text{AC})}{dx} = 0 \Rightarrow a - \frac{c}{x^2} = 0 \Rightarrow x = \sqrt{\frac{c}{a}}$$

Now $\frac{d(\text{AC})}{dx} = \frac{2c}{x^3} > 0$ for $x = \sqrt{\frac{c}{a}}$

Average cost is minimum at $x = \sqrt{\frac{c}{a}}$

Minimum average cost

$$= a\sqrt{\frac{c}{a}} + b + c\sqrt{\frac{a}{c}} = 2\sqrt{ac} + b \qquad (1)$$

$$\therefore \text{MC} = \frac{dc}{dx} = \frac{d}{dx}\left(ax^2+bx+c\right) = 2ax+b$$

$$\therefore \text{MC at } x = \sqrt{\frac{c}{a}} = 2a\sqrt{\frac{c}{a}} + a = 2\sqrt{ac} + b \qquad (2)$$

From eqs. (1) and (2) we get

MC = AC at $x = \frac{\sqrt{c}}{a}$

Illustration 8 The manufacturing cost of an article involves a fixed overhead of Rs. 100 per day, Rs. 0.50 for material and $\frac{x^2}{100}$ per day for labor and machinery to produce x articles. How many articles should be produced per day to minimize the average cost per article?

Solution

$$c(x) = 100 + 0.5x + \frac{x^2}{100}$$

$$\text{Average cost } A(x) = \frac{c(x)}{x} = \frac{100 + 0.5x + (x^2/100)}{x}$$

$$= \frac{100}{x} + 0.5 + \frac{x}{100}$$

For $A(x)$ to be minimum $\frac{d}{dx}A(x) = 0$ and $\frac{d^2A(x)}{dx^2} > 0$

Now $\frac{dA(x)}{dx^2} = 0 \Rightarrow \frac{-100}{x^2} + \frac{1}{100} = 0 \Rightarrow x = 100$ articles

Also $\frac{d^2A(x)}{dx^2} = \frac{200}{x^3} > 0$ $(a + x = 100 \text{ articles})$

Also $\frac{d^2A(x)}{dx^2} = \frac{200}{x^3} > 0$ $(at\ x = 100) \Rightarrow A(x)$ is minimum here;
100 articles should be produced to minimize the cost.

Illustration 9 A Company charges Rs. 550 for a transistor set on order of 50 or less sets. The charges are reduced by Rs. 5 per set for each set ordered in excess of 50. Find the largest size of order the company should allow so as to receive maximum revenue.

Solution

Revenue from the sale of each transistor $p = 550 - 5(x - 50)$ if $x \geq 50$

$\therefore$ Total revenue $= R(x) = P \times x$

$$= [550 - 5(x - 50)]x$$

$$\therefore\ R(x) = 800x - 5x^2$$

Now $\frac{dR}{dx} = 800 - 10x$

$\therefore\ \frac{dR}{dx} = 0 \Rightarrow 800 - 10x = 0 \Rightarrow x = 80$

Also $\frac{d^2R}{dx^2} = 10 < 0$ (for the value $x = 80$)

$\Rightarrow$ Revenue is maximum.

Illustration 10 A manufacturer can sell x items per day at a price P rupees each, where $p = 125 - \left(\frac{5}{3}\right)x$. The cost of production for x items is $500 + 13x + 0.2x^2$. Find how much he should produce to have a maximum profit assuming that all items produced can be sold. What's the maximum profit?

Solution

Let c = Total cost for production of x items

R = Total revenue obtained by selling x items

C = 500 + 13x + 0.2x^2

Now profit $p = R - c$

$$= \left(125x - \frac{5}{3}x\right) - (500 + 13x + 0.2x^2)$$

$$= 112x - 500 - \frac{28}{15}x^2$$

The profit p will be maximum if $\frac{dp}{dx} = 0$ and $\frac{d^2p}{dx^2} < 0$

$$\therefore \frac{dp}{dx} = 112 - \frac{56}{15}x$$

$$\therefore \frac{dp}{dx} = 0 \Rightarrow 112 - \frac{56}{15}x = 0 \Rightarrow x = 30$$

Now $\left(\frac{d^2p}{dx^2}\right)_{a+x=30} = -\frac{56}{15} < 0 \Rightarrow$ The profit will be maximum when $x = 30$.

Hence 30 units must be produced every day to get the maximum profit.

Maximum profit $= 112(30) - 500 - \frac{28}{15}(30)^2$

= Rs. 1,180

Illustration 11 The demand function for a particular commodity $y = 15e^{-x/3}$, $0 \le x \le 8$ where y is price per unit and x is the number of units demand. Determine the price and the quantity for which the revenue is maximum.

Solution

$$R = xy = 15xe^{-x/3}$$

$$\frac{dR}{dx} = 15e^{x/3} + 15xe^{-x/3}\left(-\frac{1}{3}\right)$$

$$= 15e^{-x/3} - 5xe^{-x/3}$$

$$\frac{dR}{dx} = 5e^{-x/3}(3 - x)$$

$\therefore$ for maximum revenue $\frac{dR}{dx} = 0 \Rightarrow 5e^{-x/3}(3 - x) = 0$

$x = 3(e^{-x/3} = 0 \Rightarrow x = \infty \text{ not possible})$

$$\therefore \frac{d^2R}{dx^2} = 5\left[e^{-x/3}(-1) + (3 - x)e^{-x/3}\left(-\frac{1}{3}\right)\right]$$

$$= 5e^{-x/3}\left(\frac{x}{3} - 2\right)$$

Now $\left(\frac{d^2R}{dx^2}\right)_{x=3} = 5e^{-1}(1-2) = \frac{-5}{e} < 0$

Revenue R is maximum at $x = 3$

Also the price for which revenue is maximum is

$y = 15e^{-1} = \frac{15}{e}$

Illustration 12 The amount of fuel consumed per hour in running an engine is proportional to the cube of the speed. When the speed is 25 km/h, the fuel consumed is 25 liter per hour at Rs. 5 a liter. Other expenses total Rs. 2,000 per hour. Find the most economical speed.

Solution

Let v be the speed and c be the cost.

Also speed is proportional to the cube of cost.

$\Rightarrow c = kv^3$ k = constant

$v = 25$ km/h the cost $= 25 \times 5 = 125$

$\therefore\ 125 = k(25)3 \Rightarrow k = \frac{1}{125}$

$\therefore\ c = \frac{1}{125}v^3$

$\therefore$ total running cost $= 2000 + \frac{v^3}{125}$

If the distance covered be s kms with speed v km/h, the number of hours consumed in the journey $= \frac{S}{v}$

Let T be the total cost of the journey

$$\text{T} = \frac{S}{v}\left(2000 + \frac{v^3}{125}\right)$$

$$= S\left(\frac{2000}{v} + \frac{v^2}{125}\right)$$

Treating S to be constant

$$\therefore\ \frac{dT}{dv} = S\left(\frac{-200}{v^2} + \frac{v^2}{125}\right)$$

$$\therefore\ \frac{dT}{dv} = 0 \Rightarrow S\left(\frac{-2000}{v^2} + \frac{2v}{125}\right) = 0$$

$v^3 = 125000$ $v = 50$ km/h

Now $\frac{d^2T}{dv^2} = \frac{4000}{v^3} + \frac{2}{125} > 0$, at $x = 50$

The most economical speed is 50 km/h.

Illustration 13 A firm finds that it can sell all that it produced (within limits) The demand function is $p = 260 - 3x$ where p is the price per unit at which it can sell x units. The cost function $c = 500 + 20x$ where x is the number of units produced. Find x so that profit is maximum.

Solution

$$R(x) = p\,x = (260 - 3x)x = 260x - 3x^2$$

$$\therefore \text{ Profit function } p(x) = R(x) - c(x)$$
$$= (260x - 3x^2) - (500 + 20x)$$
$$= -3x^2 + 240x - 500$$

$$\therefore \frac{dp}{dx} = -6x + 240$$

$$\therefore \frac{dp}{dx} = 0 \Rightarrow -6x + 240 = 0 \Rightarrow x = 40$$

Now $\frac{d^2p}{dx^2} = -6 < 0$ When $x = 40$

$\therefore$ Profit is maximum

Hence profit is maximum when $x = 40$.

Illustration 14 The total cost of a daily output of q tons of coal is Rs. $\left(\frac{1}{10}q^3 - 3q^2 + 50q\right)$. What is the value of q when average cost is minimum? Verify at this level, AC = MC.

Solution

$$c = \frac{1}{10}q^3 - 3q^2 + 50q$$

$$\therefore \text{AC} = \frac{c}{q} = \frac{1}{10}q^2 - 3q + 50$$

$$\therefore \frac{d}{dq}(\text{AC}) = \frac{1}{5}q - 3$$

For maximum or minimum $\frac{d}{dq}(\text{AC}) = 0$

$$\Rightarrow \frac{1}{5}q - 3 = 0 \Rightarrow q = 15$$

More over $\frac{d^2}{dq^2}(\text{AC}) = \frac{1}{5} > 0 \Rightarrow$ AC is minimum at $q = 15$

$$(\text{AC})_{at\ q=15} = \frac{1}{10}(15)^2 - 3(15) + 50 = \text{Rs. } \frac{55}{2}$$

$$\text{Also MC} = \frac{dc}{dq} = \frac{3q^2}{10} - 6q + 5$$

$$(\text{MC})_{at\ q=15} = \frac{3}{10}(15)^2 - 6(15) + 50 = \text{Rs. } \frac{55}{2}$$

From the above result we can say that

$$\mathrm{AC}(x) = \mathrm{MC}(x)_{at\ q\ =\ 15}$$

Illustration 15 A manufacturer can sell x items per day at a price p rupees each, where $p = 125 - \frac{5}{3}x$. If the cost of production for x items is $500 + 15x + 0.2x^2$ then

(i) Find how much he should produce to have the maximum profit, assuming all items produced are sold.

(ii) What is the maximum profit?

Solution

(i) Revenue function $R(x) = px$

$$= \left(125 - \frac{5}{3}x\right)x$$

$$\therefore\ R(x) = 125x - \frac{5}{3}x^2$$

$$= 125x - \frac{5}{3}x^2 - \left(500 + 13x + 0.2x^2\right)$$

$$p(x) = 112x - \frac{28}{15}x^2 - 500$$

$$\therefore\ p^1(x) = 112 - \frac{56x}{15}$$

Here $p^1(x) = 0 \Rightarrow 112 - \frac{56x}{15} = 0 \Rightarrow x = 30$

and $p^{11}(x) = \frac{-56}{15} < 0$ (at $x = 30$)

$\therefore$ profit is maximum at $x = 30$

(ii) Maximum profit $= 112(30) - \frac{28}{15}(30)^2 - 500 = 1180$

Illustration 16 The sum of two positive numbers is 16. Find the number if their product is to be maximum.

Solution

Let two numbers be x and y.

$\therefore$ sum $S = x + y = 16\ y = 16 - x$

$\therefore$ product $p = xy = x(16 - x) = 16x - x^2$

$\therefore$ For the product p to be maximum, we have $\frac{dp}{dx} = 16 - 2x$

$$\therefore\ \frac{dp}{dx} = 0 \Rightarrow 16 - 2x = 0 \Rightarrow x = 8$$

Also $\frac{d^2p}{dx^2} = -2 < 0_{(at\ \ x=8)} \Rightarrow p$ is maximum

Hence the two numbers are $x = 8$ and $y = 16 - x = 16 - 8 = 8$

Illustration 17 Find two positive numbers whose product is 100 and whose sum is as small as possible.

Solution

Let the two numbers be x and y : $x > 0$, $y > 0$

Product $p = x\,y \Rightarrow 100 \Rightarrow x + \dfrac{100}{x}$

sum $S = x + y = x + \dfrac{100}{x}$

$$\therefore \frac{dS}{dx} = 1 - \frac{100}{x^2}$$

$$\therefore \frac{dS}{dx} = 0 \Rightarrow 1 - \frac{100}{x^2} = 0 \Rightarrow x^2 = 100 \Rightarrow x = 10(x > 0)$$

Now $\dfrac{d^2S}{dx^2} = -(-2)\dfrac{100}{x^3} = \dfrac{200}{x^3} > 0$

$$\therefore \left(\frac{d^2S}{dx^2}\right)_{(x=100)} = \frac{200}{(10)^3} = \frac{200}{1000} = \frac{2}{10} > 0$$

$\therefore$ S is minimum at $x = 10$

Illustration 18 If the demand function of the monopolist is given by $p = 200 - 0.5q$ and its cost function is given by $c = 100 + 5q + 7q^2$ then find

(1) Value of q at which the profit is maximum
(2) The price at that output
(3) The maximum profit
(4) The output at which the total cost is minimum

Solution

$$\begin{aligned}\text{Total revenue} &= \text{price} \times \text{output}\\ &= p \times q\\ &= (200 - 0.5q)q\\ &= 200q - 0.5q^2\end{aligned}$$

(1) Profit $= p = R(x) - c(x)$

$$= (200q - 0.5q^2) - (100 + 5q + 7q^2)$$

$$= 195q - \frac{15}{2}q^2 - 100$$

$$\therefore \frac{dp}{dq} = 195 - 15q$$

For p to be maximum

$$\frac{dp}{dq} = 0 \Rightarrow 195 - 15q = 0 \Rightarrow q = 13$$

Again $\dfrac{d^2p}{dq^2} = -15 < 0$

$\therefore \frac{d^2p}{dq^2} < 0$ at $q = 13$

Hence $q = 13$ gives maximum profit

(2) The price at $q = 13$ is $p = 200 - 0.5(13)$

(3) Maximum profit at $q = 13$ is

$$p = (195)13 - \left(\frac{15}{2}\right)(13)^2 - 100 = 1167.50$$

(4) For the total cost to be minimum we must have

$$\frac{dx}{dq} = 0 \text{ and } \frac{d^2c}{dq^2} > 0$$

Here $c = 100 + 5q + 7q^2$

$$\therefore \frac{dc}{dq} = 5 + 14q$$

$$\therefore \frac{dc}{dq} = 0 \Rightarrow 5 + 14q = 0 \Rightarrow q = -\frac{5}{14}$$

Also $\frac{d^2c}{dq^2} = 14 > 0$ which means that the cost is minimum at $q = -\frac{5}{14}$

Illustration 19 A steel plant is capable of producing x tons per day of a low grade steel and y tons of high grade steel, where $y = \frac{40 - 5x}{10 - x}$. If the fixed market price of low grade steel is half of the high grade, show that about 5.5 tons of low grade level steel are produced per day at maximum total revenue.

Solution

Low grade steel = x tons per-day

High grade steel = y tons per-day and $y = \frac{40 - 5x}{10 - x}$

Let the market price of low grade steel = p

Let the market price of high grade steel = $2p$

Revenue per day of low grade steel = xp

Revenue per-day of high grade steel = $2yp$

Firm's Total revenue = $xp + 2yp$

$$\therefore R(x) = xp + 2p\left(\frac{40 - 5x}{10 - x}\right)$$

$$R^1(x) = p - \frac{20p}{(10 - x)^2} = p\left[1 - \frac{20}{(10 - x)^2}\right]$$

For maximum or minimum

$$R^1(x) = 0$$

$$\therefore p\left[1-\frac{20}{(10-x)^2}\right]=0$$

$$\therefore 1-\frac{20}{(10-x)^2}=0$$

$$\Rightarrow x=\frac{29}{2} \text{ or } x=\frac{11}{2}$$

$$\Rightarrow R^{11}(x)=-\frac{40p}{(10-x)^3}$$

Also $\left[R^{11}(x)\right]_{at\ x=29/2}=\frac{320}{729},\ p<0(\text{as } p>0)\Rightarrow R$

Thus $R(x)$ is minimum $at\ x=\frac{11}{2}=5.5$ tons and $R(x)$ is minimum at $x = \frac{29}{2}$.

Illustration 20 The manager of a printing firm plans to include 200 sq cm of actual printing matter in each page of book under production. Each page should have 2.5 cm margin along the top and bottom and 2.00 cm wide margin along the sides. What are the most economical dimensions of each printed page?

Solution

Let the length of the printing matter in each page be x and the breadth be y

$\therefore$ Area of printed matter $= xy = 200$

$$\therefore y=\frac{200}{x}$$

Since each page should have a 2.5 cm margin along the top and bottom, the length of the page will be

$x + 2.5 + 2.5 = x + 5$

As there should be 2.0 cm again along the sides so the total breath of the page will be $y + 2 + 2 = y + 4$.

$$\therefore \text{Area of page A} = (x+5)(y+4)$$

$$= (x+5)\left(\frac{200}{x}+4\right)$$

$$= 4x+\frac{1000}{x}+220$$

For most economical dimensions A should be minimum for which

$$\frac{d\text{A}}{dx}=0\Rightarrow 4-\frac{1000}{x^2}=0\Rightarrow x=5\sqrt{10}$$

and $\frac{d^2\text{A}}{dx^2}=4+\frac{2000}{x^3}$

$$\therefore \left(\frac{d^2A}{dx^2}\right)_{x=5\sqrt{10}} = 4+\frac{2000}{\left(5\sqrt{10}\right)^3} = 4+\frac{8}{4\sqrt{10}} > 0$$

$\therefore$ Area is minimum at $x = 5\sqrt{10}$

$$y = \frac{200}{5\sqrt{10}} = 4\sqrt{10}$$

The most economical dimensions are $x + 5 \Rightarrow 5\sqrt{10}+5$ and $y+4 = 4\sqrt{10}+4$

Illustration 21 A store can sell 100 tennis rackets a year at Rs. 30 each. For each Rs. 2 drop in price it can sell 10 more rackets. What sale price would produce the greatest revenue?

Solution

Let x be the price at which the rackets are to be sold. The number of multiples of 2 that the reduced price is below Rs. 30.

The reduction in price is $(30 - x)$. Multiples of 2 in $(30 - x)$ is $(30 - x)/2$. For each such multiple the store sells 10 additional rackets. Hence number of rackets sold will be

$$100+10\frac{(30-x)}{2} = 250-5x$$

Since the price is x per racket so the revenue is

$$R = (250 - 5x)x = 250x - 5x^2$$

To obtain maximum revenue

$$\frac{dR}{dx} = 0 \text{ and } \frac{d^2R}{dx^2} < 0$$

$$\frac{dR}{dx} = 0 \Rightarrow 250-10x = 0 \Rightarrow x = 25$$

but $\frac{d^2R}{dx^2} = -10 < 0$

which is negative and thus the revenue is maximum.

Hence the selling price per racket should be Rs. 25.

Illustration 22 The total profit y (in rupees) of a company from the manufacture and sale of x bottles is given by $y = -\frac{x^2}{400}+2x-80$.

(1) How many bottles must the company sell to achieve maximum profit?

(2) What is the profit per bottle when this maximum is achieved?

Solution

(1) $\frac{dy}{dx} = \frac{-2x}{400}+2 \Rightarrow \frac{dy}{dx} = 0 \Rightarrow x = 400$

$$\therefore \frac{d^2y}{dx^2} = \frac{-2}{400} < 0$$

$\Rightarrow$ Profit is maximum where $x = 400$

(2) Maximum profit

$$= -\frac{(400)^2}{400} + 2(400) - 80 = 320$$

Required profit per bottle

$$\therefore \ = \text{Rs. } \frac{320}{400} = \text{ Rs. } 0.80$$

Illustration 23 A tour operator charges Rs. 136 per passenger for 100 passengers with a discount of Rs. 4 for each 10 passengers in excess of 100. Determine the number of passengers that will maximize the amount of money the tour operator receives.

Solution

Let $R(x)$ be revenue from each passenger, then

$$R(x) = 136 - \frac{4}{10}(x - 100),\ x \geq 100$$

$$\therefore \text{TR} = x\left[130 - \frac{4}{10}(x - 100)\right]$$

$$= 176x - \frac{2}{5}x^2$$

$$\frac{d}{dx}\text{TR} = 0 \Rightarrow 176 - \frac{2}{5}(2x) \Rightarrow x = 220$$

$$\therefore R^1(x) = \frac{d^2\text{TR}}{dx^2} = \frac{-4}{5} < 0$$

$R^1(x)$ is maximum at $x = 220$

Illustration 24 For a firm under perfect competition it is given that $p = 19$ $c = \frac{x^2}{3} - 5x^2 + 28x + 27$ where p is the price per unit, x is the unit of output and c is the total cost of x units.

(1) Find the quantity produced at which profit will be maximum and also the amount of maximum profit.

(2) What happens to equilibrium output and maximum profit if $p = 12$?

Solution

(1) Profit $p = R - c = 19x - \left(\frac{x^2}{3} - 5x^2 + 28x + 27\right)$

$$= -\frac{x^3}{3} + 5x^2 - 9x - 27$$

$$\therefore \frac{dp}{dx} = 0 \Rightarrow -x^2 + 10x - 9 = 0 \Rightarrow x = 1 \text{ or } 9$$

$$\frac{d^2p}{dx^2} = -2x + 10 < 0 \ at \ x = 9$$

and $\frac{d^2p}{dx^2} > 0 \ at \ x = 1$ which is rejected here

Hence $x = 9$

(2) $p = R - x = 12x - \left(\frac{x^3}{3} - 5x^2 + 28x + 27\right)$

for maximum profit $\frac{dp}{dx} = 0$ and $\frac{d^2p}{dx^2} < 0$

$$\frac{dp}{dx} = 0 \Rightarrow -x^2 + 10x - 16 = 0 \Rightarrow x = 2, 8$$

$\frac{d^2p}{dx^2} = -2x + 10$ and it will be –ve if $x = 8$ only.

Illustration 25 Find the greatest value of $\left(\frac{1}{x}\right)^x$

Solution

Let $y = \left(\frac{1}{x}\right)^x$

$$\Rightarrow \log y = x\log\left(\frac{1}{x}\right) = x(\log 1 - \log x)$$

$$\Rightarrow \log y = -x\log x$$

$$\frac{1}{y}\frac{dy}{dx} = -x\frac{1}{x} + \log x(-1)$$

$$\Rightarrow \frac{dy}{dx} = -y(1 + \log x)$$

$$\frac{dy}{dx} = -\left(\frac{1}{x}\right)^x (1 + \log x)$$

Now $\frac{dy}{dx} = 0 \Rightarrow 1 + \log x = 0 \Rightarrow \log x = -1$

$$\Rightarrow x = e^{-1} = \frac{1}{e}$$

$$\Rightarrow x = \frac{1}{e}$$

but $\frac{dy}{dx} = -\left(-\frac{1}{x}\right)^x (x + \log x)$

$$\frac{d^2y}{dx^2} = \left[\left(-\frac{1}{x}\right)^x \left(0 + \frac{1}{x}\right) + (1 + \log x)^2 \left(-\frac{1}{x}\right)^x\right]$$

Now $\left(\frac{d^2y}{dx^2}\right)_{x=1/e} = -\left[-(e)^{1/e}(1-1)^2 + (e)^{1/e+1}\right]$

$= -e^{1/e+1} < 0$

$\Rightarrow y$ is maximum $a + x = \frac{1}{e}$ and maximum value of y is $(e)^{1/e}$.

Applied Problems of Maxima and Minima Concept and Formulae

Rectangle of dimensions x and y

Area $= x\,y$, perimeter $= 2(x + y)$

→ Circle of radius r

Area $= \pi r^2$, circumfrence $= 2\pi r$

→ Cuboid of dimensions x, y and z

Volume $= x\,y\,z$, surface Area $= 2(xy + yz + zx)$

→ Right circular cylinder of height h and radius of base r

Volume $= \pi r^2 h$, surface Area $= 2\pi rh + 2\pi r^2$

Right circular cone of height h and radius of base r

Slant height $= \sqrt{r^2 + h^2}$, volume $= \frac{1}{3}\pi r^2 h$

→ Sphere of radius r

Volume $= \frac{4}{3}\pi r^3$, surface area $= 4\pi r^2$

EXAMPLES

Illustration 1 A spherical soap bubble is expanding so that its radius is increasing at the rate of 0.02 cm/sec. As what rate is the surface area increasing when its radius is 4 cm?

Solution

Here the soap bubble is in the form of a sphere. Let at any instant time t, the radius of the spare be r cm and the surface area be 5 sq. cm. Then $\frac{dr}{dt} = 0.02\ \text{cm/sec}$

but $s = 4\pi r^2 \Rightarrow \frac{ds}{dt} = 8\pi r\,\frac{dr}{dt} = 8\pi r \times 0.02 = 0.16\pi r$

$= (0.16 \times 3.14r)$ cm²/sec when $r = 4$ cm

$= 0.16 \times 3.14 \times 4 = 2.0096$ cm²/sec

∴ Rate of increase of the surface area of the bubble = 2.0096 cm²/sec

Illustration 2 The radius of an air bubble is increasing at the rate of 0.5 cm/sec. At what rate is the volume of the bubble increasing when the radius is 1 cm?

Solution

Here $\frac{dr}{dt} = 0.5$, Also $v = \frac{4}{3}\pi r^3$

$$\therefore \frac{dv}{dt} = \frac{4}{3}\pi r^2\, 3\frac{dr}{dt} = 4\pi r^2\, \frac{dr}{dt} = 4\pi r^2 (0.5)$$

$$\therefore \frac{dv}{dt} = 4\pi(1)(0.5) = 2\pi \text{ cm}^2/\text{sec}$$

Illustration 3 The radius of a circular plate is increasing at the rate of 0.02 cm/sec. At what rate is the area increasing when the radius of the plate is 25 cm?

Solution

Let r be the radius and A be the area of the circle at any instant t. When $A = \pi r^2$, the rate of increase of area A is given by $\frac{dA}{dt}$

$$A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r\, \frac{dr}{dt}$$

$$= 2 \times 3.14 \times 25 \times 0.02 \qquad \text{Here } \frac{dr}{dt} = 0.02$$

$$= 3.14 \text{ sq. cm/sec}$$

Illustration 4 A balloon which always remains spherical, is being inflated by pumping in 900 cubic cm of gas per second. Find the rate at which the balloon is increasing when the radius is 15 cm.

Solution

Let r be the radius of the sphere and v be the volume at time t.

$$\text{Now } v = \frac{4}{3}\pi r^3 \Rightarrow \frac{dv}{dt} = \frac{4}{3}\pi\, 3r^2\, \frac{dr}{dt} = 4\pi r^2\, \frac{dr}{dt}$$

$$\therefore \frac{dv}{dt} = 4\pi r^2\, \frac{dr}{dt}$$

$$\therefore \frac{dr}{dt} = \left(\frac{dv}{dt}\right)\frac{1}{4\pi r^2}$$

$$(900)\frac{1}{4\pi(15)^2} \qquad at = 15 \quad \frac{dv}{dt} = 900$$

$$\frac{1}{\pi} = \frac{7}{22}$$

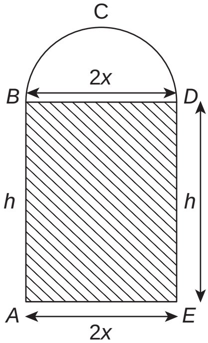
Hence the balloon is increasing at the rate of $\frac{7}{22}$ cm/sec when $r = 15$ cm.

Illustration 5 A window is in the form of a rectangle surrounded by a semi circular opening. The total perimeter of the window is 10 cm. Find the dimensions of the rectangular part of the window to admit light through the whole opening.

Solution

Let $AB = ED = h$ and $AE = BC = 2x$

The perimeter of the window is given by $2x + h + \pi x + h = 10$



$2h = 10 - 2x - \pi x$

or

The area of the window is given by

$$A = 2x \times h + \frac{1}{2}\pi x^2$$

$$= x(2h) + \frac{1}{2}\pi x^2$$

$$= x(10 - 2x - \pi x) + \frac{1}{2}\pi x^2$$

$$A = 10x - 2x^2 - \pi x^2 + \frac{1}{2}\pi x^2$$

$$= 10x - 2x^2 - \frac{1}{2}\pi x^2$$

$\therefore$ A will be maximum when

$$\frac{dA}{dx} = 0 \text{ and } \frac{d^2A}{dx^2} < 0$$

Now $\frac{dA}{dx} = 10 - 4x - \pi x = 0$

$$\Rightarrow x(4+\pi) = 10 \Rightarrow x = \frac{10}{4+\pi}\text{m}$$

but $2h = 10 - (2 + \pi)x$

$$= 10 - (2-\pi)\frac{10}{4+\pi} = \frac{40}{\pi+4}$$

$$\therefore\ h = \frac{10}{\pi+4}\text{m}$$

further $\frac{d^2A}{dx^2} = -(4+\pi) < 0$

$\therefore$ A is maximum when $h = \frac{120}{\pi+4}$m and $x = \frac{20}{\pi+4}$m

Illustration 6 A square piece of tin of side 18 cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to form a box. What should be the side of the square to be cut off so that the volume of the box is maximum? Also find the maximum volume.

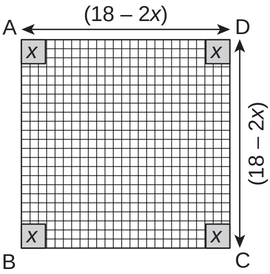
Solution

Let x be the side of the square to be cut off. Then the dimensions of the box are $18 - 2x$, $18 - 2x$ and x.

$\therefore$ volume of the box $v = (18 - 2x)^2 \times x$

$$\therefore \frac{dv}{dx} = (18-2x)^2 \times 1 - 4x(18-2x)$$

$$= (18 - 2x)(18 - 6x)$$



For extreme value $\frac{dv}{dx} = 0$

$\Rightarrow (18 - 2x) = 0$ or $(18 - 6x) = 0$

$\Rightarrow x = 9$ or $x = 3$

But $x = 9$ is rejected as the trial side is 18 cm

Now $\frac{d^2v}{dx^2} = (18-2x)(-6) + (-2)(18-6x)$

$= 24x - 144$

$\left(\frac{d^2v}{dx^2}\right)_{x=3} = 24(3) - 144 = -72 < 0$

$\Rightarrow$ Volume is maximum when $x = 3$ cm and maximum volume $= (18 - 6)^2 \times 3 = 432$ cubic cm.

Illustration 7 Show that of all the rectangles of given area, the square has the smallest perimeter.

Solution

Let the sides of the rectangle by x and y then $A = xy \Rightarrow y = \frac{A}{x}$

But perimeter $p = 2x + 2y = 2\left(x + \frac{A}{x}\right)$

$\therefore p = 2\left(x + \frac{A}{x}\right)$

Now $\frac{dp}{dx} = 0 \Rightarrow 2\left(1 - \frac{A}{x^2}\right) = 0 \Rightarrow x = \sqrt{A}$

but $\frac{d^2p}{dx^2} = \frac{4A}{x^3} > 0$ when $x = \sqrt{A} \Rightarrow$ perimeter is minimum when $x = \sqrt{A}$

$\therefore x = \sqrt{A} \Rightarrow y = \frac{A}{x} = \frac{A}{\sqrt{A}} = \sqrt{A}$

Hence we can say that the rectangle becomes square when the perimeter is least.

Illustration 8 Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is maximum.

Solution

Here product $p = x^2y^5$ $\qquad$ $x + y = 35$

$\therefore p = x^2(35 - x)^5$ $\qquad$ $\Rightarrow y = 35 - x$

For p to be maximum

$\frac{dp}{dx} = 0$

$\Rightarrow \frac{dp}{dx} = 2x(35 - x)^5 + x^2\,5(35 - x)^4\,(-1)$

$= 2x(35 - x)^5 - 5x^2(35 - x)^4$

$\therefore\ x(35 - x)^4(70 - 7x) = 0 \qquad \text{Here } x \in (0, 35)$

$\Rightarrow x = 10$

To determine if $x = 10$ is indeed a point of maximum

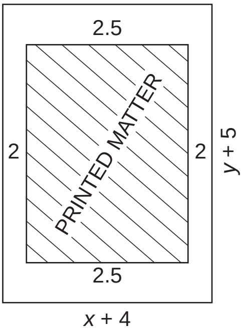
If $x < 10$ slightly the $\dfrac{dp}{dx} > 0$ and if $x > 10$ slightly, the $\dfrac{dp}{dx} < 0$

$\therefore\ x = 10 \Rightarrow y = 35 - x = 35 - 10 = 25.$

Illustration 9 The production manager of a company plans to include 180 sq. cm of actual printed matter in each page of a book under production. Each page should have a 2.5 cm wide margin along the top and bottom and 2.0 cm side margin along the sides. What are the most economical dimensions of each printed page?

Solution

Let x and y be the dimensions of actual printed matter included in each page of the book under production. Then according to the problem, dimension of each page of the book will be $x + y$ and $y + 5$, respectively.



The Area of each page

$A = (x + 4)\ (y + 5)$

$= xy + 5x + 4y + 20 \qquad \text{Here } xy = 180$

$\therefore\ A = 180 + 5x + 4\left(\dfrac{180}{x}\right) + 20 \qquad \therefore y = \dfrac{180}{x}$

$$A = 200 + 5x + \frac{700}{x}\text{s}$$

$$\therefore \frac{dA}{dx} = 0 + 5 - \frac{720}{x^2} \Rightarrow \frac{d^2A}{dx^2} = \frac{1440}{x^3} > 0$$

For A to be minimum $\dfrac{dA}{dx} = 0$ and $\dfrac{d^2A}{dx^2} > 0$

$$\therefore \frac{dA}{dx} = s - \frac{720}{x^2} = 0 \Rightarrow x^2 = \frac{720}{5} = 144 \Rightarrow x = 12$$

When $x = 12$ then $\dfrac{d^2A}{dx^2} > 0$

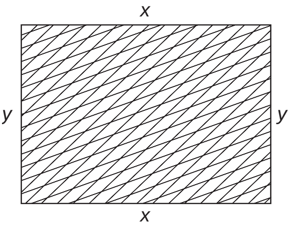
Thus A is minimum when $x = 12$

but $xy = 180 \Rightarrow y = \dfrac{180}{x} = \dfrac{180}{12} = 12$

Hence we can say that most economical dimensions of each printed page should be $12 + 4 = 16$ cm and $15 + 5 = 20$ cm.

Illustration 10 One side of a rectangle enclosure is formed by the hedge, the total length of fencing available for the other three sides is 200 yards Obtain an expression for the area of the enclosure, A sq. yards in terms of its length x yards and hence deduce the maximum area of the enclosure.

Solution



Let y represent the breadth of one side of the required rectangle

We wish to maximize the area of the rectangular enclosure given by A = xy but subject to condition

$$x + 2y = 200 \Rightarrow y = 100 - \frac{x}{2}$$

$$\therefore \text{A} = x\left(100 - \frac{x}{2}\right) = 100x - \frac{x^2}{2} \Rightarrow \frac{d\text{A}}{dx} = 100 - x$$

For maximum $\frac{d\text{A}}{dx} = 0 \Rightarrow 100 - x = 0 \Rightarrow x = 100$

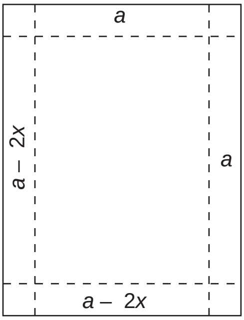
and $\frac{d^2\text{A}}{dx^2} < 0$

Hence A is maximum when x = 100 and maximum value of A is

$$100(100) - \frac{(100)^2}{2} = 5000 \text{ sq. yards.}$$

Illustration 11 A square sheet of tin each side = a cm is to be used to make an open top box by cutting a small square of tin from each corner and then folding up the sides. Find how large a square should be cut from each corner if the volume of the box is to be a maximum.

Solution



Let x be the dimension of the side of the square cut from each corner. Then the dimensions of the box obtained by folding up the sides are $a - 2x$, $a - 2x$ and x respectively.

Let v be the volume of the box, so we obtain

$v = (a - 2x)^2 x = 4x^3 - 4ax^2 + a^2x$

We wish to find x so that v is maximum. For v to be maximum we must have

$$\frac{dv}{dx} = 12x^2 - 8ax + a^2 = (6x - a)(2x - a)$$

For maximum $\frac{dv}{dx} = 0$

$$\Rightarrow x = \frac{a}{6} \text{ or } x = \frac{a}{2}$$

Now $\frac{d^2v}{dx^2} = 24x - 8a$

when $x = \frac{a}{2}$, $\left(\frac{d^2v}{dx^2}\right)_{x=a/2} = 4a > 0$ $\therefore$ Value $\frac{a}{2}$ is rejected

$$x = \frac{a}{6}, \left(\frac{d^2v}{dx^2}\right)_{x=a/6} = -4a < 0 \text{ so that } v \text{ is maximum when } x = \frac{a}{6}$$

Illustration 12 Show that a closed right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

Solution

Let r and h be the radius and height of the given cylinder. Then the volume v and the surface area s are given by $V = \pi r^2 h$, $S = 2\pi rh + 2\pi r^2$.

Now we wish to maximize V subject to the condition that S is fixed

$$\therefore S = 2\pi rh + 2\pi r^2 \Rightarrow h = \frac{s - 2\pi r^2}{2\pi r} \tag{1}$$

$$\text{and } \mathrm{V} = \pi r^2 h = \pi r^2\left(\frac{S - 2\pi r^2}{2\pi r}\right) = \frac{Sr}{2} - \pi r^3$$

$$\therefore \frac{dv}{dr} = \frac{S}{2} - 3\pi r^2$$

For v to be maximum $\frac{dv}{dr} = 0$

$$\therefore \frac{s}{2} - 3\pi r^2 = \frac{s}{2} \Rightarrow r^2 = \frac{s}{6\pi}$$

$$\Rightarrow r = \sqrt{\frac{s}{6\pi}} \text{ and } \frac{d^2v}{dr^2} = 0 - 6\pi r = -6\pi r < 0$$

which is always –ve because r is +ve

Thus v is maximum when $r = \sqrt{\frac{s}{6\pi}}$

Now by substituting $r = \sqrt{\frac{s}{6\pi}}$ in (1) we obtain

$$h = \frac{S - 2\pi(S/6\pi)}{2\pi\sqrt{S/6\pi}} = \frac{(2/3)S}{2 - \sqrt{\pi}\sqrt{s}}\sqrt{6} = 2\sqrt{\frac{6}{6\pi}} = 2r$$

Hence we can say that volume is maximum when the height is equal to the diameter of the base.

Illustration 13 A wire of 40 m length is to be cut into two pieces. One of the pieces is to be made into a square and other into a circle What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

Solution

Let the length of one piece be x m which is made into a square. Then the length of the other piece is $(40 - x)$ m which is made into a circle. If x be the radius of the circle so made, then

$$2\pi r = 40 - x \Rightarrow r = \frac{40 - x}{2\pi}$$

Also each side of the square $= \dfrac{x}{4}$

Let A be the combined area of the square and the circle. Then

$$A = \left(\frac{x}{4}\right)^2 + \pi\left(\frac{40 - x}{2\pi}\right)^2 = \frac{x^2}{16} + \frac{(40 - x)^2}{4\pi} = 0$$

We wish to find x so that A is minimum $x \in (0, 40)$

Now $\dfrac{dA}{dx} = \dfrac{x}{8} - \dfrac{(40 - x)}{2\pi}$

For minimum A, $\dfrac{dA}{dx} = 0 \Rightarrow \dfrac{x}{8} - \dfrac{(40 - x)}{2\pi} = 0$

$$\Rightarrow x = \frac{160}{\pi + y}$$

Also $\dfrac{d^2A}{dx^2} = \dfrac{1}{8} + \dfrac{1}{2\pi} > 0$

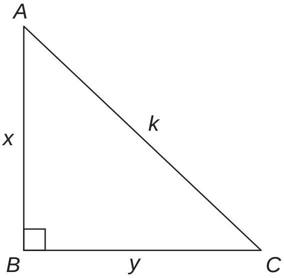
$\therefore$ When A is minimum then $x = \dfrac{160}{\pi + 4}$ m

In other words the combined area is minimum when the lengths of the two pieces are

$$x = \frac{160}{\pi + 4} \text{ m and } 40 - \frac{160}{\pi + 4} = \frac{40\pi}{\pi + 4} \text{ m.}$$

Illustration 14 Prove that the length of the hypotenuse of a right triangle proves that its area is maximum when it is an isosceles right triangle.

Solution



In ΔABC m$\angle$B = 90, AC = k > 0

If AB = x, BC = y then area

$A = \dfrac{1}{2}xy,\ x, y, A > 0$

and $x^2 + y^2 = k^2$ now $A^2 = \dfrac{1}{4}x^2y^2$

$$A^2 = \frac{1}{4}x^2\left(k^2 - x^2\right)$$

$$A^2 = \frac{1}{4}\left(k^2x^2 - x^4\right)$$

$$\therefore 2A\frac{dA}{dx} = \frac{1}{4}\left(2k^2x - 4x^3\right) \quad (1)$$

But for A to be maximum

$$\frac{dA}{dx} = 0 \Rightarrow 2k^2x - 4x^3 = 0$$

$$\Rightarrow 2k^2 = 4x^2 \Rightarrow x^2 = \frac{k^2}{2} \Rightarrow x = \frac{k}{\sqrt{2}}$$

but $y^2 = k^2 - x^2 = k^2 - \frac{k^2}{2} = \frac{k^2}{2} \Rightarrow y = \frac{k}{\sqrt{2}}$

Moreover differentiating (1) with respect to '*x*' again

$$2A\,\frac{d^2A}{dx^2} + 2\left(\frac{dA}{dx}\right)^2 = \frac{1}{4}\left(2k^2 - 12x^2\right)$$

For $A > 0,\ \frac{dA}{dx} = 0$ and $x = \frac{k}{\sqrt{2}}$

$$\therefore 2A\frac{d^2A}{dx^2} = \frac{1}{4}\left(2k^2 - 12x^2\right) < 0$$

$$\therefore \frac{d^2A}{dx^2} < 0 \text{ for } x = y = \frac{k}{\sqrt{2}}$$

$\therefore$ A is maximum.

$\therefore$ When the area of ΔABC is maximum, the triangle is isosceles.

Illustration 15 Sand is being collected in the form of a cone at the rate of 10 cubic m/sec. The radius of the base is always half of the height. Find the rate at which the height increases when the height is 5 m.

Solution

Let *v* be the volume of the cone, *h* be the height of the cone and *r* be the radius of the cone at time *t*.

$$\therefore v = \frac{1}{3}\pi r^2 h$$

Here $\frac{dv}{dt} = 10$ m^3/sec and $r = \frac{h}{2}$

$\frac{dh}{dt} = ?$ when $h = 5$ m

$$v = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

$$v = \frac{1}{12}\pi h^3$$

$$\frac{dv}{dt} = \frac{1}{12}\pi\, 3h^2\,\frac{dh}{dt}$$

$$\therefore 10 = \frac{1}{12}\pi 3\left(5\right)^2\frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{8}{5\pi}\text{ m/sec}$$

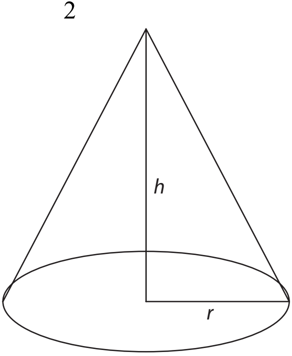
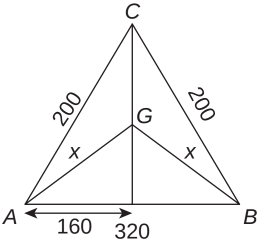


Illustration 16 A firm has a branch store in each of the three cities A, B and C. A and B are 320 km apart and C is 200 km from each of them. A godown is to be built equidistant from A and B. In order to minimize the time of transportation it should be located so that sum of distances from the godown to each of the cities is minimum. Where should the godown be built?

Solution



Let the godown be built at G So that AG = BG = x km. Let CG produced meet AB at D. Since C and G are equidistance from A and B, therefore, CG (CD) is the bisector of the line segment AB and as such AD = BD = 160 km.

Since ΔADC is a right angled triangle

$\therefore AC^2 = AD^2 + CD^2$

$\therefore CD^2 = AC^2 - AD^2 = 200^2 - 160^2$

$\therefore CD^2 = 14400 \Rightarrow CD = 120$

Similarly using right angled rectangle ADG

$GD^2 = \sqrt{x^2 - (160)^2}$ and $CG = CD - GD = 120\sqrt{x^2 - (160)^2}$

Let's denote the sum of distances from the godown to each of the three cities. Then

$S = GA + GB + GC = x + x + 120 - \sqrt{x^2 - (160)^2}$

$S = 2x + 120 - \sqrt{x^2 - (100)^2}$

Now we wish to find x so that S is minimum

$$\therefore \frac{dS}{dx} = 2 + 0 - \frac{(2x)}{2\sqrt{x^2 - (160)^2}} = 2 - \frac{x}{\sqrt{x^2 - (160)^2}}$$

$$\text{but } \frac{dS}{dx} = 0 \Rightarrow 2 - \frac{x}{\sqrt{x^2 - (160)^2}} = 0$$

$$\Rightarrow 2 = \frac{x}{\sqrt{x^2 - (160)^2}}$$

$$\Rightarrow 4\left[x^2 - (160)^2\right] = x^2$$

$$\Rightarrow 3x^2 = 4(160)^2 \Rightarrow x = \frac{320}{\sqrt{3}}$$

$$\text{Also } \frac{d^2S}{dx^2} = \frac{160}{\left[x^2 - (160)\right]^{3/2}} < 0 \text{ when } x = \frac{320}{\sqrt{3}}$$

$$\therefore S \text{ is minimum when } x = \frac{320}{\sqrt{3}} = 184.75 \text{ km}$$

Illustration 17 A square tank of capacity 250 cubic m has to be dug out. The cost of land is Rs. 50 per sq. m. The cost of digging increases with the depth and for the whole tank is 400 (depth)^2 rupees. Find the dimensions of the tank for the least total cost.

Solution

Let each side of the square base of tank be x metres and its depth be h metres.

Capacity of tank $= x^2h = 250$ cubic m $x^2 = \frac{250}{h}$

Cost of land $=$ Rs $50x^2$ and the cost of digging $=$ Rs $400h^2$

$\therefore$ Total cost $= c =$ Rs. $\left(50x^2 + 400h^2\right)$

$$c = \frac{12500}{h} + 400h^2$$

$$\therefore \frac{dc}{dh} = -\frac{12500}{h^2} + 800h$$

For c to be maximum $\frac{dc}{dx} = 0$

$$\Rightarrow \frac{-12500}{h^2} + 800h = 0 \Rightarrow h^3 = \frac{125}{8} \Rightarrow h = \frac{5}{2}$$

Also $\frac{d^2c}{dx^2} = \frac{25000}{h^3} + 800h = 0$ when $h = \frac{5}{2}$

$\therefore$ The cost c is minimum when $h = \frac{5}{2}$ m and $x = \sqrt{\frac{250 \times 2}{5}} = 10$ m

Illustration 18 Twenty metres of wire is available to fence off a flower bed in the form of a circular sector. What must be the radius of the circle if we wish to have a flower bed with the greatest possible surface area?

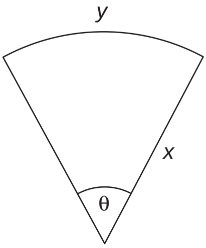
Solution

Let the length of the arc be y m and its radius x m

$\therefore x + x + y = 20$

$\therefore 2x + y = 20$

$\therefore y = 20 - 2x$



Let A be the area of the circular sector then

$$A = \frac{1}{2}xy = \frac{1}{2}x(20 - 2x) = 10x - x^2$$

Now, we wish to find A is maximum

$$\frac{dA}{dx} = 10 - 2x$$

$$\therefore \frac{dA}{dx} = 0 \Rightarrow 10 - 2x = 0 \Rightarrow x = 5$$

Also $\frac{d^2A}{dx^2} = -2 < 0$

Thus A is maximum when $x = 5$

In order to have the flower bed with the greatest possible surface area, the radius of the circle should be 5 m.

Illustration 19 Find the sides of the rectangle of greatest area which can be inscribed in the ellipse $\frac{x^2}{8} + \frac{y^2}{4} = 1$

Solution

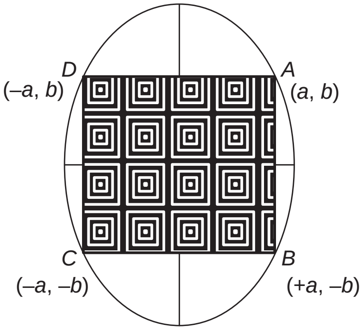
Equation of ellipse is $\frac{x^2}{8} + \frac{y^2}{4} = 1$

Consider the figure. Let ABCD be the rectangle of maximum area drawn inside the ellipse.

Also Let AB = CD = $2a$ and

AD = BC = $2b$

CD coordinator of A, B, C, D are $(a,\ b)$, $(+a,\ -b)$, $(-a,\ -b)$ and $(-a,\ b)$ respectively.



$$\therefore\ \frac{a^2}{8} + \frac{b^2}{4} = 1 \Rightarrow a^2 + 2b^2 = 8 \qquad (1)$$

Also area of rectangle ABCD $= \frac{1}{2} 2(2a)(2b) = 4ab$

From (1) we can say that

$$2b^2 = 8 - a^2 \Rightarrow b^2 = \frac{8-a^2}{2} \Rightarrow b = \sqrt{\frac{8-a^2}{2}}$$

$$\therefore \text{Area } A = 4ab = 4a\sqrt{\frac{8-a^2}{2}} = \frac{4}{\sqrt{2}} a\sqrt{8-a^2}$$

$$\therefore\ A = 2\sqrt{2}a\,\sqrt{8-a^2}$$

For maximum area we can say that

$$\frac{dA}{da} = 2\sqrt{2}\left[1\,\sqrt{8-a^2} + \frac{a\,(-2a)}{2\sqrt{8-a^2}}\right]$$

$$= 2\sqrt{2}\left[\sqrt{8-a^2} + \frac{a^2}{\sqrt{8-a^2}}\right]$$

$$= 2\sqrt{2}\left[\frac{8-a^2-a^2}{\sqrt{8-a^2}}\right]$$

$$= 4\sqrt{2}\left[\frac{4-a^2}{\sqrt{8-a^2}}\right] = 0$$

but $\frac{dA}{da} = 0 \Rightarrow 4\sqrt{2}\left(\frac{4-a^2}{\sqrt{8-a^2}}\right) = 0$

$$\Rightarrow 4 - a^2 = 0 \Rightarrow a = 2 \Rightarrow b\sqrt{\frac{8-4}{2}} = \sqrt{2}$$

$$\text{Now } \frac{d^2A}{da^2} = 4\sqrt{2}\left[\frac{\sqrt{8-a^2}\,(-2a) - \left(4-a^2\right)\left(-2a/2\sqrt{8-a^2}\right)}{\left(8-a^2\right)}\right]$$

$$= 4\sqrt{2}\left[\frac{-2a\left(8-a^2\right)+a\left(4-a^2\right)}{\left(8-a^2\right)^{3/2}}\right]$$

$$= 4\sqrt{2}\left[\frac{a\left(-16+a^3+4-a^2\right)}{\left(8-a^2\right)^{3/2}}\right]$$

$$\therefore \left(\frac{d^2\mathrm{A}}{da^2}\right)_{(a=2)} = 4\sqrt{2}\left[\frac{a\left(a^3-a^2-12\right)}{\left(8-a^2\right)^{3/2}}\right] < 0$$

Area A is maximum when $a = 2$ and $b = \sqrt{2}$

Elasticity of Demand at Price

$$n_d = -\frac{p}{x}\frac{dx}{dp}$$

ILLUSTRATIONS

Illustration 1 Suppose for a company, the price function is given by $x = f(p) = 900\left(\frac{1}{3}\right)^p$, $0 < p \le 3$. Find the elasticity of demand when $p = 1$ and $p = 2$.

Solution

$$x = 900\left(\frac{1}{3}\right)^p$$

$$\Rightarrow \frac{dx}{dp} = 900\left(\frac{1}{3}\right)^p \log_e\left(\frac{1}{3}\right)$$

$$\therefore nd = \frac{-p}{x}\frac{dx}{dp}$$

$$= \frac{-p}{900(1/3)^p}\, 900\left(\frac{1}{3}\right)^p \log_e\left(\frac{1}{3}\right)$$

$$\frac{+p\log_{10}^{3}}{\log_{10}^{e}} = \frac{0.4001}{0.4343} = 1.098$$

Illustration 2 If the demand curve is of the form $p = ae^{-kx}$ where p is the price and x is the demand, prove that the elasticity of demand $\frac{1}{kx}$. Hence deduce the elasticity of demand for the curve $p = 10e^{x/2}$

Solution

$$p = a e^{-kx}$$

$$\frac{dk}{dx} = \frac{1}{(dp/dx)} = \frac{-1}{ake^{-kx}}$$

$$\therefore nd = \frac{-p}{x}\frac{dx}{dp}$$

$$= \frac{-ae^{-kx}}{x}\frac{1}{ake^{-kx}}$$

$$= \frac{1}{kx}$$

Hence the elasticity of demand curve $p = ae^{-kx}$ is $\frac{1}{kx}$. To deduce the elasticity of demand for the curve $p = 10e^{x/2}$ we compare the function $p = 10e^{x/2}$ and find that $a = 10,\ k = \frac{1}{2}$. Consequently the elasticity of demand for the curve $p = 10e^{-x/2} = \frac{1}{1/2x} = \frac{2}{x}$.

Illustration 3 Verify that $n_d \frac{\text{AR}}{\text{AR} - \text{MR}}$ for the linear demand law $p = a + bx$

Solution

$$p = a + bx$$

$$0 + b\frac{dx}{dp} = 1$$

$$\Rightarrow \frac{dx}{dp} = \frac{1}{b}$$

But $n_d = \frac{-p}{x}\frac{dx}{dp}$

$$= -\frac{(a+bx)}{x}\frac{1}{b}$$

$$n_d = -\frac{(a+bx)}{bx}$$

$$R = px = (a + bx)x = ax + bx^2$$

$$\therefore\ \text{AR} = \frac{R}{x} = \frac{ax+bx^2}{x} = a+bx = p$$

$$\text{and MR } = \frac{d\text{R}}{dx} = a+2bx$$

$$\therefore\ \frac{AR}{AR-MR} = \frac{a+bx}{a+bx-a-2bx}$$

$$= -\frac{(a+bx)}{bx} = n_d$$

Illustration 4 Show that for the total cost function $c = \sqrt{ax+b}$, where a and b are positive constants the elasticity of total cost increases but remains less than unity as x increases.

Solution

Elasticity of total cost function is

$$nc = \frac{x}{c}\frac{dc}{dx} = \frac{x}{\sqrt{ax+b}}\,\frac{a}{2\sqrt{ax+b}} = \frac{ax}{2(ax+b)}$$

Differentiate with respect to ‘x’

$$\frac{d}{dx}(n_c) = \frac{a}{2}\left[\frac{(ax+b)1-ax}{(ax+b)^2}\right] = \frac{ab}{2(ax+b)^2} > 0$$

nc increases with an increase in the output x, however

$$nc = \frac{ax}{2(ax+b)} = \frac{1}{2}\left(\frac{ax+b-b}{ax+b}\right)$$

$$= \frac{1}{2}\left(1-\frac{b}{ax+b}\right)$$

APPLICATIONS OF INTEGRATION

- To find the total revenue function and the demand function from a given marginal revenue function.
- The marginal revenue MR is the derivative of the revenue function R. Therefore, to find R, we integrate the marginal revenue function Thus
  $R = \int MR\ dr + k$,
  where k is the constant of integration to be determined by using the fact that $R = 0$ when $x = 0$
- Once R is obtained, the corresponding demand function can be obtained by using the equation $px = R$, or $p = R/x$.
- Total revenue from the sale of a specific number of units of a commodity can be determined by making use of definite integral

  Total revenue from d units sold $= \int_0^d MR\ dx$

ILLUSTRATIONS

Illustration 1 The marginal revenue (in thousand of rupees) function for a particular commodity is $4 + r^{-0.03x}$, where x denotes the number of units sold. Determine the total revenue from the sale of 100 units. It is given that $e^{-3} = 0.05$(app.)

Solution

If R denotes the revenue function, then we are given that

$\text{MR} = 4 + e^{-0.03x}$

$$\Rightarrow R = \int \left(4 + e^{-0.03x}\right) dx$$

$$= 4x - \frac{e^{-0.03x}}{.03} + k$$

where k is the constant of integration. However, $R = 0$ when $x = 0$

$$\therefore 0 = 0 - \frac{1}{.03} + k \Rightarrow k = 33.33$$

Hence the revenue from the sale of x units is given by

$$R(x) = 4x - \frac{1}{0.03} e^{-0.03x} + 33.33$$

In particular, the revenue from the sale of 100 units is

$$\begin{aligned} R(100) &= 400 - \frac{e^{-3}}{0.03} + 33.33 \\ &= 400 - \frac{0.05}{0.03} + 33.33 \\ &= 400 - 1.66 + 33.33 \\ &= 431.67 \text{ thousands of rupees} = \text{Rs. } 4{,}31{,}670 \end{aligned}$$

Illustration 2 If the marginal revenue function for a commodity is $MR = 9 - 4x^2$, find the demand function.

Solution

Since MR is the derivative of the revenue function R, we have

$$R = \int MR\, dx$$

$$= \int \left(9 - 4x^2\right) dx = 9x - \frac{4}{3}x^3 + k$$

where k is the constant of integration. However, $R = 0$ when $x = 0$. Therefore,

$0 = 0 - 0 + k \Rightarrow k = 0$

Thus $R = 9x - \frac{4}{3}x^3$

But $R = px$, where p is the price per unit

Hence $px = 9x - \frac{4}{3}x^3$

$\Rightarrow\ p = 9 - \frac{4}{3}x^2$, which is the required demand function.

Illustration 3 Marginal revenue function of a firm is $\frac{ab}{(x-b)^2} - c$. Prove that the demand law is $p = \frac{a}{b-x} - c$

Solution

If R denotes the revenue function, then we are given that

$$R = \int\left[\frac{ab}{(x-b)^2} - c\right]dx$$

$$= -\frac{ab}{x-b} - cx + k$$

where k is an arbitrary constant. However, $R = 0$, when $x = 0$.

$$\therefore\ 0 = -\frac{ab}{-b} - 0 + k \Rightarrow k = -a$$

Hence the revenue function is

$$R = -\frac{ab}{x-b} - cx - a$$

But $R = px$, where p is the price per unit. Thus

$$px = -\frac{ab}{x-b} - cx - a$$

$$\Rightarrow p = -\frac{ab}{x(x-b)} - c - \frac{a}{x}$$

$$= \frac{-ab - ax + ab}{x(x-b)} - c$$

$$= \frac{a}{b-x} - c$$

which is the required demand law.

Illustration 4 Given the marginal revenue function $\frac{4}{(2x+3)^2} - 1$, show that the average revenue function is $p = \frac{4}{6x+9} - 1$.

Solution

Given $MR = \frac{4}{(4x+3)^2} - 1$

$$R = \int\left[\frac{4}{(2x+3)^2} - 1\right]dx$$

$$= 4\frac{(2x+3)^{-1}}{(-1)2} - x + k$$

where k is an arbitrary constant. Thus

$$R = -\frac{(2)}{(2x+3)} - x + k$$

However, $R = 0$ when $x = 0$. Therefore

$$0 = \frac{2}{2(0)+3} - 0 + k$$

$$\Rightarrow k = \frac{2}{3}$$

Hence the revenue function is given by

$$R = -\frac{2}{(2x+3)} - x + \frac{2}{3}$$

The average revenue function is given by

$$p = AR = \frac{R}{x} = -\frac{2}{x(2x+3)} - 1 + \frac{2}{3x}$$

$$= \frac{-6+4x+6}{3x(2x+3)} - 1$$

$$= \frac{4}{6x+9} - 1$$

Thus $p = \dfrac{4}{6x+9} - 1$ is the desired average revenue function.

Illustration 5 A firm's marginal revenue function is $MR = 20^{-x/10}\left(1-\frac{x}{10}\right)$. Find the corresponding demand function.

Solution

We have $MR = 20^{-x/10}\left(1-\frac{x}{10}\right)$

Integrating both sides with respect to x, we obtain

$$\text{R} = MR = 20^{-x/10}\left(1-\frac{x}{10}\right)$$

$$= 20\int e^{-x/10}dx - 2\int xe^{-x/10}dx$$

$$= 20\int e^{-x/10}dx - 2\left[-10xe^{-x/10} - \int -10e^{-x/10}\, dx\right]$$

$$= 20\int e^{-x/10}\, dx + 20xe^{-x/10} - 20\int e^{-x/10}\, dx$$

$$= 20xe^{-x/10} + k$$

where k is an arbitrary constant. However, $R = 0$, where $x = 0$.

$0 = 0 + k \Rightarrow k = 0$.

Thus the revenue function is given by

$R = 20xe^{-x/10}$

But $R = px$, where p is the price per unit. Hence

$px = 20xe^{-x/10}$

$p = 20e^{-x/10}$

which is the required demand function.

Illustration 6 A manufacturer's marginal revenue function is MR = 275 − x − $0.3x^2$. Find the increase in the manufacturer's total revenue if production is increased form 10 to 20 units.

Solution

The marginal revenue is $\text{MR} = 275 - x - 0.3x^2$, i.e., $\frac{dR}{dx} = 275 - x - 0.3x^2$

We have to find total in revenue when x increases from 10 to 20 units. That is, we have to find $R(20) - R(10)$. But

$$\begin{aligned}
R(20) - R(10) &= \int_{10}^{20} \frac{dR}{dx} dx \\
&= \int_{10}^{20} \left(275 - x - 0.3x^2\right) dx \\
&= \left[275x - \frac{x^2}{2} - 0.1x^3\right]_{10}^{20} \\
&= \left[275(20) - \frac{(20)^2}{2} - 0.01(20)^3\right] - \left[275(10) - \frac{(10)^2}{2} - 0.1(10)^3\right] \\
&= (5500 - 200 - 800) - (2750 - 50 - 100) \\
&= 1900
\end{aligned}$$

To Find the Maximum Profit if Marginal Revenue and Marginal Cost Functions are Given

If p denotes the profit function, then

$$\frac{dP}{dx} = \frac{d}{dx}(R - c) = \frac{dR}{dx} - \frac{dC}{dx} = \text{MR} - \text{MC}$$

Integrating both sides with respect to x gives

$$P = \int \frac{dP}{dx} = \int (\text{MR} - \text{MC}) dx + k$$

where k is the constant of integration. However, if we are given additional information, such as fixed cost or loss at zero level of output, we can determine the constant k. Once P is known, it can be maximized by using the concept of maxima and minima.

Remark: There is an alternative method for finding the maximum profit that avoids using the concept of maxima and minima. It uses the fact that maximum profit is obtained when "MR = MC". The maximum profit is then given by

$$P_{\max} = \int_{0}^{x_0} (\text{MR} - \text{MC}) dx$$

where x_0 is obtained by solving MR = MC. Actually Eq. (1) gives the additional profit only. So loss at zero level of output must be subtracted from this for obtaining the actual profit.

The following examples will illustrate.

Illustration 7 The marginal cost $C'(x)$ and the marginal revenue $R'(x)$ are given by $C'(x) = 20 + \frac{x}{20}$ and $R'(x) = 30$

The fixed cost is Rs. 200. Determine the maximum profit and the number of items produced for this profit.

Solution

Let P denote the profit function, then

$$\frac{dP}{dx} = \frac{dR}{dx} - \frac{dC}{dx} = 30 - 20 - \frac{x}{20} = 10 - \frac{x}{20}$$

Integrating both sides with respect to x, we get

$$p = \int \left(10 - \frac{x}{20}\right) dx = 10x - \frac{x^2}{40} + k$$

where k is the constant of integration. However, we are given that fixed cost is 200. That is

$P = -200$ when $x = 0$.

$-200 = k$

Thus $P = 10x - \frac{x^2}{40} - 200$ is the profit function.

For P to be maximum, it is necessary that

$$\frac{dP}{dx} = 0, \text{ i.e. } 10 - \frac{x}{20} = 0 \Rightarrow x = 200$$

Also $\frac{d^2P}{dx^2} = -\frac{1}{20} < 0$. Thus the output $x = 200$ indeed gives the maximum profit. The maximum profit is given by

$$P_{\max} = 10(200) - \frac{(200)^2}{40} - 200 = 800$$

Hence maximum profit is Rs. 800 which is obtained when 200 items are produced.

Illustration 8 A company suffers a loss of Rs. 121.50 if one of its products does not sell at all. Marginal revenue and marginal cost functions for the product are given by MR = $30 - 6x$ and MC = $-24 + 3x$ Determine the total profit function and the profit maximizing level of output.

Solution

If P denotes the profit function, then

$$\frac{dP}{dx} = \text{MR} - \text{MC} = (30 - 6x) - (-24 + 3x) = 54 - 9x$$

Integrating both sides with respect to x, we have

$$P = \int (54 - 9x)\,dx$$

$$= 54x - \frac{9}{2}x^2 + k$$

where k is an arbitrary constant. Since the company suffers a loss of Rs. 121.50 if the product does not sell at all, i.e. $P = -121.50$ when $x = 0$. Therefore

$$-121.50 = 54(0) - \frac{9}{2}(0)^2 + k \Rightarrow k = -121.50$$

Thus the total profit function is given by

$$P = 54x - \frac{9}{2}x^2 - 121.50$$

For P to be maximum, it is necessary that

$$\frac{dP}{dx} = 0 \text{ i.e., } 54 - 9x = 0 \ \Rightarrow x = 6$$

Also, $\frac{d^2P}{dx^2} = -9 < 0$

This shows that profit is maximum when $x = 6$.

Illustration 9 The marginal cost of producing x units of a commodity in a day is given as MC = $16x - 1591$. The selling price is fixed at Rs. 9 per unit and the fixed cost is Rs. 1,800 per day. Determine (1) Cost function, (2) Revenue function, (3) profit function, and (4) maximum profit that can be obtained in one day.

Solution

(1) The marginal cost function is given as

MC = $16x - 1591$

Integrating, we obtain

$$C = \int (16x - 1591)\,dx$$

where k is an arbitrary constant. However, since the fixed cost is Rs. 1,800, *i.e,* $C = 18{,}000$ when $x = 0$, we have

$$1800 = 8(0)^2 - 1591(0) + k \Rightarrow k = 1800.$$

Therefore the cost function is given by

$$C = 8x^2 - 1591x + 1800$$

(2) Since the selling price is fixed at Rs. 9 per unit, the revenue function is given by

$$R = 9x$$

(3) The profit function is given by subtracting the cost function from the revenue function. Thus

$$P = R - C = 9x - (8x^2 - 1591x + 1800)$$

$$= -8x^2 + 1600x - 1800$$

(4) For maximum profit, it is necessary that

$$\frac{dP}{dx} = 0$$

$$i.e,\ -16x+1600=0 \Rightarrow x=100$$

$$\frac{d^2P}{dx^2}=-16<0$$

This shows that profit is maximum when $x = 100$. The corresponding maximum profit is

$$P = (-8)(100)^2 + 1600(100) - 1800 = \text{Rs. } \mathbf{78{,}200}$$

To Find the Demand Function When the Price Elasticity of Demand is Given

If x is the number of units demanded at price p per unit of a certain product, then we know that the price elasticity of demand is given by

$$\eta d = -\frac{p}{x}\frac{dx}{dp}$$

The following examples illustrate how demand function could be obtained, using techniques of integration, if elasticity of demand is known.

Illustration 10 Derive the demand function which has the unit price elasticity of demand throughout.

Solution

It is given that

$$\eta d = 1$$

$$i.e,\ -\frac{p}{x}\frac{dx}{dp}=1$$

$$\Rightarrow \frac{dx}{x}+\frac{dp}{p}=0$$

Integrating, we have

$\log x + \log p = \log k$

where k is an arbitrary constant. Thus

$\log(xp) = \log k$

$\Rightarrow xp = p$

which is the required demand function.

Illustration 11 Obtain the demand function for a commodity whose elasticity of demand is given by $e_{xp} = bp$, where a, b are constants and p denotes the price per unit of the commodity.

Solution

The elasticity of demand is given by

$\exp = a - bp$

$$\text{i.e. } -\frac{p}{x}\frac{dx}{dp}=a-bp$$

$$\Rightarrow \frac{dx}{x}+\left(\frac{a}{p}-b\right)dp=0$$

Integrating, we have

$\log x + a \log p - bp = c$

where c is an arbitrary constant. Thus

$\log x + \log p^a = bp + c$

$\log\left(xp^a\right) = bp + c$

$\log xp^a = e^{bp + c}$

i.e. $x = p^{-a}e^{bp + c}$

which is the required demand function.

Illustration 12 The price elasticity of demand for a commodity is $\frac{p}{x^3}$. Find the demand function if the demand is 3 when the price is 2.

Solution

We are given

$$\eta_d = -\frac{p}{x}\frac{dx}{dp} = \frac{p}{x^3}$$

This gives $x^2dx + dp = 0$. Integrating, we get

$$\frac{x^3}{3} + p = k$$

where k is an arbitrary constant. However, it is given that $x = 3$ when $p = 2$.

Thus $\frac{(3)^3}{3} + 2 = k \Rightarrow k = 11$

Therefore $\frac{x^3}{3} + p = 11.$ Solving for p, we get

$$p = 11 - \frac{x^3}{3}$$

which is the required demand function.

Illustration 13 The elasticity of demand of a commodity with respect to price is calculated to be $\frac{5p}{(p+3)(p-2)}$ (where p is price). Find the demand function, if it is known that the quantity demanded is 5 units at $p = 3$.

Solution

Let e_d denote the elasticity of demand. Then

$$e_d = -\frac{p}{x}\frac{dx}{dp}$$

i.e. $\frac{5p}{(p+3)(p-2)} = -\frac{p}{x}\frac{dx}{dp}$

$$\Rightarrow \int \frac{5p}{(p+3)(p-2)}dp = -\int \frac{1}{x}dx$$

$$\Rightarrow \int \frac{-1}{p+3}dp + \int \frac{1}{p-2}dp = -\int \frac{1}{x}dx$$

(decomposing into particle fractions)

$$\Rightarrow -\log(p + 3) + \log(p - 2) = -\log x + \log k$$

where k is an arbitrary constant. Thus

$$\log \frac{p-2}{p+3} = \log \frac{k}{x}$$

$$\Rightarrow \frac{p-2}{p+2} = \frac{k}{x} \Rightarrow x = \frac{k(p+3)}{p-2}$$

It is given that $x = 5$ at $p = 3$. Hence

$$5 = \frac{k(3+3)}{3-2} \Rightarrow k = \frac{5}{6}$$

Thus the required demand function is given by

$$x = \frac{5(p+3)}{6(p-2)}$$

Illustration 14 The elasticity of cost is given by $\frac{3x}{2(3x+4)}$. Find the total cost function given that fixed cost is Rs. 20.

Solution

The elasticity of cost is given by

$$\eta_c = \frac{3x}{2(3x+4)}$$

i.e. $\frac{x}{C}\frac{dC}{dx} = \frac{3x}{2(3x+4)}$

$$\Rightarrow \frac{dC}{C} = \frac{3}{2}\frac{dx}{3x+4}$$

Integrating, we obtain

$$\log C = \frac{1}{2}\log(3x+4) + \log k$$

where k is an arbitrary constant. Thus

$$\log C = \log\sqrt{3x+4} + \log k$$

$$\Rightarrow \log C = \log k\sqrt{3x+4}$$

$$\Rightarrow C = k\sqrt{3x+4}$$

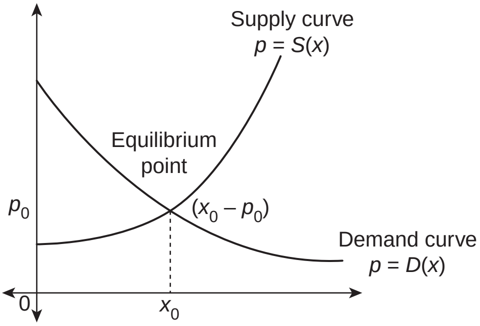
It is given that fixed cost is 20. That is, $C = 20$ when $x = 0$

$\therefore\ 20 = k\sqrt{0+4} \Rightarrow k = 10$

Hence $C = 10\sqrt{3x+4}$, which is the required cost function.

CONSUMER'S AND PRODUCER'S SURPLUS

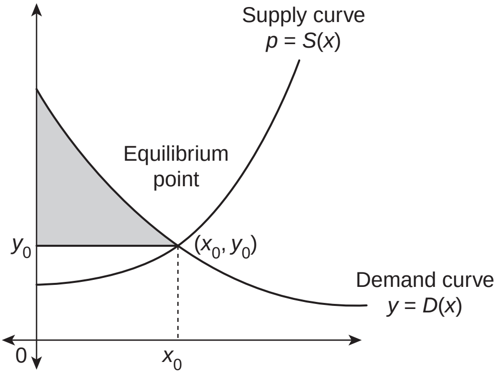
Determining the area of a region has application in economics. The figure shows the demand and supply curves for a product.



The intersection of a demand and supply curve is known as the equilibrium point. It is the point where market equilibrium is attained and occurs at the price where the quantity demanded is equal to the quantity supplied.

Let us assume that market is at equilibrium and the price per unit of the product is p_0. According to the demand curve, there are consumers who would be willing to pay more than p_0. These consumers will benefit by paying the lower equilibrium price. The total of their benefits is called the **consumer's surplus** (abbreviated *CS*) and is represented by the area between the line $p = p_0$ and the demand curve $p = D(x)$ from $x = 0$ to $x = x_0$ where x_0 is market demand. Thus

$$CS = \int_0^{x_0} D(x)dx - p_0 \times x_0$$



Consumer's Surplus Under Perfect Competition

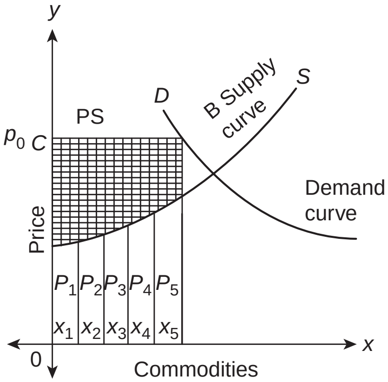
Under perfect competition, the equilibrium demand x_0 is obtained by equating the demand and supply functions. Once x_0 is obtained, the corresponding market price p_0 is obtained by substituting the value of x_0 in the demand function or the supply function.

TOPIC E

Producer's Surplus

Let x be the commodity. In *a* free market economy, sometimes a producer would be willing to sell the commodity x at a price below the market price p_0 that the consumer actually pays. But suddenly he sells at the prevailing market price p_0 and thus the producer is benefited.

This gain is called the producer's surplus. In other words the difference between the revenue actually received by the producer and the revenue he would have been willing to receive is called the producer's surplus. It is denoted by P_s.



Let the producer be willing to surplus all different prices P_1, P_2…P_n at less than market price to suppose that the producer supplies all these quantities at the price P_o. In the above diagram, the producer's surplus is the shaded portion.

Producer's Surplus

(Area of rectangle OABC) − (Area below the supply function term 0 to x_0)

$$\therefore PS = x_0 p_0 - \int_0^{x0} p\,dx.$$

ILLUSTRATIONS

Illustration 1 If the supply curve is $p = \sqrt{9+x}$ and quantity sold in the market is 7, find the producer's surplus. Can you find consumer's surplus?

Solution

We are given that the quantity sold is 7 units i.e. $x_0 = 7$. Substituting this value in supply function

$$p = \sqrt{9+x} \Rightarrow po = \sqrt{9+7} = 4$$

The produer's surplus is $PS = x_0 p_0 - \int_0^{x0} pdx$

$$= 7 \times 4 - \int_0^7 \sqrt{9 \times x}\ dx$$

$$= 28 - \left[\frac{2}{3}(9+x)^{3/2}\right]_0^7 = \frac{10}{3}$$

Hence producer's surplus $= \dfrac{10}{3}$

Illustration 2 The demand and supply functions under perfect competition are $y = 16 - x^2$ and $y = 2x^2 + 4$ respectively. Find the market price, consumer's surplus and producer's surplus.

Solution

We have demand function $y = 16 - x^2$ (1)

Supply function $y = 2x^2 + 4$ (2)

Subtracting (1) from (2) we have

$12 - 3x^2 = 0\ x = \pm 2$

Now $x = -2$ is discarded being meaningless

Thus $x = 2$

When $x = 2 = x_0$ then $y = 16 - (2)^2 = 12 = p_0$

Thus when the quantity demanded or supplied is 2 units the price is 12.

Consumer's surplus

$$\int_0^{x0} (\text{Demand function})\, dx - p_0 x_0 = \int_0^2 (16 - x^2)\, dx - 2 \times 12$$

$$\left(16x - \frac{x^3}{3}\right) - 24 = \frac{16}{3}$$

Now producer's surplus $= p_0 x_0 - \int_0^2 (\text{Supply function})\, dx$

$$= 2 \times 12 - \int_0^2 (2x^2 + 4)\, dx$$

$$= 24 - \left(\frac{2x^3}{3} + 4x\right)_0^2 = \frac{32}{3}$$

Illustration 3 Find consumer's surplus and producer's surplus defined by the demand curve $D(x) = 20 - 5x$ and the supply curve $S(x) = 4x + 8$

Solution

We know that the market equilibrium, under perfect competition can be obtained by equating the demand and supply curve

i.e. by putting $P(x) = S(x)$

$$\Rightarrow 20 - 5x = 4x + 8 \Rightarrow x = \frac{4}{3}(\text{say } x_0)$$

$$p_0 = D(x_0) = 20 - 5\left(\frac{4}{3}\right) = \frac{40}{3}$$

$$\therefore \text{ Consumer's surplus} = \int_0^{x0} D(x)dx - p_0x_0$$

$$= \int_0^{4/3} (20 - 5x)dx - \frac{40}{3}\frac{4}{3} = \left(20x - \frac{5x^2}{2}\right)_0^{4/3} - \frac{160}{9} = \frac{40}{9}$$

$$\text{Also producer's surplus} = x_0p_0 - \int_0^{x0} s(x)dx$$

$$= \frac{4}{3}\times\frac{40}{3} - \int_0^{4/3}(4x+8)dx = \frac{160}{9} = \left(\frac{4x^2}{2}+8x\right)_0^{4/3} = \frac{32}{9}$$

Illustration 4 Under perfect competition for a commodity, the demand and supply laws are $p_d = \frac{8}{x+1} - 2$ and $p_s = \frac{x+3}{2}$, respectively. Find the consumer's surplus.

Solution

We know that market equilibrium under perfect competition can be obtained by equating demand and supply.

$\therefore p_d = p_s \Rightarrow \frac{8}{x+1} - 2 = \frac{x+3}{2} \Rightarrow x^2 + 8x - 9 = 0$

$\Rightarrow (x - 1)(x + 9) = 0 \quad x = 1$ or $x = -9$

Now $x = -9$ is discarded being meaningless.

Thus $x = 1 = x_0$(say)

When $x = 1 \ (=x_0)$ then $p_0 = \frac{8}{1+1} - 2 = 2$

$$\text{Consumer's surplus} = \int_0^{x0} p_d\, dx - p_0x_0$$

$$= \int_0^1 \left(\frac{8}{x+1} - 2\right)dx - 2\times 1$$

$$= \left(8\log|x+1| - 2x\right)_0^1 - 2 = \log 2 - 4$$

Illustration 5 The demand and supply functions under perfect competition are $p_d = 1600 - x^2$ and $p_s = 2x^2 + 400$ respectively. Find the consumer's surplus and producer's surplus.

Solution

Under perfect competition

$pd = ps$

$\Rightarrow 1600 - x^2 = 2x^2 + 400 \Rightarrow 3x^2 = 1200$

$\Rightarrow x^2 = 400 \Rightarrow x = 20$

$x_0 = 20$ and $p = 1600 - (20)^2 = 1200$

Hence $x_0 = 20$ and $p_0 = 1200$ is the point equilibrium

$$\therefore \text{ Consumer's surplus } = \int_0^{x0} (p_d)\,dx - x_0 p_0$$

$$= \int_0^{20} (1600 - x^2)\,dx - 20 \times 1200 = \left(1600x - \frac{x^3}{3}\right)_0^{20} - 2400$$

$$= \frac{16{,}000}{3}$$

$$\text{Producer's surplus } = x_0 p_0 - \int_0^{x0} (p_s)\,dx$$

$$= 20 \times 1200 - \int_0^{20} (2x^2 + 400)\,dx$$

$$= 24{,}000 - \left(\frac{2x^3}{3} + 400x\right)_0^{20} = \frac{32000}{3}$$

Illustration 6 If the demand function = $D(x) = (6 - x)^2$ and supply function is $S(x) = 14 + x$, find the consumer's surplus under monopoly. The supply function is identified with marginal cost.

Solution

Here total revenue = TR = $D(x) \times x$

$= (36 - 12x + x^2)\ x$

$\therefore \text{MR} = \frac{d}{dx}\text{TR} = 36 - 24x + 3x^2$

Now the supply function is given by $S(x) = 14 + x$. Since the supply function is identified with MC

$\therefore$ MC = $S(x) = 14 + x$

From consumer's surplus under monopoly. i.e., to maximize the profit, we must have

MR = MC

$\Rightarrow 36 - 24x + 3x^2 = 14 + x$

$\Rightarrow x = 1 \text{ or } x = \frac{22}{3}$

Case–I When $x = 1 = x_0$ then $p_0 = 36 - 12 + 1 = 25$

$$\therefore \text{ Consumer's surplus } = \int_0^{1} (36 - 24x + 3x^2)\,dx - 1 \times 25$$

$$= (36x - 12x^2 + x^3)_0^1 - 25 = 0$$

Case–II When $x = \frac{22}{3} = x_0$ and

$$p_0 = 36 - 24\left(\frac{22}{3}\right) - 3\left(\frac{22}{3}\right)^2 = \frac{328}{3}$$

Consumer's surplus under monopoly

$$\int_0^{22/3} \left(36 - 12x + x^2\right) dx - \left(\frac{22}{3}\right)\left(\frac{328}{3}\right)$$

$$= \left(36x + 6x^2 + \frac{x^3}{3}\right)_0^{22/3} - \left(\frac{22}{3}\right)\left(\frac{328}{3}\right)$$

$$= \frac{4951}{81} \text{ units}$$

ANALYTICAL EXERCISES

1. The demand function for a commodity is given by $p = 100 - 8x$. Find the consumer's surplus corresponding to $p = 4$.
2. The supply function of a commodity is $10p - 2\sqrt{x + 300} = 0$ and the market price is Rs. 8. Find the producer's surplus.
3. If the demand function for a commodity is $10p = \sqrt{900 - x}$ and the demand is fixed at 500. Find the consumer surplus by two methods (*a*) integrate with respect to x (*b*) integrate with respect to p
4. The demand function for a monopolist is $3x = 60 - 10p$ and his average cost function is $\frac{20}{x} + 1 + 0.2x$ where p and x refer to the price and quantity of the commodity respectively. Determine the effect on consumer's surplus if the monopolist decides to maximize sales instead of profits.
5. When transistors were priced at Rs. 400 by a firm, 20 per week were sold. When the price was reduced to Rs. 100 the sale was 120 per week. The supply curve is $p = 2x$. Determine consumer's surplus
6. For a monopolist, the demand law is $p = 50 - x^2$ and marginal cost $MC = 1 - x^2$ Determine the consumer's surplus at the price which monopolist will like to fix. If there is a perfect competition in this case what will be the consumer's surplus at equilibrium prices?
7. The demand equation for a product is $x = \sqrt{100 - p}$ and the supply equation is $x = \frac{p}{2} - 10$. Determine the consumer's surplus and producer's surplus under market equilibrium.
8. The demand and supply functions for a commodity are given by $p = 24\, e^{-x}$ and reply. Determine the consumer's and producer's surplus.

ANSWERS

(1) Rs. 576
(2) 3.33
(3) 266.67
(4) 3.75
(5) 12696
(6) $\frac{343}{3}$
(7) CS = 341.33. PS = 64
(8) CS = 18 − 6log 4, *ps* = 6log 4 − 6

TOPIC F

Revenue Function and Demand Function When Elasticity of Demand is Given

Revenue Function

Revenue function $R = px$

ILLUSTRATION

Illustration 1 Obtain the demand function for a commodity for which the elasticity of demand is constant '$\propto$' throughout.

Solution

Now Elasticity of demand $= \dfrac{-p}{x}\dfrac{dx}{dp} = \propto$

$$\Rightarrow \frac{dx}{x} + \propto \frac{dp}{p} = 0$$

$$\Rightarrow \int \frac{dx}{x} + \propto \int \frac{dp}{p} = \int 0$$

$$\Rightarrow \log x + \propto \log p = \log k$$

$$\Rightarrow \log x + \log p^{\propto} = \log k$$

$$\Rightarrow \log\left(xp^{\propto}\right) = \log k$$

$$\Rightarrow xp^{\propto} = k$$

which is the required demand function.

Demand Function

Elasticity of demand $\eta_d = \dfrac{-p}{x}\dfrac{dx}{dp}$ (i)

$$\therefore \frac{-dp}{p} = \frac{dx}{\eta_d x} \qquad \text{(ii)}$$

when η_d is a function of x and integrate (2) with respect to variables

$$-\int \frac{dp}{p} = \int \frac{dx}{\eta_d x}$$

which gives us p is a function of x. It is the required demand function.

ILLUSTRATIONS

Illustration 1 The elasticity of demand with respect to price for a commodity is given by $\left(4 - x\right)/x$ where p is the price when demand is x. Find the demand function when price is 4 and the demand is 2. Also find the revenue function.

Solution

$$\eta_d = \frac{-p}{x}\frac{dx}{dp} = \frac{4-x}{x}$$

$$\Rightarrow \frac{-dx}{4-x} = \frac{dp}{p}$$

$$\Rightarrow -\int \frac{dx}{4-x} = \int \frac{dp}{p}$$

$$\Rightarrow \log(4-x) = \log p + \log k$$

$$\Rightarrow \log(4-x) = \log pk$$

$$\Rightarrow (4-x) = pk$$

when $x = 2\ p = 4$

$$\Rightarrow 4-2=4k \Rightarrow k=\frac{1}{2}$$

∴ demand function

$$p = \frac{4-x}{k} = \frac{4-x}{(1/2)} = 8-2x$$

Also revenue function

$$R = px = (8 - 2x)x$$

$$\therefore R = 8x - 2x^2$$

Illustration 2 The price elasticity of a commodity is $\eta_d = \frac{3p}{(p-1)(p+2)}$. Find the corresponding demand function when quantity demanded is 8 units and the price is Rs. 2.

Solution

We have $\eta_d = \frac{3}{(p-1)(p+2)}$

$$\Rightarrow \frac{-p}{x}\frac{dx}{dp} = \frac{3p}{(p-1)(p+2)} \Rightarrow \frac{-dx}{x} = \frac{3}{(p-1)(p+2)}dp$$

$$\Rightarrow -\int \frac{dx}{x} = \int \frac{3dp}{(p-1)(p+3)}$$

$$\Rightarrow -\int \frac{dx}{x} = \int \left(\frac{1}{p-1} - \frac{1}{p+2}\right)dp$$

$$\Rightarrow \log x + \log|p - 1| - \log|p + 2| = \log c$$

when $p = 2, x = 8$

$$\therefore \log 8 + \log 1 - \log 4 = \log c \Rightarrow \log c = \log 2 \Rightarrow c = 2$$

∴ The required demand function is

$$\log x + \log|p - 1| - \log|p + 2| = \log 2$$

$$\therefore \log\frac{x(p-1)}{p+2} = \log_{\text{¢}}^{2} \Rightarrow x = \frac{2(p+2)}{p-1}$$

Illustration 3 The elasticity of cost is given by $\frac{3x}{2(3x+4)}$. Find the total cost function given that the fixed cost is Rs. 20

Solution

$$\eta_c = \frac{3x}{2(3x+4)} \qquad \left(\because \eta_c = \frac{x}{c}\frac{dc}{dx}\right)$$

$$\therefore \frac{x}{c}\frac{dx}{dc} = \frac{3x}{2(3x+4)} \Rightarrow \frac{dc}{c} = \frac{3}{2}\frac{dx}{(3x+4)}$$

$$\Rightarrow \int \frac{dc}{c} = \frac{3}{2}\int \frac{dx}{3x+4}$$

$$\therefore \log c = \frac{1}{2}\log|3x+4| + \log k$$

$$\therefore C = k\sqrt{3x+4}$$

When $x = 0$ $c = 20$

$$\therefore 20 = k\sqrt{4} \Rightarrow k = 10$$

$$\therefore \text{cost } c(x) = 10\sqrt{3x+4}$$

TOPIC G

Capital Formation

Capital formation is the process of adding to a given stock of capital. The rate of capital formation can thus be expressed as $\frac{dk}{dt}$ where k is a function of t. But rate of capital formation is same as rate of net flow of investment $\text{I}(t)$.

$$\text{I}(t) = \frac{dk}{dt}$$

$$\therefore \int \text{I}(t)dt = \int\left(\frac{dk}{dt}\right)dt = k(t)$$

$$k(t) = \int \text{I}(t)dt + c$$

Where c is the content of integration and it is a required capital growth equation.

ILLUSTRATIONS

Illustration 1 If the investment flow is given by $\text{I}(t) = 5t^{1/4}$ and the capital stock at $t = 0$ is k_0, find the time path of capital k.

Solution

We know that the rate of capital formation = rate of net investment flow.

$$\therefore \frac{dk}{dt} = \text{I}(t)$$

Time path of capital $k = \int \mathrm{I}(t)\,dt$

$$\therefore k = \int 5t^{1/4}dt = 5\left(\frac{t^{5/4}}{5/4}\right) + c$$

$$\therefore k = 4t^{5/4} + C$$

But when $t = 0,\ k = k_0 \Rightarrow c = k_0$

$$\therefore k = 4t^{5/4} + k_0$$

which is the required time path of capital.

Illustration 2 If $\mathrm{I}(t) = 3t^{1/3}$ crores of rupees per year, what will be the capital formation in the time period 5 years and during the last year of the plan period i.e. 5th year? Also find the capital growth equation or time path of capital when $k_{(0)} = 25$.

Solution

(I) Given $\mathrm{I}(t) = 3t^{1/3}$

$$\therefore k(t) = \int_4^5 3t^{1/3}dt = 3\left(\frac{t^{4/3}}{4/3}\right)_4^5 = \frac{9}{4}\left(t^{4/3}\right)_4^5$$

$$= \frac{7}{4}\left(3\sqrt{625} - 3\sqrt{256}\right)$$

in the 5th year

(II) Capital growth equation

$$k(t) = \int 3t^{1/3}dt = \frac{9}{4}t^{4/3} + c$$

It is also given that initial stock of capital is 25.

$$\therefore k(0) = 25 = 0 + c \Rightarrow c = 25$$

$$k(t) = \frac{9}{4}t^{4/3} + 25$$

is the required capital growth equation.

TOPIC H

Rate of Growth or Sale

If the rate of growth or sale of a function is a known function of t say $f(t)$ where t is a time measure then the total growth or sale of the product over a time period t is

$$\text{Total Sale} = \int_0^T f(t)\,dt$$

Suppose the rate of new product is given by $f(x) = 100 - 90e^{-x}$ where x is the number of days the product is on the market. Find the total sales during the first 4 days.

Solution

$$\text{Total sale} = \int_0^4 \left(100 - 90e^{-x}\right) dx$$

$$= \left(100x + 90e^{-x}\right)_0^4$$

$$= \left(100x + 90e^{-4}\right) - \left(0 + 90e^{0}\right)$$

$$= 310 + 90\left(0.018\right) = 311.62 \text{ units}$$

ILLUSTRATIONS

Illustration 1 A company has current sales of Rs. 1,000 per month and profit to the company averages 10% of sales. The company's past experience with a certain advertising strategy is that sales increase by 2% per month continuously over the length of the advertising campaign (12 months). If the advertisement cost is Rs. 130 in a year, determine if the company should embark on a similar campaign when the market rate of interest is 12% per annum.

Solution

The monthly rate of sales during the advertising campaign is a growth curve $= \mathrm{A}e^{rt}$

where A = current rate = 1000

r = rate of growth = 2%

The actual sales during the interval $\left(0, 12\right)$ (Length of advertising campaign) is

$$\text{Total Sales } \int_0^t \mathrm{A}\, e^{rt} dt = \int_0^{12} 1000\, e^{0.02t}\ dt$$

$$\frac{1000}{0.02}\left(e^{0.02t}\right)_0^{12} = 50{,}000\left(e^{24} - 1\right)$$

Rs. 13,500

$\therefore$ The profit to the company is 10% of sales, hence the profit due to increase in sales is

$0.10\left(13560 - 12000\right) = \text{Rs. } 156$

Net profit due to advertising

$= \text{Rs. } \left(1560 - 130\right) = \text{Rs. } 26$

Interest on advertising cost

$$= \text{Rs.}\left(\frac{100 \times 12 \times 1}{100}\right)$$

Since Rs. 56 is more than Rs. 12 so the company should continue with the advertising.

Illustration 2 The purchase price of a car is Rs. 1,75,000. The rate of repairing cost of the car is given by $c = 8000\left(1 - e^{-0.5t}\right)$ where t represents years of use since purchase and c denotes the cost. Determine the cumulative repair cost at the end of 5 years. Also find the equation to give the time in years at which the cumulative repair cost equals the original cost of the car.

Solution

The cumulative repair cost at the end of 5 years is

$$\text{Repair cost} = \int_0^5 c(t)dt = \int_0^5 8000\left(1 - e^{-0.5t}\right)at$$

$$= 8000\left(t + \frac{e^{-0.5t}}{0.5}\right)_0^5$$

$$= 8000\left(5 + 2e^{-2.5} - 2\right)$$

$$= 8000\left[3 + 2(0.0821)\right]$$

$$= \text{Rs. } 25{,}313.60$$

Obviously these costs are not enough to offset the purchase price of Rs. 1,75,000. To find how long it would take for the cumulative repair cost to equal the original cost, we need to find t such that

$$\int_0^t 8000\left(1 - e^{-0.5t}\right)at = 175000$$

$$\Rightarrow \left(t + 2e^{-0.5t}\right)_0^t = \frac{175}{8}$$

$$\Rightarrow t + 2e^{-0.5t} - 2 = \frac{175}{8}$$

$$\Rightarrow 8t + 16e^{-0.5t} - 191 = 0$$

which is the required equation.

Illustration 3 A firm has current sales of Rs. 50,000 per month. The firm wants to embark on a certain advertising campaign that will increase the sales by 2% per month (compounded continuously) over the period of the campaign which is 12 months. Find the total increase in sales as a result of the campaign $\left(e^{0.24} = 1.272\right)$.

Solution

Since the advertising (campaign increases the sale by 2% per month (compounded continuously) so the rate of increase in sales per month follows a growth curve of the form

$$\mathrm{I}(t) = 50{,}000e^{0.02t}$$

$$\therefore \text{Total share after 12 months} = \mathrm{I}(t) = \int_0^{12} 50{,}000e^{-0.02t}\ at$$

$$= \frac{50,000}{0.02}\left(e^{0.02t}\right)_0^{12} \frac{50,000,00}{2}\left(e^{0.24} - 1\right) = 6,80,000$$

∴ Total increase in sales as a result of the campaign

$= 6,80,000 - (50,000)12 =$ Rs. 80,000

TOPIC I

Inventory

Given the inventory on hand $I(x)$ and the unit holding cost H_c, the total inventory carrying cost is given by

$$\text{Total inventory carrying cost} = \text{H}c \int_0^T I(x)dx$$

where T is the time period under consideration.

ILLUSTRATIONS

Illustration 1 A company receives a shipment of 200 cars every 30 days. From experience it is known that the inventory on hand is related to the number of days. Since the last shipment by $\text{I}(x) = 200 - 0.2x$, find the daily holding cost for maintaining inventory for 30 days.

Solution

Here $I(x) = 200 - 0.2x$

$H_c =$ Rs. 3.5

$T = 30$

∴ Total inventory carrying cost

$$= H_c \cdot \int_0^T I(x)dx = 3.5\int_0^{30}(200 - 0.2x)dx$$

$$= 3.5\left(200x - \frac{0.2x^2}{2}\right)_0^{30}$$

$= 20,685$

Illustration 2 A company receives a shipment of 500 scooters every 30 days. From experience it is known that the inventory on hand is related to the number of days x. Since the last shipment by $I(x) = 500 - 0.03x^2$, the daily holding cost for one scooter is Rs. 0.3. Determine the total cost for maintaining inventory for 30 days.

Solution

Total inventory carry cost for 30 days

$$= H_c \int_0^T I(x)dx = 0.3\int_0^{30}(500 - 0.03x^2)dx$$

$$= 0.3\left(500x - 0.03\frac{x^3}{3}\right)_0^{30} = 0.3(15,000 - 270)$$

$= \text{Rs. } 4,419$

TOPIC J

Amount of an Annuity

The amount of an annuity is the sum of all payments made plus all interest accumulated. Let an annuity consist of equal payments of Rs. P and let the interest rate of *r* per cent annually be compounded continuously.

$$\text{Amount of annuity after N payments } \text{A} = \int_0^N P e^{rt}\, at$$

ILLUSTRATIONS

Illustration 1 Mr. Amit places Rs. 10,000 in ABC Bank each year, which pays an interest of 10% per annum compounded continuously for 5 years. How much amount will there be after 5 years? $\left(e^{0.25} = 1.284\right)$

Solution

$p = 10,000,\ r = 0.10,\ \text{N} = 5$

$$\text{Annuity } \int_0^5 10,000e^{0.10t}\, at$$

$$= \frac{10,000}{0.01}\left(e^{0.10t}\right)_0^5 = 1,00,000\left(e^{0.50} - 1\right)$$

$= 1,00,000\left(0.0487\right) = \text{Rs. } 64,870$

Illustration 2 An account fetches interest at the rate of 5% per annum compounded continuously. An individual deposits Rs. 1,000 each year in the account. How much amount will there be in the account after 5 years?

Solution

$$\text{Amount } \int_0^5 1000e^{0.05t} \text{ at}$$

$$= \frac{1000}{0.05}\left(e^{0.25t}\right)$$

$20,000\left(e^{0.05t}\right)$

$= 20,000 \times 0.284$

$= \text{Rs. } 5,680$

ANALYTICAL EXERCISES OF TOPICS F, G, H, I, J

1. Obtain the demand function for which the elasticity of demand is
2. Obtain the demand function for a commodity where elasticity of demand is given by $n_d = a - bp$ where a and b are constants and p denotes the price per units of commodity
3. The elasticity of demand with respect to price is $\eta d = \frac{5p}{(p+3)(p-2)}$ (where p is price). Find the demand function if it is known that the quantity demanded is 5 units at $p = 3$.
4. The price elasticity of demand of a commodity is given by $\eta_d = bp - a$. Where a and b are given constants show that the demand law is $x = p^a e^{-bc(p+c)}$, c being an arbitrary constant.
5. Obtain the demand function for a commodity where price elasticity a is given by $\frac{1}{kx}$ where k is cost constant.
6. If the net investment has been occurring since the beginning of history at the rate of $I(t) = 100e^{0.2t}$ per year, then find the total capital formation from the present. $\left(\text{Hint } k(t) = \int_{-\infty}^{0} 100e^{0.2t}\, dt = 500\left(e^{0} - e^{-\infty}\right) = 500\right)$
7. A company has a current sales of Rs. 10,00,000 per month and profit to the company averages 10% of sales. The company's past experience with a certain advertising strategy is that sales will increase by 1.5% per month over the length of the advertising campaign which is 12 months. If the advertisement cost is Rs. 1,30,000, determine if the company should embark on a similar campaign when the market rate of interest is 10% per month.
8. A sum of Rs. 800 is invested in a machine whose life is 10 years. If the interest is reckoned continuously at 100% per year, show that the investment is not profitable if it gives an income stream of Rs. 100 per year for 10 years. $\left(\text{Hint } Pv_{10} \int_{-0}^{10} 100e^{0.1t}\, dt = \text{Rs. } 260.40\right)$
9. After an advertising campaign a product has sales rate $f(t)$ given $f(t) = 100e^{-0.5t}$ where t is the number of months since the close of the campaign. Find (a) The total cumulative sales after 3 months, (b) sales during the fourth month, (c) total sales as a result of the campaign. $\left(\text{Hint: } (a) \int_{0}^{3} 100e^{-0.5t}\, dt = 1{,}554 \text{ units}\right.$

$(b)\ \int_{3}^{4} 100e^{-0.5t}\, dt = 176 \text{ units}$

$\left.(c)\ \int_{0}^{\infty} 100e^{-0.5t}\, dt = 2{,}000 \text{ units}\right)$

10. If the rate of sales of a new product is given by $f\left(x\right) = 200 - 90\, e^{-x}$, where x is the number of days the product is in the market, find the total sales during the first four days.

ANSWERS

(1) $xp = k$

(2) $x = k \cdot p^{-a}e^{bp}$

(3) $x = \dfrac{5\left(p-3\right)}{6\left(p-2\right)}$

(5) $\dfrac{p}{c} = e^{-kc}$

(7) Increases in sale = 11,33,333,
profit due to increase in sale = 1,13,333.

(10) 311.62 units

14

Point

LEARNING OBJECTIVES

After studying this chapter, the student will be able to understand:

- Co-ordinate system of R^2
- Distance formulae in R^2
- Properties of distance formulae in R^2
- Circumcentre of a triangle
- Shifting the origin without changing the direction of axes
- Division of line segment
- Centroid of a triangle
- Incentre of a triangle

INTRODUCTION

DISTANCE FORMULAE IN R^1

$$A(x),\ B(y) \in R^1 \text{ then } d(A,\ B) = |AB| = |x - y|$$

DISTANCE FORMULAE IN R^2

$$A\ (x_1,\ y_1),\ B\ (x_2,\ y_2) \in R^2 \text{ then}$$
$$d(A,\ B) = AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

DISTANCE FUNCTION

$$d : R^2 \times R^2 \rightarrow R^+ \cup (0);$$
$$d\left[(x_1,\ y_1),\ (x_2,\ y_2)\right] = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\ ;\ d$$

is said to be the *distance function* or *matric function* over $R^2 \times R^2$ and $(R^2,\ d)$ is called the *matric space*.

Properties of Distance Function

If A $(x_1,\ y_1)$, B $(x_2,\ y_2)$, C $(x_3,\ y_3) \in R^2$ then

1. $d\ (A,\ B) \geq 0$
2. $d\ (A,\ B) = 0 \Leftrightarrow A = B$

3. $d\left(\mathrm{A, B}\right) = d\left(\mathrm{B, A}\right)$
4. $d\left(\mathrm{A, B}\right) + d\left(\mathrm{B, C}\right) \geq d\left(\mathrm{A, C}\right)$

Transformation of Co-ordinates

Change of Origin and Translation of Axes

If the origin 'O' $(0, 0)$ is shifted to 0' (h, k) such that the directions of new co-ordinate axes are the same as those of the old axes then the relation between the old co-ordinates (x, y) and new co-ordinates (x_1, y_1) of a point P is

$$(x, y) = (x_1 + h, y_1 + k)$$

or

$$(x_1, y_1) = (x - h, y - k)$$

CIRCUMCENTRE OF A TRIANGLE

The point of concurrence of the $\perp^{er}$ bisectors of all the three sides of a triangle is called the *circumcentre* of the triangle.

Area of ΔABC

Let A (x_1, y_1), B (x_2, y_2), C (x_3, y_3) be three non-collinear points in R^2 then the area of ΔABC is

$$\Delta = \frac{1}{2}|D| \text{ where } D = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Condition for Collinearity

If A (x_1, y_1), B (x_2, y_2); C (x_3, y_3) be collinear then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Division of a Line Segment

$\overline{\mathrm{AB}}$ is a given line segment and $\mathrm{P} \in \overleftrightarrow{\mathrm{AB}} - \{\mathrm{A, B}\}$ then point P is said to be dividing $\overline{\mathrm{AB}}$ in the ratio

(1) $\dfrac{\mathrm{AP}}{\mathrm{PB}}$ if A − P − B (2) $\mathrm{P} \in \overleftrightarrow{\mathrm{AB}} - \{\mathrm{A, B}\}$ if P − A − B or A − B − P.

Moreover if the ratio of division of line segment is denoted by λ then

(1) $\lambda > 0 \Leftrightarrow \mathrm{A - P - B}$
(2) $-1 < \lambda < 0 \Leftrightarrow \mathrm{P - A - B}$
(3) $\lambda < -1 \Leftrightarrow \mathrm{A - B - P}$

Case (1) is said to be internal division of the line segment and cases (2) and (3) are said to be external divisions of the line segment.

If $\mathrm{P}(x, y)$ divides the $\overline{\mathrm{AB}}$ in ratio λ A (x_1, y_1), B (x_2, y_2) and $\lambda \neq -1$ from A $(\mathrm{P} \neq \mathrm{A}, \mathrm{P} \neq \mathrm{B})$ then

(1) $\mathrm{P}\ (x, y) = \left(\dfrac{\lambda x_2 + x_1}{\lambda + 1}; \dfrac{\lambda y_2 + y_1}{\lambda + 1}\right)$

(2) If $\lambda = \dfrac{m}{n}$; $n \neq 0$ then

$$P\left(x, y\right) = \left(\frac{mx_2 + nx_1}{m+n}; \frac{my_2 + ny_1}{m+n}\right)$$

(3) The midpoint of $\overline{AB} = P\left(x, y\right) = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

CENTROID OF A TRIANGLE

The point of concurrence of the three medians of a triangle is called the *centroid*. The centroid of the triangle is denoted by G.

If A $\left(x_1, y_1\right)$, B $\left(x_2, y_2\right)$ and C $\left(x_3, y_3\right)$ are three vertices of ΔABC then

$$G\left(x, y\right) = \left(\frac{x_1 + x_2 + x_3}{3}; \frac{y_1 + y_2 + y_3}{3}\right)$$

INCENTRE OF A TRIANGLE

The point of concurrence of the bisectors of the three angles of a triangle is called the *incentre* of the triangle and is denoted by I.

If A $\left(x_1, y_1\right)$, B $\left(x_2, y_2\right)$ and C $\left(x_3, y_3\right)$ are three vertices of ΔABC and if AB = c, BC = a, AC = b then

$$I\left(x, y\right) = \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}; \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

ILLUSTRATIONS

Illustration 1 $\left(a + b, b + c\right), \left(a - b, c - b\right)$

Solution

Let A $\left(a + b, b + c\right)$ and B $\left(a - b, c - b\right)$

$$\therefore d\left(\text{A, B}\right) = |\text{AB}| = \sqrt{\left[\left(a+b\right) - \left(a-b\right)\right]^2 + \left[\left(b+c\right) - \left(c-b\right)\right]^2}$$

$$= \sqrt{\left(a + b - a + b\right)^2 + \left(b + c - c + b\right)^2}$$

$$= \sqrt{\left(2b\right)^2 + \left(2b\right)^2}$$

$$= \sqrt{8b^2}$$

$$= 2\sqrt{2}\,|b|$$

$$\therefore d\left(\text{A, B}\right) = |\text{AB}| = 2\sqrt{2}\,|b|$$

Illustration 2 If P $\left(x, y\right)$ is equidistance from A $\left(a + b, b - a\right)$ and B $\left(a - b, a + b\right)$ then prove that $bx = ay$.

Solution

Here P is equidistance from A and B

$\therefore$ PA = PB

$\therefore$ $PA^2 = PB^2$

$\therefore \left[x - (a+b)\right]^2 + \left[y - (b-a)\right]^2 = \left[x - (a-b)\right]^2 + \left[y - (a+b)\right]^2$

$\therefore x^2 - 2x(a+b) + (a+b)^2 + y^2 - 2y(b-a) + (b-a)^2 =$

$x^2 - 2x(a-b) + (a-b)^2 + y^2 - 2y(a+b) + (a+b)^2$

$\therefore -2x(a+b) - 2y(b-a) = -2x(a-b) - 2y(a+b)$

$\therefore 2x(a+b) + 2y(b-a) = 2x(a-b) + 2y(a+b)$

$\therefore 2ax + 2bx + 2by - 2ay = 2ax - 2ax - 2bx + 2ay + 2by$

$\therefore 4bx = 4ay$

$\therefore bx = ay$

Illustration 3 Show that $\left(1, -\frac{3}{2}\right), \left(-3, -\frac{7}{2}\right), \left(-4, -\frac{3}{2}\right)$ are the vertices of a right triangle.

Solution

Let A $\left(1, -\frac{3}{2}\right)$ B $\left(-3, -\frac{7}{2}\right)$ and C $\left(-4, -\frac{3}{2}\right)$

$\therefore AB^2 = (1+3)^2 + \left(-\frac{3}{2} + \frac{7}{2}\right)^2 = 16 + 4 = 20$

$\therefore BC^2 = (-3+4)^2 + \left(-\frac{7}{2} + \frac{3}{2}\right)^2 = 1 + 4 = 5$

and $AC^2 = (1+4)^2 + \left(-\frac{3}{2} + \frac{3}{2}\right)^2 = 25$

$\therefore$ we can say that

$AB^2 + BC^2 = AC^2$

$\therefore m\angle B = \frac{\pi}{2}^R$

$\therefore$ ΔABC is a right angled triangle. Hence above given vertices are the vertices of a right angled triangle.

Illustration 4 Prove that $(2, 3)$, $(4, 5)$, $(3, 2)$ are the vertices of a right triangle.

Solution

Let A $(2, 3)$, B $(4, 5)$ and C $(3, 2)$

$\therefore AB^2 = (2-4)^2 + (3-5)^2 = 4 + 4 = 8$

$BC^2 = (4-3)^2 + (5-2)^2 = 1 + 9 = 10$

$AC^2 = (2-3)^2 + (3-2)^2 = 1 + 1 = 2$

$\therefore$ We can say that

$AB^2 + AC^2 = BC^2$

$\therefore m\angle A = \frac{\pi}{2}$

$\therefore$ ΔBAC is a right-angled triangle. Hence above given are the vertices of a right angled triangle.

Illustration 5 If $(2, 3)$ is the circumcentre of the triangle whose vertices are $(a, 6)$, $(5, 1)$, abd $(4, b)$, determine the value of a and b.

Solution

Let A $(a, 6)$, B $(5, 1)$ and C $(4, b)$ be the vertices of ΔABC. P $(2, 3)$ is the circumcentre of ΔABC.

$PA^2 = (a - 2)^2 + (6 - 3)^2 = a^2 - 4a + 13$

$PB^2 = (2 - 5)^2 + (3 - 1)^2 = 13$

$PC^2 = (2 - 4)^2 + (3 - b)^2 = b^2 - 6b + 13$

Since P $(2, 3)$ is the circumcentre of ΔABC

$\therefore PA = PB = PC \Rightarrow PA^2 = PB^2 = PC^2$

$\therefore a^2 - 4a + 13 = 13 = b^2 - 6b + 13$

$\therefore a^2 - 4a = 0$ and $b^2 - 6b = 0$

$\therefore a(a - b) = 0$ and $b(b - 6) = 0$

$\therefore a = 0$ or $a = 4$ and $b = 0$ or $b = 6$

$\therefore$ The point A $(a, 6)$ = A $(0, 6)$ or A $(4, 6)$

and B $(4, b) \Rightarrow$ B $(4, 0)$ or B $(4, 6)$

$\therefore$ It is clear that when $a = 4$, $b \neq 6$ $[\therefore A\ (4, 6) \neq B\ (4, 6)]$

$\therefore$ If $a = 0$ then $b = 0$ or 6 and

if $a = 4$ then $b = 0$.

Illustration 6 P $(at^2, 2at)$, Q $\left(\frac{a}{t^2}; \frac{-2a}{t}\right)$ and S $(a, 0)$ are three points. Show that $\frac{1}{SP} + \frac{1}{SQ} = t$ is independent of t.

Solution

$SP^2 = (at^2 - a^2) + (2at - 0)^2$

$= a^2\left[(t^2 - 1)^2 + 4t^2\right] = a^2(t^2 + 1)^2$

$\therefore SP = |\,a\,|\,(t^2 + 1)$

$$\text{and similarly } SQ = \left(a - \frac{a}{t^2}\right)^2 + \left(0 + \frac{2a}{t}\right)^2$$

$$= a^2\left(1 - \frac{1}{t^2}\right)^2 + \frac{4a^2}{t^2}$$

$$= a^2\left(1 - \frac{2}{t^2} + \frac{1}{t^4} + \frac{4}{t^2}\right)$$

$$= a^2\left(1 + \frac{1}{t^2}\right)^2$$

$$\therefore SQ = |a|\left(1 + \frac{1}{t^2}\right)$$

$$\text{Now } \frac{1}{SP}+\frac{1}{SQ} = \frac{1}{|a|\left(t^2+1\right)}+\frac{1}{|a|\left[1+\left(1/t^2\right)\right]}$$

$$= \frac{1}{|a|\left(t^2+1\right)}+\frac{t^2}{|a|\left(t^2+1\right)}$$

$$= \frac{1+t^2}{|a|\left(t^2+1\right)} = \frac{1}{|a|}$$

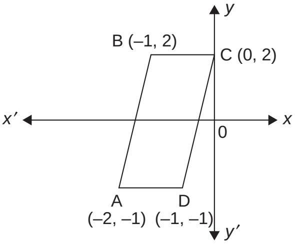
which is independent of *t*.

Illustration 7 Show that $\left(-2, -1\right)$, $\left(-1, 2\right)$, $\left(0, 2\right)$ and $\left(-1, -1\right)$ are the vertices of a parallelogram.

Solution

Let A $\left(-2, -1\right)$, B $\left(-1, 2\right)$, C $\left(0, 2\right)$ and D $\left(-1, -1\right)$ be the vertices of $\square^m$ABCD.

$\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ are the sides of $\square^m$ABCD.



$$\therefore AB = \sqrt{\left(-2+1\right)^2+\left(-1-3\right)^2} = \sqrt{1+9}=\sqrt{10}$$

$$BC = \sqrt{1+9}=\sqrt{10} = \sqrt{1}$$

$$CD = \sqrt{\left(0+1\right)^2+\left(2+1\right)^2} = \sqrt{1+9}=\sqrt{10}$$

$$DA = \sqrt{\left(-1+2\right)^2+\left(-1+1\right)^2} = \sqrt{1}$$

$\therefore$ AB = CD and BC = DA

$\therefore$ Sides of both pairs are opposite

Sides of $\square^m$ABCD are congruent

$\therefore$ $\square$ABCD is a $\square^m$

$\therefore$ The given points are the vertices of a parallelogram.

Illustration 8 If P $\left(am^2, 2am\right)$, Q $\left(an^2, 2an\right)$, S $\left(a, 0\right)$ and $mn = -1$, then P = T. Prove that $\frac{1}{SP}+\frac{1}{SQ}=\frac{1}{a}$.

Solution

$$SP^2 = \left(am^2 - a^2\right) + \left(2am - 0\right)^2$$

$$= a^2\left[\left(m^2 - 1\right)^2 + 4m^2\right]$$

$$= a^2\left(m^2 + 1\right)^2$$

$\therefore$ SP = $a\left(m^2 + 1\right)$

Similarly SQ = $a\left(n^2 + 1\right)$

$$\therefore \frac{1}{SP}+\frac{1}{SQ} = \frac{1}{a\left(m^2+1\right)}+\frac{1}{a\left(n^2+1\right)}$$

$$= \frac{1}{a}\left[\frac{n^2+1+m^2+1}{\left(m^2+1\right)\left(n^2+1\right)}\right]$$

$$= \frac{1}{a}\left(\frac{m^2 + n^2 + 2}{m^2n^2 + m^2 + n^2 + 1}\right)$$

$$= \frac{1}{a}\left(\frac{m^2 + n^2 + 2}{m^2 + n^2 + 2}\right) \qquad \left(\because mn = -1 \Rightarrow m^2n^2 = 1\right)$$

$$= \frac{1}{a}$$

Illustration 9 Prove that the area of the ΔPQR is $a^2 \left|\left(t_1 - t_2\right), \left(t_2 - t_3\right), \left(t_3 - t_1\right)\right|$, where P $\left(at_1^2, 2at_1\right)$, Q $\left(at_2^2, 2at_2\right)$, R $\left(at_3^2, 2at_3\right)$.

Solution

$$\text{Area of } \Delta\text{PQR} = \frac{1}{2}|D|$$

$$D = \begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix} = 2a^2 \begin{vmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \end{vmatrix} \quad \begin{matrix} ; c_1\left(\dfrac{1}{a}\right) \\ ; c_2\left(\dfrac{1}{2a}\right) \end{matrix}$$

$$= 2a^2 \begin{vmatrix} t_1^{\,2} - t_2^{\,2} & t_1 - t_2 & 0 \\ t_2^{\,2} - t_3^{\,2} & t_2 - t_3 & 0 \\ t_3^{\,2} & t_3 & 1 \end{vmatrix} \quad \begin{matrix} \text{R}_{21}\left(-1\right) \\ \text{R}_{32}\left(-1\right) \end{matrix}$$

$$= 2a^2\left(t_1 - t_2\right)\left(t_2 - t_3\right) \begin{vmatrix} t_1 + t_2 & 1 & 0 \\ t_2 + t_3 & 1 & 0 \\ t_3^{\,2} & t_3 & 1 \end{vmatrix} \quad \begin{matrix} ; \text{R}_1\left(\dfrac{1}{t_1 - t_2}\right) \\ ; \text{R}_2\left(\dfrac{1}{t_2 - t_3}\right) \end{matrix}$$

$$= 2a^2\left(t_1 - t_2\right)\left(t_2 - t_3\right) \begin{vmatrix} t_1 - t_3 & 0 & 0 \\ t_2 + t_3 & 1 & 0 \\ t_3^{\,2} & t_3 & 1 \end{vmatrix} \quad \text{R}_{21}\left(-1\right)$$

$$= 2a^2\left(t_1 - t_2\right)\left(t_2 - t_3\right)\left(t_1 - t_3\right) \begin{vmatrix} 1 & 0 & 0 \\ t_2 + t_3 & 1 & 0 \\ t_3^{\,2} & t_3 & 1 \end{vmatrix}$$

$$= 2a^2\left(t_1 + t_2\right)\ \left(t_2 + t_3\right)\ \left(t_1 - t_3\right)\ \left(1\right)$$

$$\Delta = \frac{1}{2}|D| = \frac{1}{2}\left|-2a^2\left(t_1 - t_2\right)\left(t_2 - t_3\right)\left(t_3 - t_1\right)\right|$$

$$\Delta = a^2 \left|\left(t_1 - t_2\right)\ \left(t_2 - t_3\right)\ \left(t_3 - t_1\right)\right|$$

Illustration 10 If A $(2, 2)$ and B $(6, 6)$ are given, find the co-ordinates of a point P such that PA = PB and the area of ΔPAB = 4.

Solution

Let P (x, y) be given for ΔPAB.

$$D = \begin{vmatrix} x & y & 1 \\ 2 & 2 & 1 \\ 6 & 6 & 1 \end{vmatrix}$$

$$= x(-4) - y(-4) + 1\ (0) = -4x + 4y$$

Area of ΔPAB $= \dfrac{1}{2}|D|$

$\therefore 4 = \dfrac{1}{2}|-4x - 4y|$

$\therefore -4x + 4y = \pm 8$

$\therefore -4x + 4y = 8 \quad \text{or} \quad -4x + 4y = -8$

$\therefore -x + y = 2 \quad \text{or} \quad -x + y = -2$

but $PA^2 = PB^2$

$\therefore (x-2)^2 + (y-2)^2 = (x-6)^2 + (y-6)^2$

$\therefore 8x + 8y = 64$

$\therefore x + y = 8$

From $x + y = 8$ and $-x + y = 2 \Rightarrow x = 3, y = 5$

and $x + y = 8$ and $-x + y = -2 \Rightarrow x = 5, y = 3$

Hence the co-ordinate of point is $(3, 5)$ or $(5, 3)$.

Illustration 11 Show that A (x_1, y_1), B (x_2, y_2) and P $\left[tx_2 + (1-t)x_1\right], ty_2 + \left[(1-t)y_1\right]$ $(t \neq 0, t \neq 1)$ are collinear. Find the ratio in which P divides $\overline{AB}$ From A.

Solution

A (x_1, y_1), B (x_2, y_2); P $\left[tx_2 + (1-t)x_1\right], ty_2 + \left[(1-t)y_1)\right]$

$$\therefore \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ tx_2 + x_1 - tx_1 & ty_2 + y_1 - ty_1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & 0 \\ t(x_2 - x)_1 & t(y_2 - y)_1 & 0 \end{vmatrix} \begin{matrix} R_{12}(-1) \\ R_{13}(-1) \end{matrix}$$

$$= t\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_2 - x_1 & y_2 - y_1 & 0 \end{vmatrix}; R_3\left(\frac{1}{t}\right) \qquad (\because R_2 \equiv R_3)$$

$= 0$

$\therefore$ A, B, P are collinear.

Let P divide $\overline{AB}$ in the ratio $\lambda : 1$ from A, then

$$tx_2 + (1 - t)x_1 = \frac{\lambda x_2 + x_1}{\lambda + 1}$$

$$\therefore \lambda tx_2 + \lambda(1 - t)x_1 + tx_2 + (1 - t)x_1 = \lambda x_2 + x_1$$

$$\therefore \lambda(t - 1)\ (x_2 - x_1) = t(x_1 - x_2)$$

$$\therefore \lambda = \ (t \neq 0,\ t \neq 1)$$

Illustration 12 Find the circumcentre of the triangle with vertices $(-1, 1)$ $(0, -4)$, $(-1, -5)$ and deduce that the circumcentre of the triangle whose vertices are $(2, 3)$, $(3, -2)$, $(2, -3)$ is the origin.

Solution

Let A $(-1, 1)$ B $(0, -4)$, and C $(-1, -5)$ be the vertices and P (x, y) be the circumcentre of ΔABC.

$\therefore AP = PB = PC \Rightarrow AP^2 = PB^2 = PC^2$

Now $AP^2 = (x + 1)^2 + (y - 1)^2 = x^2 + y^2 + 2x - 2y + 2$

$PB^2 = (x - 0)^2 + (y + 4)^2 = x^2 + y^2 + 8y + 16$

$CP^2 = (x + 1)^2 + (y + 5)^2 = x^2 + y^2 + 2x + 10y + 26$

$\therefore AP^2 = PB^2 \Rightarrow x^2 + y^2 + 2x - 2y + 2 = x^2 + y^2 + 8y + 16$

$\Rightarrow 2x - 10y = 14 \Rightarrow x - 5y = 7$ (1)

$PB^2 = PC^2 \Rightarrow x^2 + y^2 + 8y + 16 = x^2 + y^2 + 2x + 10y + 26$

$\Rightarrow 2x + 2y = -10 \Rightarrow x + y = -5$ (2)

Solving (1), (2) we can say that $P(x, y) = (-3, -2)$

Now shifting the origin at 0^1 $(-3, -2)$ the new co-ordinates of A $(-1, 1)$, B $(0, -4)$, C $(-1, -5)$ and P $(-3, -2)$ are respectively

$(-1 + 3, 1 + 2) = (2, 3)$; $(0 + 3, -4 + 2) = (3, -2)$, $(-3 + 3, -2 + 2) = (0, 0)$.

Moreover the triangle does not change by shifting the origin and so its circumcentre will also not change.

$\therefore$ The circumcentre of triangle whose vertices are $(2, 3)$, $(3, -2)$, and $(2, -3)$ is P $(0, 0)$.

Illustration 13 Let A $(7, 8)$ and B $(-6, 11)$. Find the co-ordinates of the point that divides $\overline{AB}$ from A's side in the ratio 3 : 2.

Solution

Here $\lambda = \frac{3}{2}$ from A's side

$\therefore$ The co-ordinates of the points of division are $= \left(\frac{\lambda x_2 + x_1}{\lambda + 1}; \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$

$$= \left(\frac{(3/2)(-6) + 7}{(3/2) + 1}; \frac{(3/2)(11) + 8}{(3/2) + 1}\right) = \left(\frac{-18 + 14}{3 + 2}; \frac{33 + 16}{3 + 2}\right)$$

$= \left(-\frac{4}{5}, \frac{49}{5}\right)$

∴ The point of division is $\left(-\frac{4}{15}, \frac{49}{5}\right)$

Illustration 14 If A $(3, 3)$ and B $(6, 1)$, find P on line $\overleftrightarrow{\text{AB}}$ such that AP = 4AB.

Solution

Here $\left|\frac{\text{AP}}{\text{AB}}\right| = \frac{4}{1}$

$\therefore \frac{\text{AP}}{\text{AB}} = \pm\frac{4}{1}$

Case (1)

$\frac{\text{AP}}{\text{AB}} = 4$

∴ Point A divides $\overline{\text{PB}}$
from P in ratio λ = 4

$(3, 3) = \left(\frac{\lambda x_2 + x_1}{\lambda + 1}; \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$

$\therefore (3, 3) = \left(\frac{4(6) + x}{4 + 1}; \frac{4(1) + y}{4 + 1}\right)$

$\therefore (3, 3) = \left(\frac{x + 24}{5}; \frac{y + 4}{5}\right)$

$\therefore \frac{x + 24}{5} = 3; \frac{y + 4}{5} = 3$

$\therefore x + 24 = 15; y + 4 = 15$

$\therefore x = -9; y = 11$

$\therefore \text{P}(x, y) = (-9, 11)$

Case (2)

$\frac{\text{AP}}{\text{AB}} = -4$

∴ Point A divides $\overline{\text{PB}}$
from P in ratio λ = −4

$(3, 3) = \left(\frac{\lambda x_2 + x_1}{\lambda + 1}; \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$

$\therefore (3, 3) = \left(\frac{-4(6) + x}{-4 + 1}; \frac{-4(1) + y}{-4 + 1}\right)$

$\therefore (3, 3) = \left(\frac{x - 24}{-3}; \frac{y - 4}{-3}\right)$

$\therefore \frac{x - 24}{-3} = 3; \frac{y - 4}{-3} = 3$

$\therefore x - 24 = -9; y - 4 = -9$

$\therefore x = -9 + 24; y = -9 + 4$

$\therefore x = 15; y = -5$

$\therefore \text{P}\ (x, y) = (15, -5)$

Illustration 15 For which value of k would the points $(k, 2 - 2k)$, $(-k + 1, 2k)$ and $(-4 - k, 6 - 2k)$ be distinct and collinear?

Solution

For collinearity, we can say that $\begin{vmatrix} k & 2 - 2k & 1 \\ -k + 1 & 2k & 1 \\ -4 - k & 6 - 2k & 1 \end{vmatrix} = 0$

$\therefore k(2k - 6 + 2k) - (2 - 2k)\ (-k + 1 + 1 + 4 + k) +$
$1[(-k + 1)\ (6 - 2k)] - 2k(-4 - k) = 0$

$\therefore 8k^2 - 4k - 4 = 0$

$\therefore 2k^2 - k - 1 = 0$

$\therefore (2k - 1)(k + 1) = 0$

$\therefore k = \frac{1}{2}$ or $k = 1$

For $k = \frac{1}{2}$ the points $(k, 2 - 2k)$, and $(-k + 1, 2k)$ become the same point $\left(\frac{1}{2}, 1\right)$ but we need distinct points.

$\therefore k \neq \frac{1}{2}$

$\therefore$ For $k = -1$ the points $(-1, 4)$, $(2, -2)$, and $(-3, 8)$ are distinct.

$\therefore k = -1$

Illustration 16 Find the co-ordinates of the point of trisection of the line segment joining the points $(4, 5)$ and $(13, -4)$.

Solution

Here P divides $\overline{\text{AB}}$ from A in ratio 1 : 2

$$\therefore P\ (x_1, y_1) = \left(\frac{\lambda x_2 + x_1}{\lambda + 1}; \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$$

$$\therefore P\ (x, y) = \left(\frac{(1/2)(13) + 4}{(1/2) + 1}; \frac{(1/2)(-4) + 5}{(1/2) + 1}\right) = (7, 2)$$

Now $Q\ (x_2, y_2)$ is a mid-point of $\overline{\text{PB}}$

$$\therefore a\ (x_2, y_2) = \left(\frac{7 + 13}{2}, \frac{2 - 4}{2}\right) = (10, -1)$$

Illustration 17 P $(2, -1)$ and Q $(4, 0)$ divide $\overline{\text{AB}}$ from A's side in the ratio 2 : 1 and −2 : 1 respectively. Find the co-ordinates of A and B.

Solution

Let A (x_1, y_1) and B (x_2, y_2) be given.

Here P $(2, -1)$ divides $\overline{\text{AB}}$ from A in ratio 2 : 1

$$\therefore (2, -1) = \left(\frac{\lambda x_2 + x_1}{\lambda + 1}; \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$$

$$\therefore (2, -1) = \left(\frac{2x_2 + x_1}{2 + 1}; \frac{2y_2 + y_1}{2 + 1}\right)$$

$\therefore 2x_2 + x_1 = 6$ (1) $2y_2 + y_1 = -3$ (2)

Q $(4, 0)$ divides $\overline{AB}$ from A in ratio $-2 : 1$

$$\therefore (4, 0) = \left(\frac{\lambda x_2 + x_1}{\lambda + 1}; \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$$

$$\therefore (4, 0) = \left(\frac{-2x_2 + x_1}{-2 + 1}; \frac{-2y_2 + y_1}{-2 + 1}\right)$$

$$\therefore (4, 0) = \left(\frac{-2x_2 + x_1}{-1}; \frac{-2y_2 + y_1}{-1}\right)$$

$\therefore -2x_2 + x_1 = -4$ (3)

$-2y_2 + y_1 = 0$ (4)

Now from Eqs. (1) and (3) we can say that

$x_1 = 1; x_2 = \frac{5}{2}$

and from Eqs. (2) and (4) we can say that

$y_1 = -\frac{3}{2}; y_2 = -\frac{3}{4}$

$\therefore$ The co-ordinates of A and B are

A $\left(1, -\frac{3}{2}\right)$; and B $\left(\frac{5}{2}, -\frac{3}{4}\right)$

Illustration 18 A $(6, 7)$, B $(-2, 3)$; C $(9, 1)$ are the vertices of a triangle. Find the co-ordinates of the point where the bisector of $\angle$A meets $\overline{BC}$.

Solution

Here bisector of $\angle$A intersects $\overline{BC}$ at D

$$\therefore \frac{BD}{DC} = \frac{AB}{AC} \quad (1)$$

$AB^2 = (6 + 2)^2 + (7 - 3)^2 = 80$

$AB = 4\sqrt{5}$

$AC^2 = (6 - 9)^2 + (7 - 1)^2 = 45$

$\therefore AC = 3\sqrt{5}$

$\therefore$ Point D divides $\overline{BC}$ from D in ratio 4 : 3

$$\therefore D(x, y) = \left(\frac{mx_2 + nx_1}{m + n}; \frac{my_2 + ny_1}{m + n}\right)$$

$$= \left[\frac{4(9) + (-2)(3)}{4 + 3}; \frac{4(1) + 3(3)}{4 + 3}\right]$$

$$= \left(\frac{30}{7}, \frac{13}{7}\right)$$

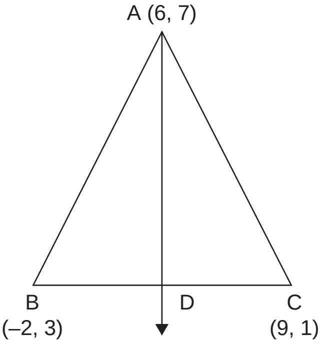
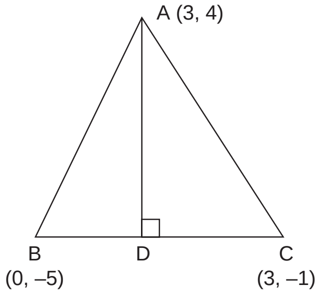


Illustration 19 A $(3, 4)$, B $(0, -5)$; C $(3, -1)$ are the vertices of ΔABC, determine the length of the altitude from A on $\overleftrightarrow{BC}$

Solution



$$BC = \sqrt{(0-3)^2 + (-5+1)^2} = 5$$

The area of $\Delta ABC = \Delta = \frac{1}{2}\overline{BC}\ P$ (1)

$$\text{but } \Delta = \frac{1}{2}|D| = \frac{1}{2}\begin{vmatrix} 3 & 4 & 1 \\ 0 & -5 & 1 \\ 3 & -1 & 1 \end{vmatrix} \text{ mod}$$

$$\Delta = \frac{1}{2}\left|3(-4) - 4(-3) + 1(15)\right|$$

$$= \frac{1}{2}|15| \qquad (2)$$

∴ From Eqs. (1) and (2) we can say that

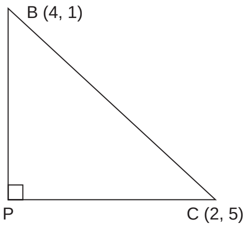
$$\frac{1}{2}|15| = \frac{1}{2}\overline{BC}\ P$$

$$\frac{1}{2}|15| = \frac{1}{2}5\ P$$

$\therefore P = 3$

Illustration 20 Point B $(4, 1)$ and C $(2, 5)$ are given. If P is a point in the plane such that $m\angle BPC = \frac{\pi}{2}$, find the set of all such points P.

Solution



Let P = P (x, y) since $m\angle BPC = \frac{\pi}{2}$

$\therefore BC^2 = BP^2 + PC^2$

$\therefore (4-2)^2 + (1-5)^2 = (x-4)^2 + (y-1)^2 + (x-2)^2 + (y-5)^2$

$\therefore x^2 + y^2 - 6x - 6y + 13 = 0$

∴ The required set of points

$\left[(x, y) \in R^2 \mid x^2 + y^2 - 6x - 6y + 13 = 0\right] - \left[(4, 1), (2, 5)\right]$

∴ The equation of the set of points

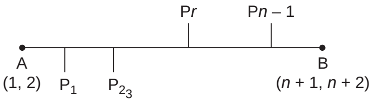
$x^2 + y^2 - 6x - 6y + 13 = 0;\ (x, y) \neq (2, 5),\ (x, y) \neq (4, 1)$

Here $(4, 1)$ and $(2, 5)$ satisfy the equation of the set but they do not satisfy the given condition. Therefore we can say that P = B or P = C and ΔPBC cannot be formed.

Illustration 21 Find the points which divide the line segment joining $(0, 0)$ and (a, b) into n equal parts.

Solution

Let $P_1, P_2, P_3 \ldots P_r, \ldots P_{n-1}$ be the points which divide $\overline{AB}$ in n equal parts.



Let $P_r = \left(x_r, y_r\right)$ then P_r divides $\overline{\text{AB}}$ in the ratio $r : n - r$ from A; $r = 1, 2, \ldots \left(n - 1\right)$ where A $\left(0, 0\right)$ and B $\left(a, b\right)$

$$P_r\ \left(x_r, y_r\right) = \left(\frac{mx_2 + nx_1}{m+n} ; \frac{my_2 + ny_1}{m+n}\right)$$

$$= \left[\frac{r(a)+(n-r)\ (0)}{r+n-r} ; \frac{r(b)+(n-r)\ (0)}{r+n-r}\right]$$

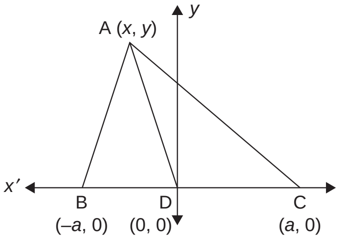
$$P_r\ \left(x_r, y_r\right) = \left(\frac{ra}{n}, \frac{rb}{n}\right); r = 1, 2, \ldots \left(n - 1\right)$$

Illustration 22 If D and G are respectively the mid-points of $\overline{\text{BC}}$ and the centriod of ΔABC, prove that

(1) $AB^2 + AC^2 = 2\left(AD^2 + BD^2\right)$

(2) $AB^2 + BC^2 + AC^2 = 3\left(GA^2 + GB^2 + GC^2\right)$

Solution



1. Here D is the origin and $\overrightarrow{\text{DC}}$ is positive direction of x-axis

   Let BD = DC = a then C = C $\left(a, 0\right)$; B = B $\left(-a, 0\right)$ and let A $\left(x, y\right)$

   $\therefore$ L.H.S. $= AB^2 + AC^2$

   $= \left(x + a\right)^2 + y^2 + \left(x - a\right)^2 + y^2$

   $= 2x^2 + 2y^2 + 2a^2$

   $= 2\left(x^2 + y^2 + a^2\right)$ (1)

   R.H.S. $= 2\left(AD^2 + BD^2\right)$

   $= 2\left[\left(x - 0\right)^2 + \left(y - 0\right)^2 + \left(-a\right)^2\right]$

   $= 2\left(x^2 + y^2 + a^2\right)$ (2)

   $\therefore AB^2 + AC^2 = 2\left(AD^2 + BD^2\right)$

2. Let G be the origin and let A $\left(x_1, y_1\right)$, B $\left(x_2, y_2\right)$ and C $\left(x_3, y_3\right)$

   $$\therefore G\ \left(0, 0\right) = G\left(\frac{x_1 + x_2 + x_3}{3} ; \frac{y_1 + y_2 + y_3}{3}\right)$$

   $\therefore x_1 + x_2 + x_3 = 0 \Rightarrow x_3 = -x_1 - x_2$

   $y_1 + y_2 + y_3 = 0 \Rightarrow x_3 = -y_1 - y_2$

   $\therefore$ C $\left(x_3, y_3\right)$ = C $\left(-x_1 - -x_2, - y_1 - y_2\right)$

   L.H.S. $= AB^2 + BC^2 + CA^2$

   $= \left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2 + \left(x_2 + x_1 + x_2\right)^2 + \left(y_2 + y_1 + y_2\right)^2 + \left(x_1 + x_1 + x_2\right)^2 + \left(y_2 + y_1 + y_2\right)^2$

   $= 6x_1^2 + 6x_2^2 + 6y_1^2 + 6y_2^2 + 6x_1x_2 + 6y_1\,y_2$

   $= 6\left(x_1^2 + x_2^2 + y_1^2 + y_2^2 + x_1x_2 + y_1y_2\right)$ (1)

$$\begin{aligned} \therefore \text{ R.H.S.} &= \text{GA}^2 + \text{GB}^2 + \text{GC}^2 \\ &= 3\left[x_1^2 + y_1^2 + x_2^2 + y_2^2 + \left(-x_1 - x_2\right)^2 + \left(-y_1 - y_2\right)^2\right] \\ &= 6\left(x_1^2 + y_2^2 + y_1^2 + y_2^2 + x_1 x_2 + y_1 y_2\right) \end{aligned} \tag{2}$$

From Eqs. (1) and (2) we can say that

$\text{AB}^2 + \text{BC}^2 + \text{CA}^2 = 3\left(\text{GA}^2 + \text{GB}^2 + \text{GC}^2\right)$

Illustration 23 If A, B, C and P are distinct non-collinear points of the plane, prove that area of ΔPAB + area of ΔPBC + area of ΔPCA ≥ area of ΔABC.

Solution

Let (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x, y) be the co-ordinates of A, B, C and P respectively.

$\therefore$ Area of ΔPAB = $\Delta_1 = \frac{1}{2}|D_1|$; area of ΔPBC = $\Delta_2 = \frac{1}{2}|D_2|$;

area of ΔPCA = $\Delta_3 = \frac{1}{2}|D_3|$; area of ΔABC = $\frac{1}{2}|D| = \Delta$

where

$$D_1 = \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}; D_2 = \begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}; D_3 = \begin{vmatrix} x & y & 1 \\ x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \end{vmatrix}$$

$$\text{and } D = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Now $D_1 + D_2 + D_3$

$$= \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} - \begin{vmatrix} x & y & 1 \\ x_3 & y_3 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} \quad R_{32} \text{ in } D_2$$

$$= \begin{vmatrix} x & y & 1 \\ x_1 - x_3 & y_1 - y_3 & 0 \\ x_2 & y_2 & 1 \end{vmatrix} - \begin{vmatrix} x & y & 1 \\ x_1 - x_3 & y_1 - y_3 & 0 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \begin{matrix} R_{23}(-1) \\ R_{32} \text{ in } D_3 \end{matrix}$$

$$= \begin{vmatrix} x & y & 1 \\ x_1 - x_3 & y_1 - y_3 & 0 \\ x_2 - x_3 & y_2 - y_3 & 0 \end{vmatrix} - \begin{vmatrix} x_3 & y_3 & 1 \\ x_1 - x_3 & y_1 - y_3 & 0 \\ x_2 - x_3 & y_2 - y_3 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_2 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = D \qquad \left[\because R_{12}\,(1);\ R_{13}\,(1)\right]$$

$\therefore D = D_1 + D_2 + D_3$

$\therefore |D| = |D_1 + D_2 + D_3|$

$\leq |D_1| + |D_2| + |D_3|$

$\therefore \ \frac{1}{2}|D| \leq \frac{1}{2}|p_1| + \frac{1}{2}|p_2| + \frac{1}{2}|p_3|$

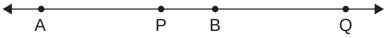
$\therefore \ \Delta \leq \Delta_1 + \Delta_2 + \Delta_3$

$\therefore \ \Delta_1 + \Delta_2 + \Delta_3 \geq \Delta$

Illustration 24 Find the co-ordinates of the points which divide segment joining A $(1, 2)$ and B $(n + 1, n + 2)$ into n equal parts and hence find the points of the trisection of $\overline{AB}$.

Solution

Let $p_1, p_{2,} \ldots p_4, \ldots p_{n-1}$ be the points dividing $\overline{AB}$ into n equal parts. Clearly p_r divides $\overline{AB}$ in the ratio $r : n - r$.



Let $p_r\ (x_r, y_r)$ then

$$x_r = \frac{r(n+1)+(n-r)(1)}{r+n-r};\ y_r = \frac{r(n+2)+(n-r)\,2}{r+n-r}$$

$\therefore\ pr\ (x_r, y_2) = p_r\ (r + 1, r + 2)$

We can find $p_1, p_{2,} \ldots p_{n-1}$ by taking $r = 1, 2, \ldots (n - 1)$.

Now for point of trisection, the ratio is 1 : 2 from A or 2 : 1 from A (That does not mean $n = 3$ is the co-ordinate of B.)

From the above figure we can say that p is

$$\left(\frac{n+1+2}{3};\frac{n+2+4}{3}\right) = \left(\frac{n+3}{3},\frac{n+6}{3}\right)$$

Similarly co-ordinates of a are

$$C\left[\frac{2(n+1)+1}{3};\frac{2(n+2)+2}{3}\right] = \left(\frac{2n+3}{3},\frac{2n+6}{3}\right)$$

Illustration 25 Find the incentre of the triangle whose vertices are $(4, 1)$, $(1, 5)$, $(-2, 1)$.

Solution

Let A $(4, 1)$, B $(1, 5)$ and C $(-2, 1)$ be the vertices of ΔABC.

Let AB = c, BC = a, CA = b

$$\therefore\ a = BC = \sqrt{(1+2)^2 + (5-1)^2} = \sqrt{9+16} = 5$$

$$b = AC = \sqrt{(4+2)^2 + 0} = \sqrt{36} = 6$$

$$c = \text{AB} = \sqrt{(4-1)^2 + (1-5)^2} = \sqrt{9+16} = 5$$

Let $I\ (x, y)$ be the incentre of ΔABC

$$\therefore I\ (x, y) = \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}; \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

$$= \left[\frac{5(4) + 6(1) + 5(-2)}{5+6+5}; \frac{5(1) + 6(5) + 5(1)}{5+6+5}\right]$$

$$\therefore I\ (x, y) = \left(1, \frac{5}{2}\right)$$

Illustration 26 If G and I are, respectively, the centroid and incentre of the triangle whose vertices are A $(-2, -1)$, B $(1, -1)$ and C $(1, 3)$, find IG.

Solution

Let G (x, y) be the centroid of the ΔABC

$$\therefore \text{G}\ (x, y) = \left(\frac{-2+1+1}{3}; \frac{-1-1+3}{3}\right) = \left(0, \frac{1}{3}\right)$$

Let $I\ (x, y)$ be the incentre of ΔABC

$$\text{AB} = c = \sqrt{(-2-1)^2 + (-1+1)^2} = 3$$

$$\text{BC} = a = \sqrt{(1-1)^2 + (-1-3)^2} = 4$$

$$\text{CA} = b = \sqrt{(1+2)^2 + (3+1)^2} = 5$$

$$\therefore I\ (x, y) = \left[\frac{4(-2) + 5(1) + 3(1)}{4+5+3}; \frac{4(-1) + 5(-1) + 3(3)}{4+5+3}\right] = (0, 0)$$

$$\therefore \text{IG} = \sqrt{(0-0)^2 + \left(0 - \frac{1}{3}\right)^2} = \frac{1}{3}$$

Illustration 27 If the distance between the centroid and the incentre of the triangle with vertices $(-36, 7)$; $(20, 7)$ and $(0, -8)$ is $\frac{25}{3}\sqrt{205}\,k$ find K.

Solution

Let G and I be the centroid and incentre of ΔABC whose vertices are A $(-36, 7)$, B $(20, 7)$, and C $(0, -8)$

$$\therefore \text{Centroid G}\ (x, y) = \left(\frac{-36+20+0}{3}, \frac{7+7-8}{3}\right) = \left(\frac{-16}{3}, 2\right)$$

$$\text{AB} = c = \sqrt{(-36-20)^2 + (7-7)^2} = \sqrt{56^2} = 56$$

$$\text{BC} = b = \sqrt{(20-0)^2 + (7+8)^2} = \sqrt{625} = 25$$

$$\text{CA} = b = \sqrt{(0+36)^2 + (-8-7)^2} = \sqrt{1521} = 39$$

and $I\left(x, y\right) = \left[\dfrac{25(-36)+39(20)+56(0)}{25+39+56}; \dfrac{25(7)+39(7)+56(-8)}{25+39+56}\right]$

$\therefore I\left(x, y\right) = \left(-1, 6\right)$

Now IG $= \dfrac{25}{3}\sqrt{205}$ K

$\therefore \sqrt{\left(-1+\dfrac{16}{3}\right)^2 + (0-2)^2} = \dfrac{25}{3}\sqrt{205}$ K

$\therefore \sqrt{\dfrac{205}{9}} = \dfrac{25}{3}\sqrt{205}$ K

$\therefore$ 1 = 25K

$\therefore$ K $= \dfrac{1}{25}$

Illustration 28 A is $\left(2, \sqrt{2}, 0\right)$ and is $\left(-2, \sqrt{2}, 0\right)$ If $|AP - PB| = 4$. Find the equation of the locus of P.

Solution

Let P $\left(x, y\right)$ be given.

$\therefore AP^2 = \left(x - 2\sqrt{2}\right)^2 + y^2 = x^2 + y^2 - 4\sqrt{2}x + 8$ and

$PB^2 = \left(x + 2\sqrt{2}\right)^2 + y^2 = x^2 + y^2 + 4\sqrt{2}x + 8$

Now $|AP - PB| = 4$

$\therefore AP - PB = \pm 4$

$\therefore AP = PB \pm 4$

$\therefore AP^2 = PB^2 \pm 8PB + 16$

$\therefore x^2 + y^2 - 4 + \sqrt{2}x\, 8 = x^2 + y^2 + 4 + \sqrt{2}x\, 8 \pm 8PB + 16$

$\therefore -8 - \sqrt{2}x\, 16 = \pm 8PB$

$\therefore \sqrt{2}x - 2 = \pm PB$

$\therefore 2x^2 + 4\sqrt{2}x + 4 = PB^2$

$\therefore 2x^2 + 4\sqrt{2}x + 4 = x^2 + y^2 + \sqrt{2}x\, 4 + 8$

$\therefore x^2 - y^2 = 4$

$\therefore$ The equation for set of point P is $x^2 - y^2 = 4$.

Illustration 29 P and Q divide $\overline{AB}$ from A's side in the ratio λ and −λ, respectively. What are the ratios in which A and B divide $\overline{PQ}$ from P's side? $\left(\lambda \neq 1, \lambda > 0\right)$

Solution

Let A as origin and $\overrightarrow{AB}$ as the positive direction of x-axis. Then A $\left(0, 0\right)$ and B $\left(x_2, y_2\right) = \left(b, 0\right)$

P divides $\overline{AB}$ from A's side in ratio λ and Q divides $\overline{AB}$ from A's side in ratio −λ.

$$\therefore\ P = P\left(\frac{\lambda b}{\lambda+1}, 0\right) \qquad Q = Q\left(\frac{-\lambda b}{-\lambda+1}, 0\right)$$

P divides $\overline{AB}$ internally and Q divides $\overline{AB}$ externally

Also $AP^2 = \dfrac{\lambda^2 b^2}{(\lambda+1)^2}$

$$\therefore\ AP = \left|\frac{\lambda}{\lambda+1}\right| AB;\ \text{and}\ AQ = \left|\frac{\lambda}{\lambda+1}\right| AB$$

$$\therefore\ \frac{AP}{AC} = \left|\frac{\lambda-1}{\lambda+1}\right|$$

Similarly $\dfrac{BP}{BQ} = \left|\dfrac{\lambda-1}{\lambda+1}\right|$

$\therefore$ One of A and B divides $\overline{PQ}$ in the ratio $\dfrac{\lambda-1}{\lambda+1}$ and the other divides $\overline{PQ}$ in ratio $\dfrac{1-\lambda}{\lambda+1}$ from P's side.

Illustration 30 Find the circumcentre and the circum radius of the triangle with vertices $(3, 0)$, $(-1, 6)$ and $(4 - 1)$.

Solution

Let A $(3, 0)$, B $(-1, -6)$ and C $(4\ -1)$ be the vertices of ΔABC and P (x, y) be the circumcentre of the triangle.

$\therefore$ AP = PB = PC

Now $AP^2 = (x-3)^2 + y^2 = x^2 + y^2 - 6x + 9$

$PB^2 = (x+1)^2 + (y+6)^2 = x^2 + y^2 + 2x + 12y + 37$

$PC^2 = (x-4)^2 + (y+1)^2 = x^2 + y^2 - 8x + 2y + 17$

$\therefore\ PA^2 = PB^2$

$\therefore\ x^2 + y^2 - 6x + 9 = x^2 + y^2 + 2x + 12y + 37$

$8x + 12y + 28 = 0$

$\therefore\ 2x + 3y + 7 = 0$ (1)

Taking $PA^2 = PC^2$

$\therefore\ x^2 + y^2 - 6x + 9 = x^2 + y^2 - 8x + 2y + 17$

$\therefore\ 2x - 2y - 8 = 0$

$\therefore\ x - y - 4 = 0$ (2)

Solving Eqs. (1) and (2)

$x = 1, y = -3$

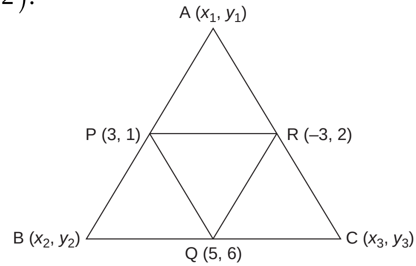
$\therefore\ P\ (x, y) = (1, -3)$

$$\therefore\ \text{Circum radius} = PA = \sqrt{(1-3)^2 + (-3-0)^2}$$

$$= \sqrt{4+9} = \sqrt{13}$$

Illustration 31 Find the vertices of a triangle if the mid-points of the sides are $(3, 1)$, $(5, 6)$ and $(-3, 2)$.

Solution



From the figure we can say that

mid-point of $\overline{\text{AB}}$ = P $(3, 1)$

$\therefore \left(\dfrac{x_1 + x_2}{2}; \dfrac{y_1 + y_2}{2}\right) = (3, 1)$

$\therefore \dfrac{x_1 + x_2}{2} = 3; \dfrac{y_1 + y_2}{2} = 1$

$\therefore x_1 + x_2 = 6$ (1)

$y_1 + y_2 = 2$ (2)

Now mid-point of $\overline{\text{BC}}$ = Q $(5, 6)$

$\therefore \left(\dfrac{x_2 + x_3}{2}, \dfrac{y_2 + y_3}{2}\right) = (5, 6)$

$\therefore \dfrac{x_2 + x_3}{2} = 5; \dfrac{y_2 + y_3}{2} = 6$

$\therefore x_2 + x_3 = 10$ (3)

$y_2 + y_3 = 12$ (4)

Mid-point of $\overline{\text{AC}}$ = R $(-3, 2)$

$\therefore \left(\dfrac{x_1 + x_3}{2}; \dfrac{y_1 + y_3}{2}\right) = (-3, 2)$

$\therefore \dfrac{x_1 + x_3}{2} = -3; \dfrac{y_1 + y_3}{2} = 2$

$\therefore x_1 + x_3 = -6$ (5)

$y_1 + y_3 = 4$ (6)

Now, by using equations (1), (3), (5) and equations (2), (4), (6) we can get vertices of A, B, C as A $(x_1, y_1) = (-5, -3)$; B $(x_2, y_2) = (11, 5)$, C $(x_3, y_3) = (-1, 7)$.

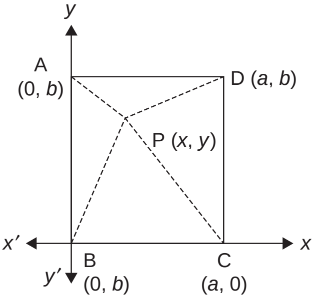


Illustration 32 If P is the interior of a rectangle ABCD prove that $PA^2 + PC^2 = PB^2 + PD^2$.

Solution

Let B $(0, 0)$, C $(a, 0)$, A $(0, b)$ and D (a, b)

$\therefore PA^2 + PC^2$

$= (x - 0)^2 + (y - b)^2 + (x - a)^2 + (y - 0)^2$

$= 2x^2 + 2y^2 + a^2 - b^2 - 2ax - 2by$ (1)

and $PB^2 + PD^2$

$= (x - 0)^2 + (y - 0)^2 + (x - a)^2 + (y - b)^2$

$= 2x^2 + 2y^2 + a^2 + b^2 - 2ax - 2by$

From (1) and (2) we can say that $PA^2 + PC^2 = PB^2 + PD^2$

Illustration 33 Area of ΔABC is $\frac{3}{2}$. The y-co-ordinate of its centroid is 8 less than three times its x-co-ordinate. If A is $(2, -3)$ and B is $(3, -2)$, find the co-ordinates of C.

Solution

Here A $(2, -3)$; B $(3, -2)$ are given.
Let C = C (x, y) and G = G (a, b)

$\therefore b = 3a - 8$

$\therefore G = G(a, b) = (a, 3a - 8)$

$$\therefore (a, 3a - 8) = \left(\frac{2 + 3 + x}{3}; \frac{-3 - 2 + y}{+3}\right)$$

$$\therefore \frac{5 + x}{3} = a\ ; 3a - 8 = \frac{y - 5}{+3}$$

$$\therefore 3\left(\frac{5 + x}{3}\right) - 8 = \frac{y - 5}{3}$$

$\therefore y = 3x - 4$ (1)

Now Area of $\Delta ABC = \frac{1}{2}|D|$
where

$$D = \begin{vmatrix} 2 & -3 & 1 \\ 3 & -2 & 1 \\ x & y & 1 \end{vmatrix}$$

$$= 2(-2 - y) + 3(3 - x) + 1(3y + 2x)$$
$$= y - x + 5$$
$$= 3x - 4 - x + 5$$
$$= 2x + 1$$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}|D|$

$$\frac{3}{2} = \frac{1}{2}|2x + 1|$$

$\therefore |2x + 1| = 3$

$\therefore 2x + 1 = \pm 3$

$2x + 1 = 3$	$2x + 1 = -3$
$2x = 2$	$2x = -4$
$x = 1$	$x = -2$
$\therefore y = 3(1) - 4$	$y = 3(-2) - 4$
$= -1$	$= -10$
$\therefore C(x, y) = (1, -1)$	$\therefore C = (x, y) = (-2, -10)$

Illustration 34 A $(4, 0)$, B $(0, -3)$ are given points. Find the equation of the set of points P such that $m\angle APB = 90°$.

Solution

Let P = $p\left(x, y\right)$; If $m\angle APB = \dfrac{\pi}{2}$

$\therefore AP^2 + BP^2 = AB^2$

$\therefore \left(x-4\right)^2 + \left(y-0\right)^2 + \left(x-0\right)^2 + \left(y+3\right)^2 = \left(4-0\right)^2 + \left(0+3\right)^2$

$\therefore x^2 + y^2 - 4x + 3y = 0$

$\therefore$ If p $\left(x, y\right)$ satisfies the condition $m\angle APB = \dfrac{\pi}{2}$ then $\left(x, y\right)$ satisfies the equation $x^2 + y^2 - 4x + 3y = 0$ does not satisfy the condition $m\angle APB = \dfrac{\pi}{2}$ see points A $\left(4, 0\right)$ and B $\left(0, 3\right)$ both satisfy $x^2 + y^2 - 4x + 3y = 0$ but in that case P = B or P = C and $\angle APB$ cannot be formed.

$\therefore$ The required point set is

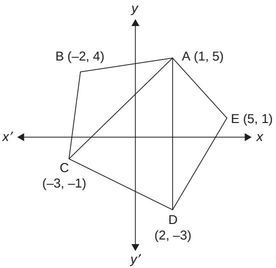
$\left[p\left(x, y\right) / x^2 + y^2 - 4x + 3y = 0\right] - \left[\left(4, 0\right), \left(0, -3\right)\right]$

The equation of point is

$x^2 + y^2 - 4x + 3y = 0;\ \left(x, y\right) \neq \left(4, 0\right);\ \left(x, y\right) \neq \left(0, -3\right)$

Illustration 35 Find the area of the pentagon whose vertices are $\left(1, 5\right)$, $\left(-2, 4\right)$, $\left(-3, -1\right)$, $\left(2, -3\right)$, $\left(5, 1\right)$.

Solution



Area of pentagon ABCDE

= Area of ΔABC + Area of ΔACD +
  Area of ΔADE

For ΔABC

$$\Delta = \frac{1}{2}|D|$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 5 & 1 \\ -2 & 4 & 1 \\ -3 & -1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left|1\left(5\right) - 5\left(1\right) + 1\left(14\right)\right|$$

$$= 7$$

For ΔACD

$$\Delta = \frac{1}{2}|D_2|$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 5 & 1 \\ -3 & -1 & 1 \\ 2 & -3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left|1\left(2\right) - 5\left(-5\right) + 1\left(9+2\right)\right|$$

$$= \frac{1}{2}|38| = 19$$

For ΔADE

$$\Delta = \frac{1}{2}|D_3|$$

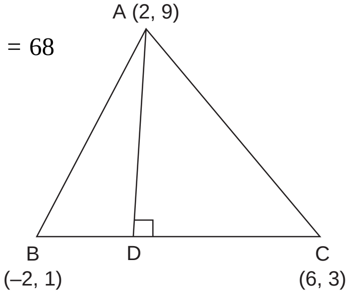
$$= \frac{1}{2}\begin{vmatrix} 1 & 5 & 1 \\ 2 & -3 & 1 \\ 5 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left|1\left(-4\right) - 5\left(-3\right) + 1\left(17\right)\right|$$

$$= \frac{1}{2}|28| = 14$$

Illustration 36 A is $(2, 9)$, B $(-2, 1)$ and C is $(6, 3)$ and the area of $\Delta A\overline{BC}$ is 28. Find the length of the perpendicular line-segment A to $\overline{BC}$.

Solution



Now $BC^2 = (-2-6)^2 + (1-3)^2 = 64 + 4 = 68$

$\therefore$ $BC = 2\sqrt{17}$

Area of $\Delta ABC = \frac{1}{2}$ BC AD

$28 = \left(\frac{1}{2}\ 2\sqrt{17}\ AD\right)$

$AD = \frac{28}{\sqrt{17}}$

Illustration 37 Prove for □^{m} ABCD $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$.

Solution

Select the intersection of diagonals as the origin $(0, 0)$
Taking A (x_1, y_1); B (x_2, y_2); C $(-x_1, -y_1)$ and D $(-x_2, y_2)$
Since CD = AB; DA = BC

$AB^2 + BC^2 + CD^2 + DA^2 = 2(AB^2 + BC^2)$

$= 2\left[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (x_1 + x_2)^2 + (y_1 + y_2)^2\right]$

$= 4(x_1^2 + x_2^2 + y_1^2 + y_2^2)$ (1)

Now $AC^2 + BD^2 = (x_1 + x_1)^2 + (y_1 + y_1)^2 + (x_2 + x_2)^2 + (y_2 + y_2)^2$

$= 4x_1^2 + 4y_1^2 + 4x_2^2 + 4y_2^2$

$= 4(x_1^2 + x_2^2 + y_1^2 + y_2^2)$ (2)

From (1) and (2) we can say that

$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$

Illustration 38 Prove that points A $(-2, 5)$, B $(4, -1)$, C $(a, 1)$ and D $(3, 7)$ are vertices of a parallalogram. L divides $\overline{AC}$ from A in the ratio 2 : 1, M is the mid-point of $\overline{BC}$. Prove that D, L, M are collinear.

Solution

Mid-point of $\overline{AC} = \left(\frac{9-2}{2}, \frac{1+5}{2}\right) = \left(\frac{7}{2}, 3\right)$

Mid-point of $\overline{BD} = \left(\frac{3+4}{2}, \frac{7-1}{2}\right) = \left(\frac{7}{2}, 3\right)$

$\therefore$ $\overline{AC}$, $\overline{BD}$ bisect each other

$\therefore$ AC, BD are diagonals of a parallelogram

$\therefore$ ABCD is a parallelogram

L divides $\overline{\text{AC}}$ from A in the ratio 2 : 1

$$\therefore L\left[\frac{2(9)+(-2)}{2+1};\frac{2(1)+5}{2+1}\right]=L\left(\frac{16}{3},\frac{7}{3}\right)$$

M is the mid-point of $\overline{\text{BC}}$

$$\therefore\ M\left(\frac{9+4}{2},\frac{-1+1}{2}\right)=M\left(\frac{13}{2},0\right)$$

$D\ (3, 7)$ is given

$$\therefore \begin{vmatrix} 3 & 7 & 1 \\ \frac{16}{3} & \frac{7}{3} & 1 \\ \frac{13}{2} & 0 & 1 \end{vmatrix} = 3\left(\frac{7}{3}\right)-7\left(\frac{16}{3}-\frac{13}{2}\right)+1\left(-\frac{91}{6}\right)=0$$

$\therefore$ L, M, D are collinear.

Illustration 39 Area of a triangle is 5 units. Two of its vertices are $(2, 1)$ and $(3, -2)$ and the y-co-ordinate of the third vertex is greater than its x-co-ordinate by 3. Find the third vertex of the triangle.

Solution

Let the third vertex of ΔABC be C $(x_1, x+3)$. A $(2, 1)$, B $(3, -2)$ are already given.

Now let us find the area of ΔABC

$$D = \begin{vmatrix} x & x+3 & 1 \\ 2 & 1 & 1 \\ 3 & -2 & 1 \end{vmatrix} = 3x + (x+3) - 7 = 4x - 4$$

but the area of ΔABC = 5

$$\therefore\ \frac{1}{2}|\text{D}| = 5$$

$$\therefore\ \frac{1}{2}|4x-4| = 5$$

$\therefore$ $2x - 2 = \pm 5$

$$2x - 2 = 5 \qquad\qquad 2x - 2 = -5$$

$$2x = 7 \qquad\qquad 2x = -3$$

$$x = \frac{7}{2} \qquad\qquad x = \frac{-3}{2}$$

$$\therefore \text{C}\ (x, x+3) = \left(\frac{7}{2},\frac{13}{2}\right) \text{ or } \left(\frac{-3}{2},\frac{3}{2}\right)$$

Illustration 40 O $(0, 0)$; A $(0, 4)$; B $(6, 0)$ are given points in R^2. Find the equations of set of all P $(x, y) \in R^2$ such that the area of ΔPOA is twice the area of ΔPOB.

Solution

Area of ΔPOA = 2 $\left(\text{Area of } \Delta\text{POB}\right)$

$$\therefore \ \frac{1}{2}|D| = 2\left(\frac{1}{2}|D^1|\right)$$

$$\text{where } D \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 0 & 4 & 1 \end{vmatrix} \text{ and } D^1 = \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 6 & 0 & 1 \end{vmatrix}$$

$D = -4x$ and $D^1 = 6y$

$\therefore \ D^2 = 4D'^2$

$\therefore \ 16x^2 = 144y^2$

or $x^2 - 9y^2 = 0$ is the required equation of set of point P.

Illustration 41 Points P and Q divide $\overline{\text{AB}}$ from A in the ratio $\underline{\lambda}$ and $-\lambda$ respectively. Find the ratios in which, A and B divide $\overline{\text{PQ}}$ from P. $\left(\lambda > 0, \lambda \neq 1\right)$

Solution

Take A $\left(0, 0\right)$ and B $\left(a, 0\right)$ on x-axis. So P and Q are also on x-axis.

$\therefore$ y-co-ordinates of P, Q, A, B are zero

$$\therefore \ \text{P} = \text{P}\left(\frac{\lambda a + 0}{\lambda + 1}, 0\right); \text{Q} = \text{Q}\left(\frac{-\lambda a + 0}{-\lambda + 1}, 0\right)$$

Let A divide $\overline{\text{PQ}}$ in the ratio K : 1 from P

$$\therefore \ x\text{-co-ordinate of A} = 0 = \frac{\text{K}\left[-\lambda a/\left(-\lambda + 1\right)\right] + \left[\lambda a/\left(\lambda + 1\right)\right]}{\text{K} + 1}$$

$$\therefore \ \frac{\lambda ak}{\lambda - 1} + \frac{\lambda a}{\lambda + 1} = 0$$

$$\therefore \ \text{K} = -\frac{\lambda - 1}{\lambda + 1} = \frac{1 - \lambda}{\lambda + 1}$$

Similarly Let B divide $\overline{\text{PQ}}$ from P in the ratio $\text{K}^1 = 1$

$$\therefore \ x\text{-co-ordinate of B} = a = \frac{\text{K}^1\left[-\lambda a/\left(-\lambda + 1\right)\right] + \left[\lambda a/\left(\lambda + 1\right)\right]}{\text{K}^1 + 1}$$

$$\therefore \ ak^1 + a = \frac{\lambda ak^1}{\lambda - 1} + \frac{\lambda a}{\lambda + 1}$$

$$\therefore \ \text{K}^1\left[\frac{a\left(\lambda - 1\right) - \lambda a}{\lambda - 1}\right] = \left(\frac{\lambda a - a - \lambda a}{\lambda + 1}\right)$$

$$\text{K}^1 = \frac{\lambda - 1}{\lambda + 1}$$

Thus A and B divide $\overline{PQ}$ from P in the ratio $\pm\left(\dfrac{\lambda-1}{\lambda+1}\right)$. A and B divide $\overline{PQ}$ internally according as λ < 1 or λ > 1 respectively.

Illustration 42 *P* divides the segment joining A $(3, 5)$ and B $(-5, 1)$ in such a way that the area of ΔPOQ is 6 units where $Q = Q\,(-2, 4)$ and O is the origin. Find the co-ordinates of *P*.

Solution

Let *P* divide $\overline{AB}$ in the ratio λ from A then

$$P = P\left(\frac{-5\lambda+3}{\lambda+1};\frac{\lambda+5}{\lambda+1}\right)$$

Area of ΔPOQ = 6 units

$$\therefore\ \frac{1}{2}|D| = 6 \qquad\qquad \therefore\ D = \pm 12$$

$$\therefore\ \begin{vmatrix} \dfrac{-5\lambda+3}{\lambda+1} & \dfrac{\lambda+5}{\lambda+1} & 1 \\ 0 & 0 & 1 \\ -2 & 4 & 1 \end{vmatrix} = \pm 12$$

$$\therefore\ 4\left(\frac{3-5\lambda}{\lambda+1}\right) + 2\left(\frac{\lambda+5}{\lambda+1}\right) = \pm 12$$

$$\therefore\ 6 - 10\lambda + \lambda + 5 = \mathrm{I}\ (6\lambda + 6)$$

$$\therefore\ -9\lambda + 11 = \pm\ (6\lambda + 6)$$

$$\therefore\ \lambda = \frac{1}{3}\ \text{ OR }\ \lambda = \frac{17}{3}$$

$$\therefore\ P\left[\frac{-5\,(1/3)+3}{(1/3)+1};\frac{(1/3)+5}{(1/3)+1}\right]\ \text{ OR}$$

$$P = P\left[\frac{-5\,(17/3)+3}{(17/3)+1};\frac{(17/3)+5}{(17/3)+1}\right]$$

$$\therefore\ P = P\ (1, 4)\ \text{ or }\ \mathrm{P}\left(\frac{-19}{5},\frac{8}{5}\right)$$

Illustration 43 Find the points at which the *x*-axis and *y*-axis divide the segment joining A $(2, 3)$ and B $(4, -7)$

Solution

Let x-axis divide $\overline{\text{AB}}$ in the ratio $\lambda = 1$ from A at $P\left(x, 0\right)$ then

$$0 = \frac{\lambda\left(-7\right) + 3}{\lambda + 1} \Rightarrow -7\lambda + 3 = 0 \Rightarrow \lambda = \frac{3}{7}$$

$$\therefore\ x = \frac{\left(3/7\right)\left(4\right) + 2}{\left(3/7\right) + 1} = \frac{12 + 14}{3 + 7} = \frac{13}{5}$$

$\therefore\ P\left(x, 0\right) = P\left(\frac{13}{5}, 0\right)$ is the point at which x-axis divides $\overline{\text{AB}}$

Let y-axis divide $\overline{\text{AB}}$ from A in ratio $\lambda^1 : 1$ at Q $\left(0, y\right)$

$$\therefore\ 0 = \frac{\lambda'\left(4\right) + 2}{\lambda' + 1} \Rightarrow 4\lambda^1 = -2 \Rightarrow \lambda^1 = \frac{-1}{2}$$

$$\therefore\ y = \frac{\left(-1/2\right)\left(-7\right) + 3}{\left(-1/2\right) + 1} = \frac{7 + 6}{1} = 13$$

$\therefore\ \left(0, y\right) = \left(0, 13\right)$ is the point at which y-axis divides $\overline{\text{AB}}$.

Illustration 44 Find the equation of set of points which are the mid-points of $\overline{\text{AB}}$ A $\left(a, 0\right)$ and B $\left(0, b\right)$ given that $a^2 + b^2 = 4c^2$

Solution

Let $P\left(x, y\right)$ be mid-point of $\overline{\text{AB}}$

where A $\left(a, 0\right)$, B $\left(0, b\right)$ and $a^2 + b^2 = 4c^2$

$\therefore\ x = \frac{a}{2}, y = \frac{b}{2}\ \therefore\ a = 2x, b = 2y$

$\therefore\ a^2 + b^2 = 4c^2$ gives us $4x^2 + 4y^2 = 4c^2$

$\therefore\ x^2 + y^2 = c^2$

Thus $x^2 + y^2 = c^2$ is the equation of set of mid–point of $\overline{\text{AB}}$.

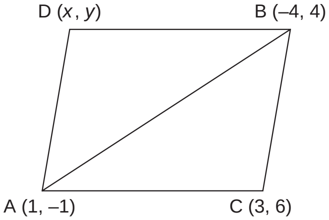


Illustration 45 If $\left(1, -1\right)$, $\left(-4, 4\right)$ and $\left(3, 6\right)$ are the vertices of a rhombus, find the co-ordinates of fourth vertex.

Solution

Let A $\left(1, -1\right)$, B $\left(-4, 4\right)$ C $\left(3, 6\right)$ then

$\text{AB}^2 = \left(1 + 4\right)^2 + \left(1 + 4\right)^2 = 50$

$\text{BC}^2 = \left(-4 - 3\right)^2 + \left(4 - 6\right)^2 = 53$

$\text{AC}^2 = \left(1 - 3\right)^2 + \left(-1 - 6\right)^2 = 53$

Since AC = BC, construct parallelogram ACBD

This will be rhombus due to AC = BD

Let $D = D\left(x, y\right)$

Now mid point of $\overline{CD}$ = mid–point of $\overline{AB}$

$$\therefore \left(\frac{x+3}{2}\ \frac{y+6}{2}\right) = \left(\frac{1-4}{2}\ \frac{1+4}{2}\right)$$

$\therefore\ x = -6,\ y = -3$

$\therefore\ D\left(x, y\right) = \left(-6, -3\right)$ is the fourth vertex of the rhombus.

ANALYTICAL EXERCISES

1. If the area of the triangle with vertices $\left(2, 3\right)$, $\left(4, 5\right)$, $\left(a, 3\right)$ is 5 determine a.
2. Find the area of the triangle whose vertices are $\left(l^2, 2l\right)$, $\left(m^2, 2m\right)$, $\left(n^2, 2n\right)$ if $l \neq m \neq n$.
3. Show that $\left(1, -2\right)$, $\left(2, 3\right)$, $\left(-3, 2\right)$ and $\left(-4, -3\right)$ are the vertices of a rhombus.
4. Find the equation of the set of points, sum of whose distances from P $\left(ae, 0\right)$, and Q $\left(-ae, 0\right)$ is $2a$.
5. Find the area of the pentagon whose vertices are $\left(4, 3\right)$, $\left(-5, 6\right)$, $\left(-7, -2\right)$, $\left(0, -7\right)$, $\left(3, 6\right)$.
6. Find the circumcentre of the triangle whose verticer are $\left(1, 2\right)$, $\left(3, 4\right)$ and $\left(2, 1\right)$.
7. For what value of a would the points $\left(0, 0\right)$, $\left(0, 1\right)$ and $\left(a, 1\right)$ be the vertices of a right angled triangle?
8. In which ratio does the x-axis divide the line segment joining A $\left(3, 5\right)$, B $\left(2, 7\right)$ from A’s side ?
9. If A $\left(1, -2\right)$, B $\left(-7, 1\right)$, find a point P on $\overleftrightarrow{AB}$ such that 3AP = 5AB.
10. P divides $\overline{AB}$ from A’s side in the ratio 2 : 1. If A $\left(3, 8\right)$ and P $\left(1, 12\right)$, find the co-ordinates of B.
11. A $\left(1, 2\right)$ and B $\left(2, 3\right)$ are given. Find C on $\overleftrightarrow{AB}$ such that 2AC = 3AB.
12. Find the equation of the locus of points which is equidistance from the point $\left(4, 3\right)$ and x-axis.
13. A $\left(-4, 0\right)$, and B $\left(4, 0\right)$ are given. Find the locus of a point P such that the difference of its distances from A and B is 4.
14. Two of the vertices of a triangle are $\left(2, 3\right)$ and $\left(-1, 4\right)$ and its area is 10 units If the third vertex is on x-axis, find its co-ordinates.
15. Prove that the mid-points of the sides of any quadrilateral are the vertices of a parallelogram.
16. Prove that the co-ordinates of all three vertices of an equilateral triangle cannot be rational numbers.
17. Prove that if $\left(a^2, y_1\right)$, $\left(b^2, y_2\right)$, $\left(c^2, y_3\right)$ are the vertices of a triangle then the centroid of the triangle cannot be on y-axis.
18. Prove that the centroid of the triangle with vertices $\left(a, b - c\right)$; $\left(b, c - a\right)$; $\left(c, a - b\right)$ is on x-axis.
19. Find a and b if $\left(1, -1\right)$, is the centroid of the triangle with vertices $\left(-5, 3\right)$, $\left(a, -1\right)$; $\left(6, b\right)$.

20. Find the co-ordinates of centroid; circumcentre and incentre of the triangle with vertices $(3, 4)$ $(0, 4)$, $(3, 0)$.

21. O is $(0, 0)$ and A is $(-2, -3)$ Find the equation of the locus of p (x, y) if OP: AP = 5 : 3. Obtain the equation of set of all such points P.

22. A is $(-4, 6)$; B is $(2, -2)$ and C $\in$ $[(x, y) / 2x - y + 1 = 0]$ prove that the equation of the locus of the centroid of ΔABC is $2x - y + 3 = 0$.

23. A line passing through a fixed point (c, d) intersects the axis in A and B. Find the locus of the mid-point of $\overline{AB}$.

24. If the points $(1, 2)$, $(2, 3)$ and (x, y) are from an equilateral triangle then find x and y.

25. Prove that $(2a, 4a)$; $(2a, 6a)$ and $(2a + \sqrt{3}a, 5a)$ are the vertices of an equilateral triangle.

26. Prove that A $(4, 1)$; B $(7, 5)$ and C $(0, 4)$ are the vertices of an isosceles right triangle.

27. A $(6, 3)$, B $(-3, 5)$ C $(4, -2)$ and P (x, y) are points in the plane and P, B, C are not collinear. Prove that the ratio of the areas of ΔPBC and ΔABC is $|x + y - 2| : 7$.

28. Prove that if a, b, c are distinct real numbers then the point (a, a^2), (b, b^2), (c, c^2) cannot be collinear.

29. If A $(3, -2)$ and B $(0, 7)$, find P $\in$ $\overleftrightarrow{AB}$ such that AP = 4AB.

30. A $(0, 1)$, B $(2, 4)$ are given. Find C $\in$ $\overleftrightarrow{AB}$ such that AB = 3AC.

31. If P is a point on the circumcentre of equilateral ΔABC then prove that $AP^2 + BP^2 + CP^2$ does not depend on the position of P.

32. P is $(-5, 1)$ and Q is $(3, 5)$. A divides $\overline{PQ}$ from P's side in the ratio K = 1. B is $(1, 5)$ and C is $(7, -2)$. Find K so that the area of ΔABC would be 2.

33. Prove that the area of the triangle formed by joining the mid-points of the sides of ΔABC is a quarter of the area of Δ ABC.

34. If B is $(4, 1)$; C is $(2, 5)$ and $m \angle BPC = \frac{\pi}{2}$ find the locus of P.

35. If the mid-points of the sides of a triangle are $(5, -2)$, $(8, 2)$, $(5, 2)$ find the incentre of the triangle.

36. If G is the centroid of ΔABC and P is any point in the plane of ΔABC then prove that $AP^2 + BP^2 + CP^2 = AG^2 + BG^2 + CG^2 + 3PG^2$.

37. Prove that, in a right triangle, the mid-point of the hypotenuse is the circumcentre of the triangle.

38. Show that if $(a, 0)$, $(0, b)$, $(1, 1)$; $a \neq 0$; $b \neq 0$ are collinear then and only then

$$\frac{1}{a} + \frac{1}{b} = 1$$

39. A $(0, 0)$ and B $(4, -3)$ are given points. Find the equation of the set of point P such that 2AP = 3PB.

40. Find the incentre, centroid and circumcentre of triangle whose vertices are A $(4, -2)$, B $(-2, 4)$ and C $(5, 5)$.

41. P $(2, 2)$ and Q $(5, -2)$ are given points. Find R on x-axis such that $\angle PRQ$ is a right angle.

42. Find the area of quadrilateral whose vertices are $(-1, -1)$, $(-4, -2)$, $(-5, -4)$, $(-2, -3)$.

43. If A $(0, 1)$ and B $(2, 9)$ are given. Find C on $\overleftrightarrow{AB}$ such that AB = 3AC.

ANSWERS

(1) $a = -3$ or 7

(2) $(l - m)(m - n)(n - l)$

(5) 97 units

(6) $\left(\frac{5}{2}, \frac{5}{2}\right)$

(7) $a \in R - (0)$

(8) $\lambda = -\frac{5}{7}\left(\frac{11}{2}, 0\right)$

(9) $\left(\frac{43}{3}, -7\right), \left(-\frac{37}{3}, 3\right)$

(10) $(6, 14)$

(11) $C\left(\frac{5}{2}, \frac{7}{2}\right)$, or $C\left(-\frac{1}{2}, \frac{1}{2}\right)$

(12) $x^2 - 8x - 6y + 25 = 0$

(13) $3x^2 - y^2 = 12,$

(14) $(3, 0)$, $(-9, 0)$

(19) $a = 2, b = -5$

(20) $G\left(2, \frac{8}{3}\right), P\left(\frac{3}{2}, 2\right), I(2, 3)$

(21) $16x^2 + 16y^2 + 100x + 150y + 325 = 0$

(23) $\frac{c}{x} + \frac{d}{y} = 2$

(24) $(x, y) = \left(\frac{3 \pm \sqrt{3}}{2}, \frac{5 \pm \sqrt{3}}{2}\right)$

(29) $(-9, 34)$, $(15, 38)$

(30) $\left(\frac{2}{3}, 2\right), \left(-\frac{2}{3}, 0\right)$

(32) $K = 70R \ \frac{31}{9}$

(34) $\left[(x, y) \mid x^2 + y^2 - 6x - 6y + 13 = 0, x, y + R\right]$
$- \left[(4, 1), (2, 5)\right]$

(39) $5x^2 + 5y^2 - 72x + 52y + 225 = 0$

(40) $\left(\frac{5}{2}, \frac{5}{2}\right), \left(\frac{7}{3}, \frac{7}{3}\right), \left(\frac{15}{8}, \frac{15}{8}\right)$

(41) $(6, 13)$

(42) 5

(43) $\left(-\frac{2}{3}, -\frac{5}{3}\right), \left(\frac{2}{3}, \frac{11}{3}\right)$

15

Straight Line

LEARNING OBJECTIVES

The paramatric equation of a line

- The cartesian equation of a line
- Slope of a line
- Angle between two intersecting lines
- Intersection of two lines
- The slope-point equation
- The equation of having slope *m* and intrecept of *y*-axis is *c*
- The equation of the form (*P* – α) of a line

INTRODUCTION

Paramatric and cartesian equation of a line
A (x_1, y_1) and B (x_2, y_2) where $A \neq B$ are given points

1. Cartesian equation of $\overleftrightarrow{AB} = \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

   If $x_1 \neq x_2, y_1 \neq y_2$ then the cartesian equation of AB is
   $$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$$

2. Paramatric equations of $\overleftrightarrow{AB}$ is
   $$x = tx_2 + (1 - t)x_1$$
   $$y = ty_2 + (1 - t)y_1, t \in \text{R}$$

If $P(x, y)$ is a point of $\overleftrightarrow{AB}$ then

1. $t = 0 \Leftrightarrow P = A$
2. $t = 1 \Leftrightarrow P = B$
3. $0 < t < 1 \Leftrightarrow A - P - B$
4. $t < 0 \Leftrightarrow P - A - B$
5. $t > 1 \Leftrightarrow A - B - P$

SLOPE OF A LINE

1. If line l makes an angle of measure θ in the upper plane of x-axis with the direction of x-axis and $\theta \neq \frac{\pi}{2}$ then tan θ is the slope of the line.
2. Slope of a line $\perp^{er}$ to y-axis is zero.
3. Slope of a line $\perp^{er}$ to x-axis is not defined.
4. Slope of the line passing through A (x_1, y_1) and B (x_2, y_2) is $\perp^{er}$ $x_1 \neq x_2$.
5. Let l_1 and l_2 be two lines having slopes m_1 and m_2 (l_1, l_2 not being vertical) then

   $\Pi \to$ $l_1 \parallel l_2 \Leftrightarrow m_1 = m_2$

   $\Pi \to$ $l_1 \perp^{er} l_2 \Leftrightarrow m_1 m_2 = -1$

   $\Pi \to$ $m_1 m_2 \neq -1$, $m_1 \neq m_2$ then the measure of angle between l_1 and l_2 is

   $$\theta = \tan^{-1}\left(\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|\right);\ \theta \in \left(0, \frac{\pi}{2}\right)$$

   If l_1 is a vertical line and l_2 makes an angle of measure α with x-axis then the measure of angle between l_1 and l_2 is $\left|\frac{\pi}{2} - \alpha\right|$ where $\tan \alpha = m_2$

Intercepts of a Line on Co-ordinate Axes

1. In l intersects x-axis at A $(a, 0)$ and y-axis at B $(0, b)$ then a is called the x-intercept of line l; b is called the y-intercept of line l.
2. If $l \parallel x$-axis then its x-intercept does not exist.
   If $l \parallel y$-axis then its y-intercept does not exist.
3. If the line passes through origin, both of its intercepts on co-ordinate axes are zero.

LINEAR EQUATION IN R^2

$: l : ax + by + c = 0\ a^2 + b^2 \neq 0$

1. $l : ax + by + c = 0$, $a^2 + b^2 \neq 0$ is called a linear equation in R^2 and its graph in R^2 is always a line.
2. If $a = 0$, l is a horizontal line. More over if $a = 0$, $c = 0$. l coincides line. More over if $b = 0$, $c = 0$, l coincides with x-axis.
3. If $b = 0$, l is a vertical line. More over if $b = 0$, $c = 0$ then l coincides with y-axis.
4. If $c = 0$, the line passes through origin.
5. Slope of $l = \frac{-a}{b}$, $b \neq 0$.
6. X-intercept and y-intercept of l are respectively $\frac{-c}{a}$ and $\frac{-c}{b}$ where $a \neq 0$, $b \neq 0$, $c \neq 0$.

Equation Representing a Family of Lines

1. The equation of the family of lines parallel to the given line $ax + by\ c = 0$; $a^2 + b^2 \neq 0$ is $ax + by + c' = 0$; $c' \in R - (c)$.

2. The equation of the family of lines $\perp^{er}$ to the given line $ax + by + c = 0$; $a^2 + b^2 \neq 0$ is $bx - ay + k = 0$; $k \in R$
3. The equation of the family of lines passing through the fixed point A (x_1, y_1) is $a(x - x_1) + b(y - y_1) = 0$; $a^2 + b^2 \neq 0$.

INTERSECTION OF TWO LINES

For the lines $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$
where $ai^2 + bi^2 \neq 0$; $i = 1, 2$.

1. If $a_1b_2 - a_2b_1 \neq 0$ then the lines intersect in the unique point whose co-ordinates are $\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} - \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}\right)$
2. If $a_1b_2 - a_2b_1 = 0$; $b_1c_2 - b_2c_1 \neq 0$ and $a_1c_2 - a_2c_1 \neq 0$ then two lines are distinct and parallel
   〉 The point of intersection is not possible solution set is ϕ.
3. If $a_1b_2 - a_2b_1 = 0$; $b_1c_2 - b_2c_1 = a_1c_2 - a_2c_1$
   i.e. $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$ then they are coincident lines and their intersection is the infinite set.
   $\left[(x, y) \mid a_1x + b_1y + c_1 = 0;\ a_1^2 + b_1^2 \neq 0\right]$.

Some Standard Forms of Cartesian Equation of Line

1. Line passing through A (x_1, y_1) and B (x_2, y_2) and

$$A \neq B, \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 \text{ or } \frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$$

2. Line passing through A (x_1, y_1) and having slope m is $(y - y_1) = m(x - x_1)$.
3. Line having slope m and y-intercept c : $y = mx + c$.
4. Line whose x-intercept and y-intercept are respectively a and b $(a \neq 0;\ b \neq 0)$: $\frac{x}{a} + \frac{y}{b} = 1$
5. Line passing through A (x_1, y_1) and making angle of measure θ with the direction of x-axis $\frac{x - x_1}{\sin\theta} = \frac{y - y_1}{\cos\theta}$; the co-ordinates of points at a distance r from A (x_1, y_1) on this line are $(x_1 \pm r\cos\theta, y_1 \pm r\sin\theta)$, where $\tan\theta$ is the slope of this line.
6. The line on which the perpendicular from the origin maker angle of measure α with +ve direction of x-axis and the length of that perpendicular is P: $x\cos\alpha + y\sin\theta = P$ $(-\pi < \alpha < \pi)$

If the equation of line is given in the from $ax + by + c = 0$ then the length of the perpendicular from the origin on this line is $\frac{|c|}{\sqrt{a^2 + b^2}}$ and

(1) $c > 0$ then $\cos\alpha = \dfrac{-a}{\sqrt{a^2+b^2}}$; $\sin\alpha = \dfrac{-a}{\sqrt{a^2+b^2}}$

(2) $c < 0$ then $\cos\alpha = \dfrac{a}{\sqrt{a^2+b^2}}$; $\sin\alpha = \dfrac{b}{\sqrt{a^2+b^2}}$

If $ax + by + c = 0$; $a^2 + b^2 \neq 0$ is the given line. Then the co-ordinate of any point on this line can be selected as

$\left(x, -\dfrac{ax+c}{b}\right)$ or $\left(-\dfrac{by+c}{a}, y\right)$.

ILLUSTRATIONS

Illustration 1 Write as a set the line passing through A $(1, -3)$, B $(5, -2)$. Also write as sets $\overline{AB}$, $\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overleftrightarrow{AB} - \overline{AB}$, $\overrightarrow{BA} - \overline{AB}$.

Solution

The paramatric equations of $\overleftrightarrow{AB}$ are

$x = 5t + (1 - t)1$ $\qquad y = t(-2) + (1 - t)(-3)$

$\therefore x = 4t + 1$ $\qquad \therefore y = t - 3 \qquad t \in R$

$$
\begin{aligned}
\text{Here } \overleftrightarrow{AB} &= \left[(x, y)/\ x = 4t + 1, y = t - 3, t \in R\right] \\
\overline{AB} &= \left[(x, y)/\ x = 4t + 1, y = t - 3, t \in (0, 1)\right] \\
\overrightarrow{AB} &= \left[(x, y)/\ x = 4t + 1, y = t - 3, t \geq 0\right] \\
\overrightarrow{BA} &= \left[(x, y)/\ x = 4t + 1, y = t - 3; t \leq 1\right] \\
\overleftrightarrow{AB} - \overline{AB} &= \left[(x, y)/\ x = 4t + 1, y = t - 3; t \in R - (0, 1)\right] \\
\overrightarrow{BA} - \overline{AB} &= \left[(x, y)/\ x = 4t + 1, y = t - 3; t < 0\right]
\end{aligned}
$$

Illustration 2 If A $(3, -2)$, B $(6, 5)$ and P (x, y) $\overline{AB}$, find the range of $2x - 3y$.

Solution

The paramatric equations as of $\overleftrightarrow{AB}$ are

$x = t(6) + (1 - t)(3) = 3t + 3; t \in R$

$y = t(5) + (1 - t)(-2) = 7t - 2; t \in R$

Now $\overline{AB} = \left[(x, y)/\ x = 3t + 3; y = 7t - 2, t \in (0, 1)\right]$

$$
\begin{aligned}
2x - 3y &= 2(3t + 3) - 3(7t - 2) \\
&= -15t + 12
\end{aligned}
$$

$\therefore$ For $t \in [0, 1]$ we have to find the minimum and maximum value of $-15t + 12$

Now $0 \leq t \leq 1$

$0 \geq -15t \geq -15$

$\therefore -15 \leq -15t \leq 0$

$\therefore -15 + 12 \leq -15t + 12 \leq 12$

$\therefore -3 \leq -15t + 12 \leq 12$

$\therefore -15t + 12 \in (-3, 12)$

Illustration 3 Find the paramatric equations of the lines passing through A $(3, -1)$, and B $(0, 3)$. Also write $\overleftrightarrow{\mathrm{AB}}, \overline{\mathrm{AB}}, \overrightarrow{\mathrm{AB}}, \overrightarrow{\mathrm{BA}}, \overleftrightarrow{\mathrm{AB}} - \overline{\mathrm{AB}}, \overrightarrow{\mathrm{BA}} - \overline{\mathrm{AB}}$ as sets.

Solution

Paramatric equations of $\overleftrightarrow{\mathrm{AB}}$ are

$$\begin{array}{l|l} x = t(0) + (1-t)3 & y = t(3) + (1-t)\,(-1) \\ \quad = 3 - 3t & y = 4t - 1 \qquad t \in R \end{array}$$

$\therefore$ In set notation

$$\begin{aligned} \overleftrightarrow{\mathrm{AB}} &= \left[(x, y)/\ x = 3 - 3t, y = 4t - 1, t \in R\right] \\ \overline{\mathrm{AB}} &= \left[(x, y)/\ x = 3 - 3t, y = 4t - 1, t \in (0, 1)\right] \\ \overrightarrow{\mathrm{AB}} &= \left[(x, y)/\ x = 3 - 3t, y = 4t - 1, t \parallel 0\right] \\ \overrightarrow{\mathrm{BA}} &= \left[(x, y)/\ x = 3 - 3t, y = 4t - 1, t \le 1\right] \\ \overleftrightarrow{\mathrm{AB}} - \overline{\mathrm{AB}} &= \left[(x, y)/\ x = 3 - 3t, y = 4t - 1, t \in R - (0, 1)\right] \\ \overrightarrow{\mathrm{BA}} - \overline{\mathrm{AB}} &= \left[(x, y)/\ x = 3 - 3t, y = 4t - 1, t < 0\right] \end{aligned}$$

Illustration 4 The paramatric equations of $\overleftrightarrow{\mathrm{AB}}$ are $x = 16t + 13$; $y = 14t + 17$, $t \in R$. If the y-co-ordinate of a point on $\overleftrightarrow{\mathrm{AB}}$ is 38 find the x-co-ordinate.

Solution

Let the point P $(a, 38)$

P $(a, 38) \in \overleftrightarrow{\mathrm{AB}}$

Paramatric equations of a line are

$x = 16t + 13$ and $y = 14t + 13$

$\therefore\ a = 16t + 13$ and $38 = 14t + 13$

$\therefore\ t = \dfrac{3}{2}$

$\therefore\ t = \dfrac{3}{2} \Leftrightarrow a = 16\left(\dfrac{3}{2}\right) + 13 = 37$

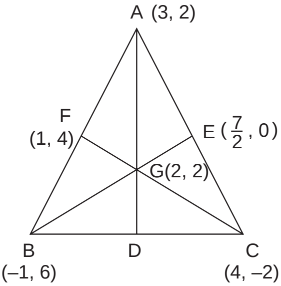
$\therefore$ x-co-ordinate of the given point is 37.

Illustration 5 Obtain the co-ordinate of the centroid of the triangle whose vertices are A $(3, 2)$; B $(-1, 6)$, C $(4, -2)$. Find the equations of all the three medians and cartesian equations of the lines containing the three medians.

Solution

Let D, E, F be the mid-points of $\overline{\mathrm{BC}}, \overline{\mathrm{AC}}, \overline{\mathrm{AB}}$, respectively. $\overline{\mathrm{AD}}$, $\overline{\mathrm{BE}}$ and $\overline{\mathrm{CF}}$ are the medians of ΔABC.

$\therefore$ Centroid G $= \left(\dfrac{3-1+4}{3}, \dfrac{2+6-2}{3}\right)$



G $(x, y) = (2, 2)$

∴ Paramatric equation of median $\overline{AD}$ are

$x = \frac{3}{2}t + (1-t)3 = 3 - \frac{3}{2}t\ ;\ t \in (0,\ 1)$

$y = 2t + (1-t)2 = 2 - \frac{3}{2}t\ ;\ t \in (0,\ 1)$

∴ $\overline{AD} = \left[(x, y/\ x = 3 - \frac{3}{2}t,\ y = 2;\ t \in \text{R}\right]$

∴ $\overline{AD}$ is parallel to x-axis and its cartesian equation is $y = 2$

Paramatric equation of median $\overline{BE}$ is

$x = \frac{7}{2}t + (1-t)(-1) = \frac{9}{2}t - 1;\ t \in [0,\ 1]$

$y = t(0) + (1-t)6 = 6 - 6t;\ t \in [0,\ 1]$

∴ $\overline{BE} = \left[(x, y/\ x = \frac{9}{2}t - 1,\ y = 6 - 6t,\ t \in (0,\ 1)\right]$

Equation of $\overline{BE} = \left[(x, y)/\ x = \frac{9}{2}t - 1;\ y = 6 - 6t,\ t \in R\right]$

∴ $x = \frac{9}{2}t - 1\ \Pi\ t = \frac{2x+2}{9}$ and $y = 6 - 6t\ \Pi\ t = \frac{6-y}{6}$

∴ $\frac{2x+2}{9} = \frac{6-y}{6}$

∴ $12x + 12 = 54 - 9y$

∴ $12x + 9y - 42 = 0$

∴ $4x + 3y - 14 = 0$ is the cartesian equation of $\overline{BE}$.

Illustration 6 If the measure of the angle between two lines is $\frac{\pi}{4}$ and if the slope of one of the lines is $\frac{3}{2}$ find the slope of the other.

Solution

Here $\alpha = \frac{\pi}{4}$ (given), $m_1 = \frac{3}{2}$

Now $\alpha = \tan^{-1}\left(\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|\right)$

∴ $\tan\frac{\pi}{4} = \left|\frac{(3/2) - m_2}{1 + (3/2)m_2}\right|$

∴ $1 = \left|\frac{3 - 2m_2}{2 + 3m_2}\right|$

∴ $\left|\frac{3 - 2m_2}{2 + 3m_2}\right| = \pm 1$

$\frac{3-2m_2}{2+3m_2}=1$ or $\frac{3-2m_2}{2+3m_2}=-1$

$\therefore\ 3-2m_2=2+3m_2$ or $3-2m_2=-2-3m_2$

$\therefore\ 5m_2=-1$ or $-m_2=5$

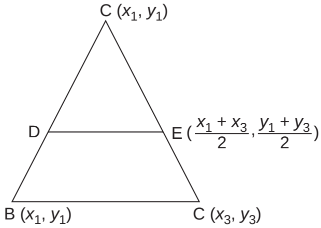
$\therefore\ m_2=\frac{-1}{5}$ or $m_2=5$

$\therefore$ The slope of the other line is $\frac{1}{5}$ or -5

Illustration 7 Prove using slopes of lines that the incensement joining the mid-points of $\overline{AB}$ and $\overline{AC}$ in ΔABC is parallel to BC .

Solution

Let A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3) are the vertices of ΔABC let D and E be respectively the mid-points of $\overline{AB}$ and $\overline{\overline{AC}}$. Now slope of $\overleftrightarrow{BC}=\frac{y_3-y_2}{x_3-x_2}$



$$\therefore \text{ Slope of } \overleftrightarrow{DE} = \frac{\left\{\left[\left(y_1+y_3\right)/2\right]-\left[\left(y_1+y_2\right)/2\right]\right\}}{\left\{\left[\left(x_1+x_3\right)/2\right]-\left[\left(2x+x_2\right)/2\right]\right\}}$$

$$=\frac{y_3-y_2}{x_3-x_2}$$

$\therefore$ Slope of $\overleftrightarrow{DE}$ = Slope of $\overleftrightarrow{BC}$

$\therefore\ \overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$

$\therefore\ \overline{DE} \parallel \overline{BC}$

Illustration 8 Obtain the lines (1) parallel to and (2) perpendicular to the line $3x+7y-1=0$ and passing through $(-2, 5)$.

Solution

The general equation of a line parallel to $3x+7y-1=0$ is $3x+7y+k=0$ which passes through $(-2, 5)$

$\therefore\ 3(-2)+7(5)+k=0$

$\therefore\ k=-29$

Hence the required equation of a line parallel to $3x+7y-1=0$ is $3x+7y=29=0$

The general equation of a line perpendicular to $3x+7y-1=0$ is $7x-3y+k=0$ which passes through $(-2, 5)$

$\therefore\ 7(-2) - 3(5) + k = 0$

$\therefore\ k = 29$

Hence the required equation of a line $\perp^{er}$ to $3x + 7y - 1 = 0$ is $7x = 3y + 29 = 0$

Illustration 9 If A is $(2, 2)$ B $(0, 4)$ and C $(3, 3)$ then find the equation of (1) the median of ΔABC through A, (2) the altitude of ΔABC through A, (3) the $\perp^{er}$ bisector of $\overline{BC}$, (4) The bisscetor of $\angle$BAC.

Solution

(1) The median of ΔABC through A passes through mid-point of $\overline{BC}$ which is $\left(\frac{0+3}{2}, \frac{4+3}{2}\right) = \left(\frac{3}{2}, \frac{7}{2}\right)$.

$\therefore$ Median A passes through $(2, 2)$ and $\left(\frac{3}{2}, \frac{7}{2}\right)$

$$\therefore \text{ Its equation is } \begin{vmatrix} x & y & 1 \\ 2 & 2 & 1 \\ \frac{3}{2} & \frac{7}{2} & 1 \end{vmatrix} = 0$$

$\therefore\ -\frac{3}{2}x - \frac{1}{2}y + 4 = 0$

$\therefore\ 3x + y = 8$

(2) Slope of $\overline{BC} = \frac{4-3}{0-3} = \frac{-1}{3}$

$\therefore$ Slope of the altitude from A on $\overline{BC}$ is 3. This altitude passes through A $(2, 2)$.

$\therefore$ Equation of the line containing the altitude from A on $\overline{BC}$ is

$(y - 2) = 3(x - 2)$

$\therefore\ 3x - y = 4$

(3) The perpendicular bisector of $\overline{BC}$ passes through the mid-point $\left(\frac{3}{2}, \frac{7}{2}\right)$ of $\overline{BC}$ and since it is perpendicular to $\overline{BC}$ its slope is 3.

$\therefore$ The equation of $\perp^{er}$ bisector of $\overline{BC}$

$\left(y - \frac{7}{2}\right) = 3\left(x - \frac{3}{2}\right)$

$\therefore\ 3x - y = 1$

(4) $AC = \sqrt{1^2 + 1^2} = \sqrt{2}$, $AB = \sqrt{4+4} = 2\sqrt{2}$

If the bisector of $\angle$A intersect $\overleftrightarrow{BC}$ in D

$\frac{BD}{DC} = \frac{AB}{AC} = \frac{2\sqrt{2}}{\sqrt{2}} = \frac{2}{1}$

$\therefore$ D divides $\overline{BC}$ in the ratio 2 : 1 from B

$$D = \left(\frac{2(3)+1(0)}{2+1}, \frac{2(3)+1(4)}{2+1}\right) = D\left(2, \frac{10}{3}\right)$$

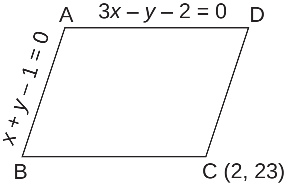
$\therefore$ Since D and A are both having same x-coordinate 2

$\overleftrightarrow{AD} \parallel y$-axis

$\therefore$ its equation is $x = 2$.

Illustration 10 Equations of two of the sides of a parallelogram are $3x - y - 2 = 0$ and $x - y - 1 = 0$ and one of its vertices is $(2, 3)$. Find the equation of the remaining sides of the parallelogram.

Solution



Lines $3x - y - 2 = 0$ and $x - y - 1 = 0$ are not parallel because their slopes are not equal and $(2, 3)$ does not satisfy any of the equations of lines.

$\therefore$ ABCD is a parallelogram as shown in the figure.

$\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$

$\therefore$ The equation of $\overleftrightarrow{CD}$ is $x - y + k = 0$ and $\overleftrightarrow{CD}$ passes through C $(2, 3)$

$\therefore$ $2 - 3 + k = 0$

$\therefore$ $k = 1$

$\therefore$ $\overleftrightarrow{CD}$: $x - y + 1 = 0$

$\overleftrightarrow{BC} \parallel \overleftrightarrow{AD}$

$\therefore$ The equation of $\overleftrightarrow{BC}$ is $3x - y + k' = 0$ and $\overleftrightarrow{BC}$ passes through < $(2, 3)$

$\therefore$ $3(2) - 3 + k' = 0$

$\therefore$ $k' = -3$

$\therefore$ $\overleftrightarrow{BC} = 3x - y - 3 = 0$

Illustration 11 If $\frac{1}{a} + \frac{1}{b} = k$, a non-zero constant, prove that for all non-zero values of a and b, line $\frac{x}{a} + \frac{y}{b} = 1$ passes through a fixed point and find the co-ordinate of that point.

Solution

$$\frac{1}{a} + \frac{1}{b} = k \Leftrightarrow \frac{1}{ak} + \frac{1}{bk} = 1$$

But the equation of line is

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\therefore \frac{x}{a} + \frac{y}{b} = \frac{1}{ak} + \frac{1}{ak}$$

$$\therefore \frac{y}{a} - \frac{1}{bk} = \frac{1}{ak} - \frac{x}{a}$$

$$\frac{1}{b}\left(y - \frac{1}{k}\right) = -\frac{1}{a}\left(x - \frac{1}{k}\right)$$

$$\therefore \left(y-\frac{1}{k}\right) = -\frac{b}{a}\left(x-\frac{1}{k}\right)$$

$$\therefore \left(\frac{1}{k},\frac{1}{k}\right) \in \frac{x}{a}+\frac{y}{b}=1$$

$\therefore$ Line $\frac{x}{a}+\frac{y}{b}=1$ passes through point $\left(\frac{1}{k},\frac{1}{k}\right)$.

Illustration 12 Find the equation of a line that is perpendicular to $\sqrt{3}x - y + 5 = 0$ given that its x-intercept is 2.

Solution

The equation of a line $\perp^{er}$ to the line

$\sqrt{3}x - y + 5 = 0$ is $x + \sqrt{3}y + k = 0$

$\therefore$ x-intercept of $x + \sqrt{3}y + k = 0$ is 2

$$\therefore \frac{-k}{1} = 2$$

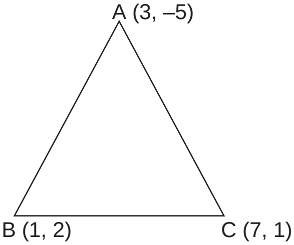
$\therefore k = -2$

$\therefore$ The equation of the reuatired lines is $x + \sqrt{3}y - 2 = 0$

Illustration 13 Find the equations of the lines containing the sides of the triangles whose vertices are $(3, -5)$, $(1, 2)$ and $(7, 1)$.

Solution

Here A $(3, -5)$, B $(1, 2)$ $\overleftrightarrow{AB}$



$\therefore$ Equation of $\overleftrightarrow{AB}$ is

$$\frac{y+5}{2+5} = \frac{x-3}{1-3}$$

$$\therefore \frac{y+5}{7} = \frac{x-3}{-2}$$

$\therefore -2y - 10 = 7x - 21$

$\therefore 7x + 2y - 11 = 0$ (1)

B $(1, 2)$, C $(7, 1)$ $\in$ $\overleftrightarrow{BC}$

$\therefore$ Equation of $\overleftrightarrow{BC}$ is

$$\frac{y-2}{-1-2} = \frac{x-1}{7-1}$$

$$\therefore \frac{y-2}{-3} = \frac{x-1}{6}$$

$\therefore 2y - 4 = -x + 1$

$\therefore x + 2y - 5 = 0$ (2)

C $(7, 1)$, A $(3, -5)$ $\overleftrightarrow{AC}$

$\therefore$ Equation of $\overleftrightarrow{AC}$ is

$$\frac{y+1}{-5+1} = \frac{x-7}{3-7}$$

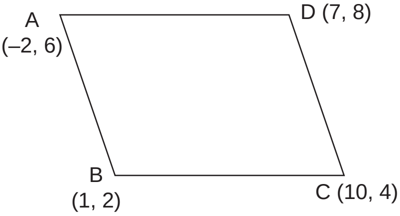
$\therefore y + 1 = x - 7$

$\therefore x - y - 8 = 0$ (3)

$\therefore$ Equations of lines containing the sides of ΔABC are $7x + 2y - 11 = 0$, $x + 2y - 5 = 0$ and $x - y - 8 = 0$.

Illustration 14 A Parallelogram has vertices $(-2,\ 6)$, $(1,\ 2)$, $(10,\ 4)$ and $(7,\ 8)$. Find the equations of the lines containing the diagonals of this parallelogrm.

Solution



Here $\overline{AC}$ and $\overline{BD}$ are the diagonals of $\square^m$ ABCD. Now $\overleftrightarrow{AC}$ containing diagonal $\overline{AC}$ passes through points A $(-2,\ 6)$ and $(10,\ 4)$.

$\therefore$ Equation of $\overleftrightarrow{AC}$ is

$$\frac{y-6}{4-6} = \frac{x+2}{10+2}$$

$$\therefore \frac{y-6}{-2} = \frac{x+2}{12}$$

$\therefore$ $6y - 36 = -x - 2$

$\therefore$ $x + 6y - 34 = 0$ (1)

Line $\overleftrightarrow{BD}$ containing diagonal $\overline{BD}$ passes through points B $(1, 2)$ and D $(7, 8)$

$\therefore$ Equation of $\overleftrightarrow{BD}$ is

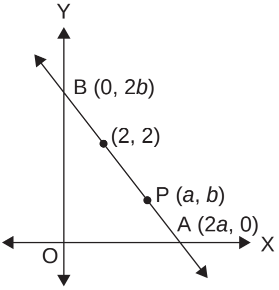
$$\frac{y-2}{8-2} = \frac{x-1}{7-1}$$

$$\therefore \frac{y-2}{6} = \frac{x-1}{6}$$

$\therefore$ $x - y + 1 = 0$ (2)

Illustration 15 Find the equations of the line given that it passes through $(2, 2)$ and that the sum of its intercepts on the axes is 9.

Solution



For the given line intercepts of x and y-axes are a and b respectively.

$\therefore$ $a + b = 9 \Rightarrow b = 9 - a$

$\therefore$ Equation of a line making intercepts a and b on x and y-axis is $\frac{x}{a} + \frac{y}{b} = 1$ and it passes through $(2, 2)$.

$$\therefore \frac{2}{a} + \frac{y}{9-a} = 1$$

$18 - 2a + 2a = 9a - a^2$

$\therefore$ $a^2 - 9a + 18 = 0$

$\therefore$ $(a-6)\,(a-3) = 0$

$\therefore$ $a = 6$ or $a = 3$

$\therefore$ $a = 6 \Rightarrow b = a = 6 = 3$ and

$a = 3 \Rightarrow b = a - 3 = 6$

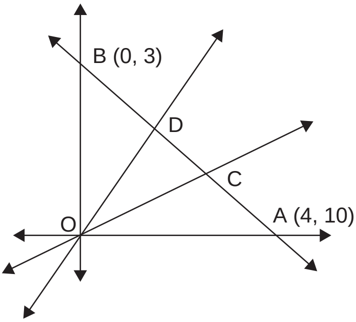
Thus the equations of the given line are

$\frac{x}{3}+\frac{y}{6}=1$ $\quad\quad$ $\frac{x}{6}+\frac{y}{3}=1$

$\therefore\ 2x + y - 6 = 0$ $\quad\quad$ $\therefore\ x + 2y - 6 = 0$

Illustration 16 The line $3x + 4y = 12$ intersects the x and $-y$-axis at A and B respectively. Find the equation of the lines joining the origin to the points of trisection of $\overline{\text{AB}}$.

Solution



$3x + 4y = 12$

$\therefore\ \frac{x}{4}+\frac{y}{3}=1$

$\therefore$ intercepts of x and y-axes are 4 and 3 respectively.

$\therefore$ A $(4, 0)$ and B $(0, 3)$

Let points C and D be points trisecting $\overline{\text{AB}}$

$\therefore$ A − C − D − B

$\therefore$ Point C divides $\overline{\text{AB}}$ from A in ratio 1 : 2

$$\therefore \text{ Co-ordinate C} = \left[\frac{(1/2)(0)+4}{(1/2)+1};\frac{(1/2)(3)+0}{(1/2)+1}\right]=\left(\frac{8}{3},1\right)$$

Now D is a mid-point of $\overline{\text{CB}}$

$$\therefore \text{ Co-ordinate of D} = \left[\frac{(8/3)+0}{2};\frac{1+3}{2}\right]=\left(\frac{4}{3},2\right)$$

Now $\overleftrightarrow{OC}$ passes through $(0, 0)$ and C $\left(\frac{8}{3},1\right)$

$$\therefore \text{ Equation of } \overleftrightarrow{OC} = \frac{y-0}{1-0}=\frac{x-0}{(8/3)-0}$$

$\therefore\ 8y = 3x$

$\therefore\ 3x - 8y = 0$ (1)

Now $\overleftrightarrow{OD}$ passes through $(0, 0)$ and D $\left(\frac{4}{3},2\right)$

$$\therefore \text{ Equation of } \overleftrightarrow{OD} = \frac{y-0}{2-0}=\frac{x-0}{(4/3)-0}$$

$\therefore\ 6x - 4y = 0$

$\therefore\ 3x - 2y = 0$

Illustration 17 The equation of the line containing one of the diagonals of a sphere is $8x - 15y = 0$. Find the equations of the sides of the square passing through the vertex $(1, 2)$ of the square and also the line containing the remaining diagonals.

Solution

Here let B $(1, 2)$ be one of the vertices of square ABCD

$(1, 2)$ does not satisfy $8x - 15y = 0$

$\therefore$ $8x - 15y = 0$ is the equation of $\overleftrightarrow{AC}$

$$\therefore \text{ Slope of } \overleftrightarrow{AC} = \frac{-8}{-15} = \frac{8}{15}$$

$$\therefore \text{ Slope of } \overleftrightarrow{BD} = \frac{-15}{8} \qquad \left(\because \overline{AC} \perp \overline{BD}\right)$$

Thus $\overleftrightarrow{BD}$ of slope $\frac{-15}{8}$ passes through B $(1, 2)$

$\therefore$ Equation of $\overleftrightarrow{BD}$

$$(y - 2) = \frac{-15}{8}(x - 1)$$

$\therefore$ $8y - 16 = -15x + 15$

$\therefore$ $15x + 8y - 31 = 0$ (1)

Now slope of $\overleftrightarrow{AC} = m_2 = \frac{8}{15}$

Suppose slope of $\overleftrightarrow{AB}$ $\left(\overleftrightarrow{BC}\right) = m_1$

$\therefore$ Measure of angle between $\overleftrightarrow{AC}$ and $\overleftrightarrow{AB}$ is $\theta = \frac{\pi}{4}$

$$\therefore \tan\frac{\pi}{4} = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\therefore 1 = \left|\frac{m_1 - (8/15)}{1 + (8/15)\, m_1}\right|$$

$$\therefore 1 = \left|\frac{15m_1 - 8}{15 + 8m_1}\right|$$

or

$$\therefore \frac{15m_1 - 8}{15 + 8m_1} = 1 \qquad\qquad \therefore \frac{15m_1 - 8}{15 + 8m_1} = -1$$

$\therefore$ $15m_1 - 8 = 15 + 8m_1$ $\qquad$ $\therefore$ $15m_1 - 8 = -15 - 8m_1$

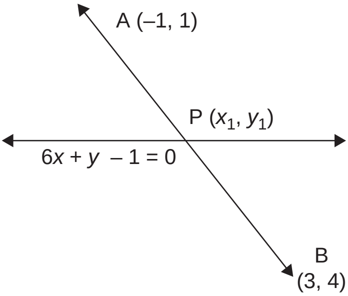
$\therefore$ $7m_1 = 23$ $\qquad$ $\therefore$ $23m_1 = -7$

$$\therefore m_1 = \frac{23}{7} \qquad\qquad \therefore m_1 = \frac{-7}{23}$$

Equation of $\overleftrightarrow{AB}$ is	Equation of $\overleftrightarrow{BC}$ is
$(y-2) = \frac{-7}{23}(x-1)$	$(y-2) = \frac{23}{7}(x-1)$
$3y - 46 = -7x + 7$	$7y - 14 = 23x - 23$
$\therefore\ 7x + 23y - 53 = 0$	$\therefore\ 23x - 7y - 9 = 0$

Illustration 18 Prove that the points A $(-1, 1)$ and B $(3, 4)$ are on opposite sides of the line $6x + y - 1 = 0$

Solution



The paramatric equations of $\overleftrightarrow{AB}$ are

$x = t(3) + (1-t)(-1) = 4t - 1$

$y = t(4) + (1-t)\,1 = 3t + 1$

Suppose $\overleftrightarrow{AB}$ intersects the line

$6x + y - 1 = 0$ in $p\ (x_1, y_1)$

Then for some $t \in R\ x_1 = 4t - 1;$

$y_1 = 3t + 1$

As P (x_1, y_1) lie on $6x + y - 1 = 0$ we get

$6(4t-1) + (3t+1) - 1 = 0$

$\therefore\ 27t - 6 = 0$

$t = \frac{6}{27} > 0 = \frac{2}{9} \in [0, 1]$

$\therefore\ p\ (x, y) \in \overline{AB}$

$\therefore$ As A – P – B and P is on the line $6x + y - 1 = 0$, we get A, B are opposite sides of the line.

Illustration 19 Let A $(3, -1)$, and B $(0, 4)$. If P $(x, y) \in \overline{AB}$, obtain the maximum and minimum value of $3y - x$.

Solution

Paramatric equations of $\overleftrightarrow{AB}$ are

$x = 0(t) + (1-t)3 = 3 - 3t$

$y = 4t + (1-t)(-1) = 5t - 1\ t \in R$

$3y - x = 3(5t-1) - (3-3t)$

$= 18t - 6$

Now $p\ (x, y) \in \overline{AB}$

$t \in [0, 1]$

$0 \leq t \leq 1$

$\therefore 0 \leq 18t \leq 18$

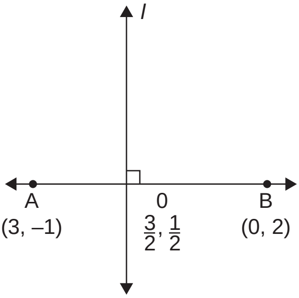
$\therefore -6 \leq 18t \leq 6 \mid 12$

$\therefore 18t - 6 \in (-6, 12)$

Illustration 20 Obtain the equation of the $\perp^{er}$ bisector of $\overline{AB}$ where A is $(3, -1)$ and B is $(0, 2)$.

Solution

Let the line l be the perpendicular bisector of $\overline{AB}$ here. Let D be the mid-point of $\overline{AB}$.



$$\therefore D = \left(\frac{3}{2}, \frac{1}{2}\right)$$

$$\therefore \text{Slope of } \overleftrightarrow{AB} = \frac{2+1}{0-3} = -1$$

$\therefore$ Slope of l is = 1 and it is passes through $\left(\frac{3}{2}, \frac{1}{2}\right)$

$$\therefore \left(y - \frac{1}{2}\right) = 1\left(x - \frac{3}{2}\right)$$

$\therefore x - y - 1 = 0$ which is the required equation of line l.

Illustration 21 If the ratio of the x-intercept and the y-intercept of a line is $\frac{3}{2}$ and if the line passes through A $(1, 2)$, find the equation of the line.

Solution

Let the intercepts of x and y-axis be a and b respectively.

$\therefore$ Equation of the line is $\frac{x}{a} + \frac{y}{b} = 1$ (1)

Here $\frac{a}{b} = \frac{3}{2} \Rightarrow a = \frac{3b}{2}$

Now substitute the value $a = \frac{3b}{2}$ in (1)

$$\therefore \frac{x}{(3b/2)} + \frac{y}{b} = 1$$

$$\therefore\ \frac{2x}{3b} + \frac{y}{b} = 1$$

$\therefore\ 2x + 3y = 3b$

But the line passes through A $\left(1,\ 2\right)$

$\therefore\ 2\left(1\right) + 3\left(2\right) = 3b$

$$\therefore\ b = \frac{8}{3}$$

$$\text{But } a = \frac{3b}{2} = \frac{3}{2}\cdot\frac{8}{3} = 4$$

Now substitute the value of a and b in (1)

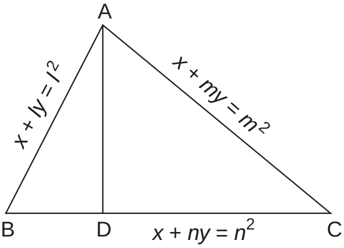
$$\therefore\ \frac{x}{4} + \frac{y}{\left(8/3\right)} = 1$$

$$\therefore\ \frac{x}{4} + \frac{3y}{8} = 1$$

$\therefore\ 2x + 3y - 8 = 0$

Illustration 22 The equation of the sides of a triangle are $x + ly = l^2$, $x + my = m^2$, $x + ny = n^2$ find the co-ordinate of orthocentre.

Solution



$x + ly = l^2$ (1)

$x + my = m^2$ (2)

$x + ny = n^2$ (3)

Equation of a line through the intersection of eqs. (1) and (2) is

$x + ly - l^2 + k\left(x + my - m^2\right) = 0$

$\therefore\ \left(1 + k\right)x + \left(1 + km\right)y - l^2 - km^2 = 0$ (4)

line (4) is perpendicular to (3) if

$$\left(-\frac{l + k}{l + mk}\right)\left(\frac{-1}{n}\right) = -1$$

$\therefore\ -l + k = -ln - nmk$

$$\therefore\ k = \frac{1 - ln}{1 + mn}$$

Now by substituting the value of k in (4) we get

$$\left(1+\frac{-1-\ln}{l+mn}\right)x+\left(l+\frac{-m-lmn}{l+mn}\right)y-l^2-km^2=0$$

$\therefore\ (mn-ln)x+(l-m)y-l^2-m^2+lm^2n-l^2mn=0$

$\therefore\ nx-y+(l+m+lmn)=0$

This is the equation of $\overleftrightarrow{AD}$

Similarly the equation of $\overline{BE}$ is

$lx-y+(m+n+lmn)=0$ (6)

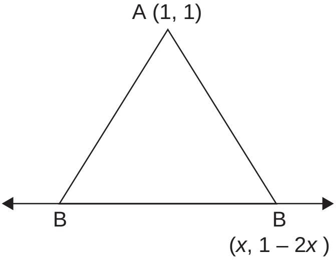
Now solving (5), (6) we get co-ordinate of orthocentre is

$x:1, y=l+m+n+lmn$

$\therefore\ (x, y)=(l, l+m+n+lmn)$

Illustration 23 Find the points on $2x+y=1$ which are at a distance 2 from $(1, 1)$.

Solution



A $(1, 1)$ does not satisfy the equation of line $2x+y=1$

$\therefore$ A $(1, 1)$ is not a point on the given line

Now equation of line is $2x+y=1$ i.e.

$y=1-2x$

$\therefore$ Any point on this line can be taken as B $(x, 1-2x)$

Now AB = 2 ⇒ $AB^2=4$

$\therefore\ (x-1)^2+(1-2x-1)^2=4$

$\therefore\ x^2-2x+1+4x^2=4$

$\therefore\ 5x^2-2x-3=0$

$\therefore\ (5x+3)(x-1)=0$

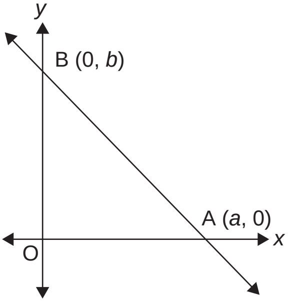
$\therefore\ x=\frac{-3}{5}$ or $x=1$

Now $x=\frac{-3}{5}\Rightarrow B\left[\frac{-3}{5}, 1-2\left(\frac{-3}{5}\right)\right]=B\left(\frac{-3}{5}, \frac{11}{5}\right)$ (1)

$x=1\Rightarrow B\,[1, 1-2(1)]=B\,(1, -1)$

Illustration 24 A line intersects x-axis at A and y-axis at B. Given that AB = 15 and $\overleftrightarrow{AB}$ makes with the two axes, a triangle with an area of 54. Find the equation of the line.

Solution



Suppose $\overleftrightarrow{AB}$ intersects x-axis at A $(a, 0)$ and y-axis at B $(0, b)$.

$\therefore$ Equation of $\overleftrightarrow{AB}$ is $\frac{x}{a} + \frac{y}{b} = 1$ (1)

Now, Area of $\Delta OAB = \frac{1}{2}\ \overline{OA}\ \overline{OB}$

$$\therefore\ 54 = \frac{1}{2} ab$$

$$\therefore\ b = \frac{108}{a} \quad (2)$$

Now AB = 15, $AB^2 = 225$, $a^2 + b^2 = 225$

$\therefore\ a^2 + \left(\frac{108}{a}\right)^2 = 225$

$\therefore\ a^2 + \frac{11664}{a^2} = 225$

$\therefore\ a^4 - 225a^2 + 11664 = 0$

$\therefore\ (a^2 - 144)\ (a^2 - 81) = 0$

$\therefore\ a^2 = 144$ or $a^2 = 81$

$\therefore\ a = \pm 12$ or $a = \pm 9$

$\therefore$ if $a = \pm 12 \Rightarrow b = \pm 9$

and if $a = \pm 9 \Rightarrow b = \pm 12$

Now substitute the values of a and b in equation (1)

$\therefore\ \frac{x}{\pm 12} + \frac{y}{\pm 9} = 1$ or $\frac{x}{\pm 9} + \frac{y}{\pm 12} = 1$

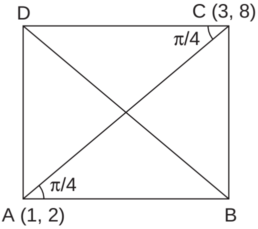
$\therefore\ \pm 3x \pm 4y = 36$ or $\pm 4x \pm 3y = 36$

Illustration 25 The points $(1, 2)$ and $(3, 8)$ are a pair of diagonally opposite vertices of a square. Find the equations of lines containing all the sides of the square.

Solution

Let A $(1, 2)$ and B $(3, 8)$ be opposite vertices of a square ABCD

$\therefore$ Slope of $\overleftrightarrow{AC} = \frac{8-2}{3-1} = 3 = m_1$



Suppose m_2 be the slope of $\overleftrightarrow{AB}$ (or $\overleftrightarrow{BC}$)

Now here angle between $\overleftrightarrow{AC}$ and $\overleftrightarrow{AB}$ $(\overleftrightarrow{BC}) = \frac{\pi}{4}$

$\therefore\ \tan\frac{\pi}{4} = \left|\frac{3 - m_2}{1 + 3m_2}\right|$

$\therefore\ \frac{3 - m_2}{1 + 3m_2} = \pm 1$

or

$$\therefore \frac{3-m_2}{1+3m_2}=1 \qquad\qquad \frac{3-m_2}{1+3m_2}=-1$$

$$\therefore 3-m_2=1+3m_2 \qquad\qquad 3-m_2=-1-3m_2$$

$$4m_2=2 \qquad\qquad 2m_2=-4$$

$$\therefore m_2=\frac{1}{2} \qquad\qquad \therefore m_2=-2$$

$\therefore$ Taking slope of $\overleftrightarrow{AB}$ $(\overleftrightarrow{BC})=-2$

$\therefore \overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

$\therefore$ Slope of $\overleftrightarrow{CD}=\frac{1}{2}$

$\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$

$\therefore$ Slope of $\overleftrightarrow{AD}=-2$

$\therefore$ Equation of $\overleftrightarrow{AB}$ having slope $\frac{1}{2}$ and passing through A $(1, 2)$

$$(y-2)=\frac{1}{2}(x-1)$$

$\therefore 2y-4=x-1$

$$\therefore x-2y=3=0 \tag{1}$$

Now equation of $\overleftrightarrow{BC}$ having slope -2 and passing through C $(3, 8)$

$$(y-8)=-2(x-3)$$

$\therefore y-8=-2x+6$

$$\therefore 2x+y-14=0 \tag{2}$$

Equation of $\overleftrightarrow{CD}$ with slope $\frac{1}{2}$ and passing through C $(3, 8)$ is

$$(y-8)=\frac{1}{2}(x-3)$$

$\therefore 2y-16=x-3$

$$x-2y+13=0 \tag{3}$$

Equation of $\overleftrightarrow{AD}$ of slope -2 and passing through A $(1, 2)$ is

$$(y-2)=-2(x-1)$$

$\therefore y-2=-2x+2$

$$\therefore 2x+y-4=0 \tag{4}$$

Illustration 26 Prove that for every $\lambda \in R$ the line $3x+\lambda y-5=0$ passes through a fixed point and find this fixed point.

Solution

$$\text{Here } \begin{cases} 3x+\lambda y-5=0 \\ (3x-5)+\lambda y=0 \end{cases}$$

$a, b_2-a_2b_1=3-0=3 \neq 0$

$\therefore$ The given line passes through the point of intersection of lines.

$3x-5=0$ and $y=0$

$\therefore$ Co-ordinates of this point of intersection = $\left(\frac{5}{3}, 0\right)$

Thus we can say that $3x + \lambda y - 5 = 0$ passes through a fixed point $\left(\frac{5}{3}, 0\right)$ for $\lambda \in R$

Illustration 27 A line intersects x and y-axes at A and B respectively. If AB = 10 and 30A = 40B then find the equation of the line.

Solution

Let A = A $(a, 0)$ and B $(0, b)$ then OA = $|a|$ and OB = $|b|$ are the intercepts of x- and y-axes, respectively.

$\therefore$ 3OA = 4OB $\Rightarrow 3\,|a| = 4\,|b| \Rightarrow a = \frac{\pm 4}{3}b$

Now AB = 10 $\Rightarrow$ AB^2 = 100

$\therefore a^2 + b^2 = 100$

$$\therefore \left(\frac{\pm 4}{3}b\right)^2 + b^2 = 100$$

$\therefore 25b^2 = 900$

$b = \pm 6 \Rightarrow a = \pm 8$

$\therefore$ From $\frac{x}{a} + \frac{y}{b} = 1$ the equation of line is $\frac{x}{\pm 8} + \frac{y}{\pm 6} = 1$

Illustration 28 A $(1, 2)$, B $(2, 1)$ and P (x, y) are collinear such that P $\in \overline{\text{AB}}$. Show that $7 \leq 2x + 3y \leq 8$.

Solution

Paramatric equations of $\overleftrightarrow{\text{AB}}$ are

$$\begin{aligned} x &= tx_2 + (1 - t)x_1 & \qquad y &= ty_2 + (1 - t)\, y_1 \\ &= t(2) + (1 - t)\,(1) & &= t(1) + (1 - t)\, 2 \\ &= t + 1 & &= 2 - t \end{aligned}$$

We know that for $t = 0$ p = A, $t = 1$, p = B

But $0 < t < 1 \Rightarrow$ A − P − B

$\therefore p \in \overline{\text{AB}} \Rightarrow 0\,|t|\,1$

Now $2x + 3y = 2(t + 1) + 3(2 - t)$

$= 8 - t$

$\therefore 0 \leq t \leq |1 \Rightarrow 0 \geq -t \geq -1$

$\Rightarrow -1 \leq -t \geq 0$

$\Rightarrow 7 \leq 8 - t \leq 8$

$\therefore 2x + 3y \in [7, 8]$

Illustration 29 l: $ax + by + c = 0$; $a^2 + b^2 \neq 0$ is the given line and A (x_1, y_1), B $(x_2, y_2) \notin l$. Line l divides $\overline{\text{AB}}$ in the ratio λ from A. Show that

$$\lambda = -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}.$$

Solution

Clearly, $p\left(x, y\right) = p\left(\frac{\lambda x_2 + x_1}{\lambda + 1}; \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$

But $p\left(x, y\right)$ is on line $l = ax + by + c = 0$

$\therefore\ a\left(\frac{\lambda x_2 + x_1}{\lambda + 1}\right) + b\left(\frac{\lambda y_2 + y_1}{\lambda + 1}\right) + c = 0$

$\therefore\ a\lambda x_2 + ax_1 + b\lambda x_2 + by_1 + c = 0$

$\lambda\left(ax_2 + by_2 + c\right) = -\left(ax_1 + by_1 + c\right)$

$\therefore\ \lambda = -\left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right)$

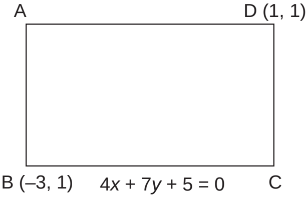
Here $ax_1 + by_1 + c \neq 0$; $ax_2 + by_2 + c \neq 0$;

since A, B $\notin l$

If $\lambda > 0 \Rightarrow$ A, B are on opposite sides of l $\lambda < 0$; $\lambda \neq -1$; A, B are on the same side of l.

Illustration 30 Two vertices of a rectangle ABCD are B $\left(-3, 1\right)$ and D $\left(1, 1\right)$ and the equation of line containing a side of the rectangle is $4x + 7y + 5 = 0$. Find the equations of lines containing other three sides.

Solution



It is clear that B $\left(-3, 1\right)$ is on line $4x + 7y + 5 = 0$ and D $\left(1, 1\right)$ is not on that line

Draw rectangle ABCD as shown in the figure

$\overleftrightarrow{BC} = 4x + 7y + 5 = 0$

$\therefore\ \overleftrightarrow{AB} = 7x - 4y + k = 0\ (\overleftrightarrow{AB} \perp \overleftrightarrow{BC})$

$\overleftrightarrow{AB}$ passes through B $\left(-3, 1\right)$

$\therefore\ -21 - 4 + k = 0$

$\therefore\ k = 25$

$\therefore\ \overleftrightarrow{AB} = 7x - 4y + 25 = 0$

$\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}\ \therefore\ \overleftrightarrow{CD} = 7x - 4y + k^1 = 0$ and it passes through D $\left(1, 1\right)$

$\therefore\ 7\left(1\right) - 4\left(1\right) + K^1 = 0$

$\therefore\ K^1 = -3$

$\therefore\ \overleftrightarrow{CD} = 7x - 4y - 3 = 0$

$\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}\ \ \overleftrightarrow{AD} : 4x + 7y + K^1 = 0$ and it passes through D $\left(1, 1\right)$

$\therefore\ 4\left(1\right) + 7\left(1\right) + K^1 = 0$

$\therefore\ K^1 = -11$

$\therefore\ \overleftrightarrow{AD} = 4x + 7y - 11 = 0$

$\therefore$ Required equations are $4x + 7y - 11 = 0$; $7x - 4y - 3 = 0$; $7x - 4y + 25 = 0$

ANALYTICAL EXERCISES

1. If P $(a, b) \in [(x, y) / x = 5t + 7, y = 6 - 7t, t \in R]$ and if $a + b = 21$ find (a, b).
2. If A $(2, 5)$, and B $(4, 7)$, show that P $(0, 3) \in \overleftrightarrow{AB}$. But P $(0, 3) \notin \overline{AB}$.
3. For A $(3, -5)$, B $(-5, 3)$ obtain the parmatric equations of $\overleftrightarrow{AB}$. If P $(x, y) \in \overline{AB}$ find the range of $4x - y$.
4. Find a point (a, b) on the line whose paramatric equations are $x = 2t + 1$, $y = 1 - t, t \in R$ such that $a + b = 1$.
5. For A $(2, 5)$ and and B $(4, 7)$ prove that $(6, 9) \in \overleftrightarrow{AB}$ but $(6, 9) \notin \overline{AB}$.
6. Prove by using slopes that the point A $(2, 5)$, B $(3, -2)$, C $(-4, 1)$ and D $(-5, 8)$ are the vertices of a parallelogram.
7. If A $(1, 2)$, B $(2, -2)$, C $(8, 2)$ and D $(4, 1)$, show that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are perpendicular.
8. Measure of the angle between two lines is $\frac{\pi}{4}$ and one of the lines has slope $\frac{1}{3}$. Find the slope of the other line.
9. For what value of k would the line through $(k, 7)$ and $(2, -5)$ have slope $\frac{2}{3}$.
10. Determine the intersection of the following pair of lines
    (1) $5x - 2y + 3 = 0; 2x + y - 1 = 0$
    (2) $5x - 2y + 3 = 0; 4y - 10x - 6 = 0$
    (3) $x \cos \alpha + y \sin \alpha = P; x \sin \alpha - y \cos \alpha = 0$
11. If A is $(1, 3)$ and B $(2, 4)$; show that (1) P $(5, 6) \notin \overleftrightarrow{AB}$, (2) Q $(0, 2) \in \overleftrightarrow{AB}$ but Q $(0, 2) \notin \overline{AB}$.
12. Find the equation of the lines passing through $(4, 5)$ and (1) parallel to and (2) $\perp^{er}$ to $2x + y = 1$.
13. The sides of a rectangle are parallel and perpendicular to $3x + 4y - 7 = 0$. If a pair of opposite vertices of the rectangle are $(2, 3)$ and $(-1, 2)$, find the equations of the four sides of the rectangle.
14. Prove that for all real values of λ, the line $\lambda x + 5y - 2 = 0$ passes through a fixed point and find the point.
15. Obtain the equation of a line the sum of whose intercepts is 15 and is parallel to the line $2x + 3y + 11 = 0$.
16. Find k, m so that the lines $kx - 2y - 1 = 0$ and $6x - 4y - m = 0$ (1) intersect in a unique point, (2) are parallel, (3) are identical.
17. Find the equation of the perpendicular bisector of $\overline{AB}$ where A is $(-3, 2)$ and B $(7, 6)$.
18. If the mid-points of the sides of a triangle are $(2, 1)$, $(-5, 7)$ and $(-5, -5)$, find the equations of the lines containing the sides of the triangle.
19. Find the equations of the lines with slope-2 and intersecting x-axis at points distance 3 units from the origin.

20. Find the equations of the lines making an angle of measure $\frac{\pi}{6}$ with the positive direction of the x-axis and intersecting y-axis at point distant 2 units from the origin.

21. A line passing through $(2, 4)$ intersects the x and y-axis at A and B respectively. Find the equation of the locus of the mid-point of $\overline{\text{AB}}$.

22. A line passing through $(3, 1)$ makes a triangle of area 8 in the first quadrant with the two axes. Find the equation of this line.

23. Find the equations of the lines inclined at an angle of measure $\frac{\pi}{4}$ with the line $x + 2y = 3$ and passing through $(3, 2)$.

24. The equations of the line containing one of the sides of an equilateral triangle is $x + y = 2$ and one of the vertices of the triangle is $(2, 3)$. Find the equations of lines containing the remaining sides of the Triangle.

25. Find the equations of the lines containing the altitudes of the triangle whose vertices are A $(7, -1)$, B $(-2, 8)$ and C $(1, 2)$.

26. Two of the vertices of a rectangle ABCD are B $(-3, 1)$ and D $(1, 1)$ and the equation of the line containing one of the sides of the rectangle is the remaining sides.

27. Write the equations of lines through $\left(3, \frac{-1}{2}\right)$ that are parallel to x and y-axis.

28. Obtain the equation of the line given that the foot $\perp^{er}$ from the origin to the line is $(2, 3)$.

29. Obtain the equation of the line passing through $(-1, 7)$ and $\perp^{er}$ to the line $y + 2 = 0$.

30. A line intersects x-axis at A and y-intercept is -3 and that is inclined at an angle whose measure is to x-axis.

31. Find the equation of the line which passes through $(3, 4)$ and which makes an angle of measure $\frac{\pi}{4}$ with the line $3x + 4y - 2 = 0$.

32. Prove that for all the lines $\lambda x + 3y = 6$ passes through a fixed point. Obtain co-ordinate.

33. Prove that if A is $(2, 3)$ and B is $(5, -3)$ then the line $2x + 3y - 9 = 0$ divides from A's side in the ratio 1 : 2.

34. Prove that the points $(3, 4)$ and $(-2, 1)$ are on the opposite sides of the line $3x - y + 6 = 0$.

35. Show that the centroid of ΔABC lies on the line $21x + 27y - 74 = 0$, if C lies on $7x + 9y - 10 = 0$ and A and B have co-ordinates $(6, 3)$ and $(-2, 1)$ respectively.

36. A $(2, 3)$, and B $(5, 9)$ are given points and p $(x, y) \notin \overline{\text{AB}}$. Find the range of $x + y$.

37. If A (a, b), B (c, d), C $(a - c, b - d)$ are not collinear then show that they form a parallelogram with origin.

38. A $(-2, 1)$, B $(6, 5)$ are given points. Show that the $\perp^{er}$ from P $(4, 1)$ on $\overline{\text{AB}}$ divide $\overline{\text{AB}}$ in the ratio 5 : 8 from A.

ANSWERS

(1) $(a, b) = (-12, 34)$,

(3) $[(x, y) \mid x = 3 - 8t, y = 8t - 5, t \in R]$, $(-23, 17)$

(4) $(-1, 2)$

(8) $\frac{1}{2}$ or 2

(10) **(1)** $\left(\frac{-1}{9}, \frac{11}{9}\right)$

(2) Lines are co-incident

(3) (p $\cos\alpha$, P $\sin\alpha$)

(12) $2x + y = 13$, $x - 2y + 6 = 0$,

(13) $3x + 4y - 5 = 0$, $4x - 3y + 10 = 0$, $3x + 4y - 18 = 0$, $4x - 3y + 1 = 0$

(15) $2x + 3y - 18 = 0$,

(16) $\left\{\frac{-11}{5}, \frac{7}{5}\right\}$, $(2, 3)$, $\varnothing$

(17) $5x - 2y - 18 = 0$

(18) $x = 2$, $6x - 7y + 79 = 0$, $6x + 7y + 65 = 0$

(19) $2x + y \pm 6 = 0$

(20) $x - \sqrt{3}y + 2\sqrt{3} = 0$ or $x - \sqrt{3}y - 2\sqrt{3} = 0$

(21) $\frac{1}{x} + \frac{2}{y} = 1$

(22) $x + 9y - 12 = 0$ or $x + y - 4 = 0$

(23) $x + 3y - 9 = 0$ or $3x - y - 7 = 0$

(24) $(2 + \sqrt{3})x - y + (2\sqrt{3} - 1) = 0$

(25) $x - 2y - 9 = 0$, $2x - y + 12 = 0$, $x - y + 1 = 0$

(26) $7x - 4y + 25 = 0$, $7x - 4y + 25 = 0$, $4x - 7y - 11 = 0$

(27) $x = 3$, $y = -\frac{1}{2}$

(28) $2x + 3y - 13 = 0$

(29) $x = -1$

(30) $8x - 9y = 42$

(31) $x - 7y + 25 = 0$, $7x - y - 25 = 0$

(36) $(5, 14)$

16 Straight Lines

LEARNING OBJECTIVES

After studying this chapter, the student will be able to understand:

- Concept of concurrent lines and its theorems
- Perpendicular distance from the point to the line
- Perpendicular distance between two parallel lines
- Equation of bisector of angle between two lines

INTRODUCTION

CONCURRENT LINES

If more than two lines intersect in exactly one common point then they are said to be concurrent lines.

Theorem 1

The necessary and sufficient conditions for three distinct lines:

$$\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \quad ai^2 + bi^2 \neq 0;\ i = 1, 2, 3 \\ a_3x + b_3y + c_3 = 0 \end{cases}$$

The conditions for three distinct lines to be concurrent are:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0\ ;\ \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \neq 0\ ;\ \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \neq 0 \text{ and}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Theorem 2

If $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ $\left(ai^2 + bi^2 \neq 0;\ i = 1, 2\right)$ are two distinct intersecting lines and *l* and *m* are any two real numbers not both zero then equation

$l(a_1x + b_1y + c_1) + m(a_2x + b_2y + c_2) = 0$ for any $l, m \in$ R; $l^2 + m^2 \neq 0$ represents a line through their point of intersection.

If

1. $l = 0 \Rightarrow a_2x + b_2y + c_2 = 0$ is the required equation
2. If $a_2x + b_2y + c_2 = 0$ is not the required line then $l \neq 0$ and in that case set $\frac{m}{l} = \lambda$ then the required line is $(a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0$.

Perpendicular Distance of a Line from a Point not on the Line

The $\perp^{\text{er}}$ distance of $ax + by + c = 0$; $a^2 + b^2 \neq 0$

From 0 $(0, 0)$ is $= \dfrac{|c|}{\sqrt{a^2 + b^2}}$

The $\perp^{\text{er}}$ distance of $ax + by + c = 0$; $a^2 + b^2 \neq 0$

From A (x_1, y_1) is $= \dfrac{|ax_1 + by_1 + c_1|}{\sqrt{a^2 + b^2}}$

Perpendicular Distance Between Parallel Lines

The $\perp^{\text{er}}$ distance between $ax + by + c = 0$ and $ax + by + c^1 = 0$; $c^1 \neq c$; $a^2 + b^2 \neq 0$ is

$$= \frac{|c - c^1|}{\sqrt{a^2 + b^2}}$$

Equation of Bisection of Angle Between Two Lines

Equation of bisection of angle between two lines $a_1x + b_1y + c_1 = 0$; $a_1^2 + b_1^2 \neq 0$ and $a_2x + b_2y + c_2 = 0$; $a_2^2 + b_2^2 \neq 0$ is

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

ILLUSTRATIONS

Illustration 1 Find k if the lines $2x + 3y - 1 = 0$; $x - y - 7 = 0$; and $4x + ky - 2 = 0$ are to be concurrent.

Solution

As the lines are concurrent, the conditions for concurrency are necessary. Thus

$a_1b_2 - a_2b_1 = 2(-1) - 1(-3) = -5 \neq 0$

$a_2b_3 - a_3b_2 = k - 4(-1) = k + 4 \neq 0$ gives $k \neq -4$

$a_1b_3 - a_3b_1 = 2(k) - 4(3) = 2k - 12 \neq 0$ gives $k \neq 6$

and $\begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -7 \\ 4 & k & -2 \end{vmatrix} = 2(2 + 7k) - 3(-2 + 28) - 1(k + 4) = 0$

$\therefore\ 13k - 78 = 0$

$\therefore\ k = 6$ But $k \neq 6$

Hence given lines are not concurrent for any value of k.

Illustration 2 Prove that if a, b, c are non-zero distinct constants then the lines $ax + a^2y = 1$; $bx + b^2y = 1$: $cx + c^2y = 1$ cannot be concurrent.

Solution

As the lines are concurrent then

$a_1b_2 - a_2b_1 = ab^2 - a^2b = ab(b - a)$; $a \neq 0, b \neq 0, a \neq b$

$a_2b_3 - a_3b_2 = bc^2 - b^2c = bc(c - b)$; $a \neq 0, c \neq 0, b \neq c$

$a_1b_3 - a_3b_1 = ac^2 - a^2c = ac(c - a)$; $c \neq 0, a \neq 0, c \neq a$

Finally we can say that

$$\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = \begin{vmatrix} a-b & a^2-b^2 & 0 \\ b-c & b^2-c^2 & 0 \\ c & c^2 & 1 \end{vmatrix} \begin{matrix} ;\ R_{21}(-1) \\ \\ ;\ R_{32}(-1) \end{matrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 1 & a+b & 0 \\ 1 & b+c & 0 \\ c & c^2 & 1 \end{vmatrix} \begin{matrix} ;\ R_1\left(\dfrac{1}{a-b}\right) \\ \\ ;\ R_2\left(\dfrac{1}{b-c}\right) \end{matrix}$$

$$= \begin{vmatrix} 0 & a-c & 0 \\ 1 & b+c & 0 \\ c & c^2 & 1 \end{vmatrix} \ ;\ R_{21}(-1)$$

$$= (a-b)(b-c)(a-c)\begin{vmatrix} 0 & 1 & 0 \\ 1 & b+c & 0 \\ c & c^2 & 1 \end{vmatrix}$$

$$= (a-b)\ (b-c)\ (c-a) \neq 0 \qquad (\because a \neq b \neq c)$$

$\therefore$ Lines are not concurrent

Illustration 3 If $l + m + n = 0$, prove that the lines $lx + my + n = 0$; $mx + ny + l = 0$; $nx + ly + m = 0$ are concurrent. Assume that $m^2 \neq ln$; $n^2 \neq ml$, and $l^2 \neq mn$.

Solution

$l_1 = lx + my + n = 0$

$l_2 = mx + ny + l = 0$

$l_3 = nx + ly + m = 0$

$\therefore\ a_1b_2 - a_2b_1 = ln - m^2 \neq 0 \qquad (\because ln \neq m^2)$

$a_2b_3 - a_3b_2 = ml - n^2 \neq 0 \qquad (\because lm \neq n^2)$

$a_1b_3 - a_3b_1 = l^2 - mn \neq 0 \qquad (\because l^2 \neq mn)$

$$\text{and } D = \begin{vmatrix} l & m & n \\ m & n & l \\ n & l & m \end{vmatrix}$$

$$= \begin{vmatrix} l+m+n & l+m+n & l+m+n \\ m & n & l \\ n & l & m \end{vmatrix} \begin{matrix} R_{21}(1) \\ \\ R_{32}(1) \end{matrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ m & n & l \\ n & l & m \end{vmatrix} = 0 \qquad (\because l + m + n = 0)$$

$\therefore$ l_1, l_2, l_3 are concurrent lines.

$\therefore$ Point of concurrency is obtained from intersection of l_1 and l_2.

$$(x, y) = \left[\frac{ml - n^2}{nl - m^2}; -\frac{(l^2 - mn)}{nl - m^2}\right]$$

$$(x, y) = \left[\frac{m(-m-n) - n^2}{(-m-n)n - m^2}; \frac{m(-l-m) - l^2}{(-m-l)l - m^2}\right]$$

$$\therefore (x, y) = \left(\frac{-m^2 - mn - n^2}{-m^2 - mn - n^2}; \frac{-m^2 - n^2 - ml}{-m^2 - n^2 - ml}\right) \qquad \left(\begin{matrix} \because l + m + n = 0 \\ m = -l - n \\ l = -m - n \end{matrix}\right)$$

$$(x, y) = (1, 1)$$

Illustration 4 Find the equation of line passing through the points of intersection of the lines $x + y + 4 = 0$ and $3x - y - 8 = 0$ and also through the point $(2, -3)$.

Solution

Let $l_1 : x + y = 4 = 0$

$l_2 : 3x - y - 8 = 0$

$(2, -3)$ does not satisfy $x + y + 4 = 0$ and $3x - y - 8 = 0$

$\therefore$ Now the equation of the line passing through the point of intersection of l_1 and l_2 is given by

$$(x + y + 4) + \lambda(3x - y - 8) = 0 \qquad (1)$$

This line passes through $(2, -3)$

$\therefore$ $(2 - 3 + 4) + \lambda(6 + 3 - 8) = 0$

$\therefore$ $\lambda + 3 = 0$

$\therefore$ $\lambda = -3$

Now $\lambda = -3$ Substitute in eq. (1)

$\therefore$ $(x + y + 4) - 3(3x - y - 8) = 0$

$\therefore\ -8x + 4y + 28 = 0$

$\therefore\ 2x - y - 7 = 0$ is the equation of line intersection of l_1 and l_2 and $(2, -3)$.

Illustration 5 Find the equation of the line that passes through the point of intersection of $x + 2y = 5$ and $3x + 7y = 17$ and is $\perp^{er}$ to the line $3x + 4y = 0$.

Solution

Slope of line $3x + 4y = 0$ is $-\frac{3}{4}$

Slope of line $3x + 7y - 17 = 0$ is $-\frac{3}{7}$

$$\therefore \left(\frac{-3}{4}\right)\left(\frac{-3}{7}\right) = \frac{9}{28} \neq -1$$

$\therefore$ Above lines are not mutually $\perp^{er}$

So $3x + 7y - 1 = 0$ is not the required line

Suppose equation of required line l is

$$l = (x + 2y - 5) + \lambda(3x + 7y - 17) = 0 \quad (1)$$

$$\therefore\ (1 + 3\lambda)x + (2 + 7\lambda)y - (5 = 17\lambda) = 0$$

$$\therefore \text{ Slope of line } l = -\frac{1 + 3\lambda}{2 + 7\lambda}$$

Slope of $3x + 4y = 0$ is $-\frac{3}{4}$

Now line l is $\perp^{er}$ to line $3x - 4y = 0$

$$\therefore \text{ Slope of line } l = \frac{4}{3}$$

$$\therefore\ -\frac{1 + 3\lambda}{2 + 7\lambda} = \frac{4}{3}$$

$$\therefore\ -3 - 9\lambda = 8 + 28\lambda$$

$$\lambda = -\frac{11}{37}$$

Now substitute $\lambda = -\frac{11}{37}$ in eq. (1)

$$(x + 2y - 5) - \frac{11}{37}(3x + 7y - 17) = 0$$

$$\therefore\ 4x - 3y + 2 = 0$$

Illustration 6 Find the equation of the line passing through the point of intersection of the lines $2x + y = 1$ and $x + 3y - 2 = 0$ and making a triangle of area $\frac{3}{8}$ with the two axes.

Solution

In line $x + 3y - 2 = 0$

Intercept of x-axis $= -\frac{c}{a} = 2$

Intercept of y-axis $= -\frac{c}{b} = -\frac{2}{3}$

$\therefore$ Area of triangle by $x + 3y - 2$ is

$$\frac{1}{2}(2)\left(-\frac{2}{3}\right)$$

$$= \frac{2}{3} \neq \frac{3}{8}$$

$\therefore$ We can say that $x + 3y - 2 = 0$ is not the required line.

Let the required line l be

$$\left(2x + y - 1\right) + \lambda\left(x + 3y - 2\right) = 0 \qquad (1)$$

$$\therefore \left(2 + \lambda\right)x + \left(1 + 3\lambda\right)y - \left(1 + 2\lambda\right) = 0$$

$$\Delta = \frac{1}{2}\left|\left(x - \text{intercept}\right)\left(y - \text{intercept}\right)\right|$$

$$\therefore \frac{3}{8} = \frac{1}{2}\left|\left[\frac{\left(1 - 2\lambda\right)}{\left(2 + \lambda\right)}\frac{\left(1 + 2\lambda\right)}{\left(1 + 3\lambda\right)}\right]\right|$$

$$\therefore \pm\frac{3}{8} = \frac{1}{2}\frac{\left(1 + 2\lambda\right)^2}{\left(2 + \lambda\right)\left(1 + 3\lambda\right)}$$

$$\therefore \pm\frac{3}{4} = \frac{\left(1 + 2\lambda\right)^2}{\left(2 + \lambda\right)\left(1 + 3\lambda\right)}$$

$\therefore -3\left(2 + \lambda\right)\left(1 + 3\lambda\right) = 4\left(1 + 2\lambda\right)^2$ or

$+3\left(2 + \lambda\right)\left(1 + 3\lambda\right) = 4\left(1 + 2\lambda\right)^2$

$\therefore -3\left(3\lambda^2 + 7\lambda + 2\right) = 4\left(1 + 4\lambda + 4\lambda^2\right)$

or

$3\left(3\lambda^2 + 7\lambda + 2\right) = 4\left(1 + 4\lambda + 4\lambda^2\right)$

$\therefore 25\lambda^2 + 37\lambda + 10 = 0$ or $7\lambda^2 - 5\lambda - 2 = 0$

$\therefore \lambda = \frac{-37 \pm \sqrt{369}}{50}$ or $\left(7\lambda + 2\right)\left(\lambda - 1\right) = 0$

$\therefore \lambda = \frac{-37 \pm \sqrt{369}}{50}$ or $\lambda = -\frac{2}{7}$ or $\lambda = 1$

Now substitute $\lambda = 1$ in eq^n (1)

$\therefore \left(2x + y - 1\right) + 1\left(x + 3y - 2\right) = 0$

$\therefore 3x + 4y - 3 = 0$

Also substitute $\lambda = -\frac{2}{7}$ in eq. (1)

$$\therefore \left(2x + y - 1\right) - \frac{2}{7}\left(x + 3y - 2\right) = 0$$

$\therefore\ 12x + y - 3 = 0$

and for any $\lambda = \dfrac{-37 \pm \sqrt{369}}{50}$, we get two other equations of the line

Illustration 7 Find the equation of the line passing through the point of intersection of the lines $x - 3y + 1 = 0$ and $2x + 5y - 9 = 0$ and that is at a distance of $\sqrt{5}$ from the origin.

Solution

Measure of $\perp^{er}$ line segment on $2x + 5x - 9 = 0$ from origin is

$$P = \frac{|-c|}{\sqrt{a^2 + b^2}} = \frac{|-9|}{\sqrt{4 + 25}} = \frac{9}{\sqrt{29}} \neq \sqrt{5}$$

$\therefore\ 2x + 5y = 9$ is not the required line

$\therefore\ l : (x - 3y + 1) + \lambda(2x + 5y - 9) = 0$ (1)

$\therefore\ (1 + 2\lambda)x + (-3 + 5\lambda)y \pm (1 - 9\lambda) = 0$

Now distance of line l from the origin $P = \dfrac{|-c|}{\sqrt{a^2 + b^2}}$

$$\therefore\ \sqrt{5} = \frac{|1 - 9\lambda|}{\sqrt{(1 + 2\lambda)^2 + (5\lambda - 3)^2}}$$

$\therefore\ 5\left[(1 + 2\lambda)^2 + (5\lambda - 3)^2\right] = (1 - 9\lambda)^2$ $(\because$ Squaring both sides$)$

$\therefore\ 64\lambda^2 - 112\lambda + 49 = 0$

$\therefore\ (8\lambda - 7)^2 = 0$

$\therefore\ \lambda = \dfrac{7}{8}$

Now Substitute $\lambda = \dfrac{7}{8}$ in eq. (1)

$\therefore\ (x - 3y + 1) + \dfrac{7}{8}(2x + 5y - 9) = 0$

$\therefore\ 2x + y - 5 = 0$

which is the required equation of line

Illustration 8 Find the equation of the line with equal intercepts on the two axes and that passes through the point of intersection of the lines $3x - y + 1 = 0$ and $x + y - 5 = 0$

Solution

Line $l_1 : x + y - 5 = 0$

$\therefore\ l_1$ made equal intercepts $(\because a = b)$

Now for line $l_2 = 3x - y + 1 = 0$

$\therefore\ l_2$ does not make equal intercepts $(\because a \neq b)$

$\therefore\ l_2$ is not the required equation of line

$\therefore\ l_2$ passes through the point of intersection of l_1 and l_2

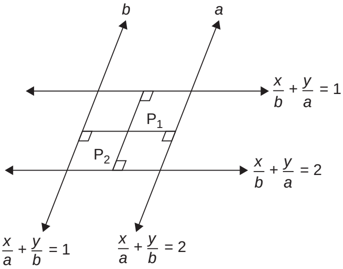
$\therefore\ l_1 : x + y - 5 = 0$ is the required equation of line

Illustration 9 Prove that the lines $\frac{x}{a}+\frac{y}{b}=1$; $\frac{x}{b}+\frac{y}{a}=1$, $\frac{x}{a}+\frac{y}{b}=2$; $\frac{x}{b}+\frac{y}{a}=2$ form a rhombus.

Solution

Here $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{a}+\frac{y}{b}=2$ are parallel and their $\perp^{\text{er}}$ distance P_1 between them is

$$P_1=\frac{|1-2|}{\sqrt{\left(1/a^2\right)+\left(1/b^2\right)}}=\frac{|ab|}{\sqrt{b^2+a^2}}$$



The other pair of lines

$\frac{x}{b}+\frac{y}{a}=1$ and $\frac{x}{b}+\frac{y}{a}=2$

are also parallel and distance P_2 between

them is $P_2=\frac{|1-2|}{\sqrt{\left(1/b^2\right)+\left(1/a^2\right)}}=\frac{|ab|}{\sqrt{a^2+b^2}}$

Thus $P_1 = P_2$

∴ The four lines form a quadrilateral. With both pairs of opposite sides parallel and such that the $\perp^{\text{er}}$ distance between them of opposite sides is equal

∴ Quadrilateral is a rhombus

Illustration 10 Obtain the equations of lines that divide the angle between the lines $3x - 4y - 2 = 0$; and $5x - 12y + 2 = 0$ and prove that the bisecting lines are $\perp^{\text{er}}$ to each other.

Solution

The lines bisecting the angle between the given lines have the equation

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

$$\therefore\ \frac{3x-4y-2}{\sqrt{3^2+(-4)^2}}=\pm\frac{5x-12y+2}{\sqrt{5^2+(-12)^2}}$$

$$\therefore\ \frac{3x-4y-2}{5}=\pm\frac{5x-12y+2}{13}$$

$\therefore\ 39x - 52y = 26 = \pm\left(25x - 60y + 10\right)$

$\therefore\ 39x - 52y - 26 = 25x - 60y + 10$

or

$39x - 52y - 26 = -25x + 60y - 10$

$\therefore\ 14x + 8y - 36 = 0$ or $64x - 112y - 16 = 0$

$\therefore$ The equations of the two bisecting lines are

$7x + 4y - 18 = 0$ and $4x - 7y - 1 = 0$

Now slope of $7x + 4y - 18 = 0$ is $m_1 = -\dfrac{7}{4}$ and

Slope of $4x - 7y - 1 = 0$ is $m_2 = \dfrac{4}{7}$

$$\therefore m_1 \cdot m_2 = \left(-\frac{7}{4}\right)\left(\frac{4}{7}\right) = -1$$

So we can say that lines are $\perp^{er}$ to each other

Illustration 11 Obtain the $\perp^{er}$ distance from the point $(4, 5)$ to the line $3x - 5y + 7 = 0$.

Solution

$\perp^{er}$ distance of point $(4, 5)$ from the line $3x - 5y + 7 = 0$ is given by

$$P = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$= \frac{|3(4) - 5(5) + 7|}{\sqrt{(3)^2 + (5)^2}}$$

$$= \frac{|-6|}{\sqrt{34}}$$

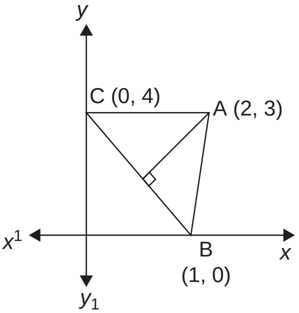
$$\therefore \text{P} = \frac{6}{\sqrt{34}}$$

Illustration 12 If A is $(2, 3)$, B $(1, 0)$ and C $(0, 4)$, find the length to the altitude through A of ΔABC.

Solution

Here $\overline{\text{AD}}$ is the altitude of ΔABC on base $\overline{\text{BC}}$

Here AD = P = $\perp^{er}$ distance from A on $\overleftrightarrow{\text{BC}}$



$$\begin{vmatrix} x & y & 1 \\ 1 & 0 & 1 \\ 0 & 4 & 1 \end{vmatrix} = 0$$

$$\therefore x(-4) - y(1) + 1(4) = 0$$

$$\therefore 4x + y - 4 = 0$$

$$\therefore P = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$= \frac{|4(2) + 1(3) + (-4)|}{\sqrt{4^2 + 1^2}}$$

$$= \frac{|8 + 3 - 4|}{\sqrt{17}} = \frac{7}{\sqrt{17}}$$

Illustration 13 The $\perp^{er}$ distance of which point on the x-axis is 4 from the line $3x - 4y - 5 = 0$?

Solution

Let $P\left(x, 0\right)$ be at a $\perp^{er}$ distance from $3x - 4y - 5 = 0$

$$\therefore\ 4 = \left|\frac{3(x) - 4(0) + (-5)}{\sqrt{3^2 + 4^2}}\right|$$

$$\therefore\ 4 = \frac{|3x - 5|}{5}$$

$\therefore\ |3x - 5| = 20$

$\therefore\ 3x - 5 = \pm 20$

$\therefore\ 3x - 5 = 20$ or $3x - 5 = -20$

$\therefore\ 3x = 25$ or $3x = -15$

$\therefore\ x = \frac{25}{3}$ or $x = -5$

$\therefore$ The required point of $P\left(x, 0\right)$ is $\left(\frac{25}{3}, 0\right)$ or $\left(-5, 0\right)$

Illustration 14 Find the $\perp^{er}$ distance between the lines $3x - 4y = 8$ and $6x - 8y + 5 = 0$.

Solution

Let $l_1 = 6x - 8y - 16 = 0$

and $l_2 = 6x - 8y + 5 = 0$

Now $\perp^{er}$ distance between l_1 and l_2 is given by

$$P = \frac{|c - c^1|}{\sqrt{a^2 + b^2}}$$

$$= \frac{\left|(-16) - (5)\right|}{\sqrt{36 + 64}} = \frac{|-21|}{\sqrt{100}} = \frac{21}{10} \text{ units}$$

Illustration 15 Obtain the equations of lines bisecting the angle between the lines $3x + 4y + 2 = 0$ and $5x - 12y + 1 = 0$ and show that the bisecting lines are $\perp^{er}$ to each other.

Solution

Let $l_1 = 3x + 4y + 2 = 0$ and

$l_2 = 5x - 12y + 1 = 0$

$$\therefore\ \frac{a_1x + b_1y + c_1}{\sqrt{a_2^2 + b_1^2}} = \pm\frac{\left(a_2x + b_2y + c_2\right)}{\sqrt{a_2{}^2 + b_2{}^2}}$$

$$\therefore\ \frac{3x + 4y + 2}{\sqrt{3^2 + 4^2}} = \pm\frac{\left(5x - 12y + 1\right)}{\sqrt{5^2 + (-12)^2}}$$

$$\therefore \frac{3x + 4y + 2}{5} = \pm \frac{(5x - 12y + 1)}{13}$$

$$\therefore 13(3x + 4y = 2) = \pm 5(5x - 12y + 1)$$

$$\therefore 39x + 52y + 26 = \pm (25x - 60y + 5)$$

$$\therefore 39x + 52y + 26 = 25x - 60y + 5 \quad \text{or} \quad 39x + 52y + 26 = -25x + 60y - 5$$

$$\therefore 2x + 16y + 3 = 0 \quad \text{or} \quad 64x - 8y + 31 = 0$$

Now slope of $2x + 16y + 3 = 0$ is $m_1 = -\frac{a}{b} = -\frac{1}{8}$ and

Slope of $64x - 8y + 31 = 0$ is $m_2 = -\frac{a}{b} = 8$

Illustration 16 Find the co-ordinates of the foot of the $\perp^{er}$ from A $(a, 0)$ to the line. $y = mx + \frac{a}{m}$; $m \neq 0$.

Solution

Let the equation of line be $y = mx + \frac{a}{m}$

$\therefore m^2x - my = a = 0$

A line $\perp^{er}$ to this line would be $mx + m^2y + k_1 = 0 \; \forall \; k \in R$, and it passes through A $(a, 0)$

$\therefore ma + k = 0$

$\therefore k = -ma$

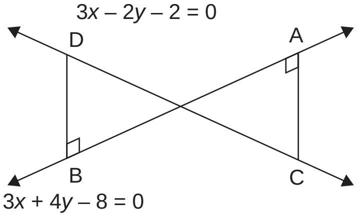
$\therefore$ The equation of the $\perp^{er}$ is $mx + m^2y - ma = 0$ or $x + my - a = 0$.

The foot of the $\perp^{er}$ is the point of intersection of $m^2x - my + a = 0$ and $x + my - a = 0$ is $x = 0$; $y = \frac{a}{m}$ (after solving)

Hence foot of the $\perp^{er}$ is $\left(0, \frac{a}{m}\right)$

Illustration 17 Find point on the line $3x - 2y - 2 = 0$ whose $\perp^{er}$ distance from the line $3x + 4y - 8 = 0$ is 3.

Solution



Let the required points be (x_1, y_1)

As it is on $3x - 2y - 2 = 0$

We have $3x_1 - 2y_1 - 2 = 0$

$$\therefore y_1 = \frac{3x_1 - 2}{2}$$

$$\therefore (x_1, y_1) = \left(x_1, \frac{3x_1 - 2}{2}\right)$$

Now the $\perp^{er}$ distance of $\left(x_1, \frac{3x_1 - 2}{2}\right)$ from the line $3x + 4y - 8 = 0$ is 3

$$\therefore \frac{\left|3x_1 + 4\left[\left(3x_1 - 2\right)/2\right] - 8\right|}{\sqrt{9+16}} = 3$$

$\therefore |9x_1 - 12| = 15$

$\therefore |3x_1 - 4| = 5$

$\therefore 3x_1 - 4 = \pm 5$

$\therefore 3x_1 - 4 = 5$ or $3x_1 - 4 = -5$

$\therefore x_1 = 3$ or $x_1 = -\dfrac{1}{3}$

$$\therefore y_1 = \frac{3\left(3\right)-2}{2} = \frac{7}{2} \quad \text{or} \quad y_1 = \frac{3\left(-1/3\right)-2}{2} = -\frac{3}{2}$$

$\therefore$ The required points are $\left(3, \dfrac{7}{2}\right)$ and $\left(-\dfrac{1}{3}, -\dfrac{3}{2}\right)$

Illustration 18 Show that quadrilateral formed by the lines $ax \pm by - c = 0$; $ax \pm by - c = 0$ is a rhombus and that its area is. $\dfrac{2c^2}{|ab|}$.

Solution

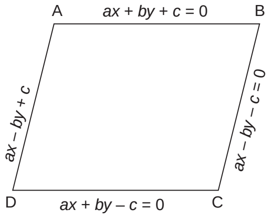
Let the equation be

$\overleftrightarrow{AB} = ax + by + c = 0$ (1)

$\overleftrightarrow{BC} = ax - by - c = 0$ (2)

$\overleftrightarrow{CD} = ax + by - c = 0$ (3)

$\overleftrightarrow{DA} = ax - by + c = 0$ (4)



Here slope of $\overleftrightarrow{AB}$ = Slope of $\overleftrightarrow{CD} = -\dfrac{a}{b}$

Slope of $\overleftrightarrow{BC}$ = Slope of $\overleftrightarrow{CA} = \dfrac{a}{b}$

$\therefore \overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{BC} \parallel \overleftrightarrow{DA}$

$\therefore$ Quadrilateral ABCD is a ▱^{m}ABCD

$\therefore \perp^{er}$ distance between $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ is given by

$$P_1 = \frac{|c - c^1|}{\sqrt{a^2+b^2}} = \frac{\left|c - \left(-c\right)\right|}{\sqrt{a^2+b^2}} = \frac{|2c|}{\sqrt{a^2+b^2}}$$

$\perp^{er}$ distance between $\overleftrightarrow{BC}$ and $\overleftrightarrow{DA}$ is given by

$$P_2 = \frac{|c - c^1|}{\sqrt{a^2+b^2}} = \frac{|c - (-c)|}{\sqrt{a^2+b^2}} = \frac{|2c|}{\sqrt{a^2+b^2}}$$

$\therefore P_1 = P_2$

$\therefore\ \square^m$ ABCD is a rhombus

Now C (x, y) can be obtained by solving eqs. (2) and (3)

$$\therefore \text{C } (x, y) = \left(\frac{bc + bc}{ab + ab}; 0\right) = \left(\frac{c}{a}, 0\right)$$

Now point D can be obtained by solving eqs. (1) and (4)

$$\therefore \text{D } (x^1, y^1) = \left(\frac{bc - bc}{-ab - ab}; \frac{-(ac + ac)}{-ab - ab}\right) = \left(0, \frac{c}{b}\right)$$

$$\therefore \text{CD} = \sqrt{\frac{c^2}{a^2} + \frac{c^2}{b^2}} = \frac{|c|\sqrt{a^2 + b^2}}{|ab|}$$

Now area of rhombus ABCD

= Base × Altitude on base

$$= \overline{\text{CD}}\text{P}_1 = \frac{|c|\sqrt{a^2 + b^2}}{|ab|}\ \frac{|2c|}{\sqrt{a^2 + b^2}}$$

$$= \frac{|2c^2|}{|ab|}$$

Illustration 19 If the lines $a_1x + b_1y = 1$; $a_2x + b_2y = 1$ and $a_3x + b_3y = 1$ are concurrent, show that (a_1, b_1), (a_2, b_2) and (a_3, b_3) are collinear.

Solution

Let $l_1 = a_1x + b_1y - 1 = 0$

$l_2 = a_2x + b_2y = 1 = 0$

$l_3 = a_3x + b_3y - 1 = 0$

Here l_1, l_2 and l_3 are concurrent

$$\therefore \begin{vmatrix} a_1 & b_1 & -1 \\ a_2 & b_2 & -1 \\ a_3 & b_3 & -1 \end{vmatrix} = 0 \Rightarrow -\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} = 0\ ,\ c_3\ (-1)$$

$$\Rightarrow \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} = 0$$

$\therefore\ (a_1, b_1)$, (a_2, b_2) and (a_3, b_3) are collinear

Illustration 20 Find the foot of the ⊥er from the origin to the line $\frac{x}{a} + \frac{y}{b} = 1$.

Solution

Here $l_1 = \frac{x}{a} + \frac{y}{b} = 1$

$\therefore$ $bx + ay - ab = 0$ (1)

$\therefore$ Equation of line containing the $\perp^{er}$ line from the origin is $ax - by = 0$ (2)

Now foot of the $\perp^{er}$ is a point of intersection of lines $bx + ay - ab = 0$ and $ax - by = 0$

By solving eqs. (1) and (2) we get

$$x = \frac{ab^2}{a^2 + b^2} \text{ and } y = \frac{a^2b}{a^2 + b^2}$$

$\therefore$ Required points of co-ordinates are

$$(x, y) = \left(\frac{ab^2}{a^2 + b^2}; \frac{a^2b}{a^2 + b^2}\right)$$

Illustration 21 Without finding the point of intersection of the lines $2x + 3y + 4 = 0$ and $3x + 4y - 5 = 0$ get the equation of the line passing this point of intersection and $\perp^{er}$ to the line $6x - 7y + 8 = 0$.

Solution

Here let $l_1 : 2x + 3y + 4 = 0$

$\therefore$ Slope of $l_1 = -\frac{2}{3}$ and $l_1 = 3x + 4y - 5 = 0$

$\therefore$ Slope of $l_2 = -\frac{3}{4}$ and slope of $6x - 7y + 8 = 0$ is $\frac{6}{7}$

$\therefore$ Slope of line $\perp^{er}$ to $6x - 7y + 8 = 0$ is $= -\frac{7}{6}$

$\therefore$ Neither l_1 nor l_2 is $\perp^{er}$ to $6x - 7y + 8 = 0$

Suppose the equation of the required line l is

$(2x + 3y + 4) + \lambda(3x + 4y - 5) = 0$

$\therefore$ $(2 + 3\lambda)x + (3 + 4\lambda)y + (4 - 5\lambda) = 0$ (1)

$\therefore$ Slope of line $l = -\frac{2 + 3\lambda}{3 + 4\lambda}$

$\therefore$ $-\frac{7}{6} = -\frac{2 + 3\lambda}{3 + 4\lambda}$

$\therefore$ $21 + 28\lambda = 12 + 18\lambda$

$\therefore$ $10\lambda = -9$

$\therefore$ $\lambda = -\frac{9}{10}$

Now substitute the value of $\lambda = -\frac{9}{10}$ in eq. (1)

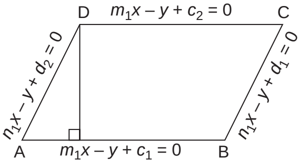
$\therefore$ $7x + 6y - 85 = 0$

Illustration 22 Equations of lines containing the sides of a parallelogram are $y = m_1x + c_1$; $y = m_1x + c_2$; $y = n_1x + d_1$ and $y = n_2x + d_2$ $(c_1 \neq c_2, d_1 \neq d_2)$. Find the area of the parallelogram.

Solution

$l_1 = m_1x - y + c_1 = 0$ $\therefore$ Slope $= m_1$

$l_2 = m_1x - y + c_2 = 0$ $\therefore$ Slope $= m_1$



$l_3 = n_1x - y + d_1 = 0$ $\qquad \therefore$ Slope $= n_1$
$l_4 = n_1x - y + d_2 = 0$ $\qquad \therefore$ Slope $= n_1$
Lines $m_1x - y + c_1 = 0$, $m_1x - y + c_2 = 0$;
$n_1x = y + d_1 = 0$ and
$n_1x - y + d_2 = 0$ are the pairs of opposite sides of ▱^{m} ABCD

$\therefore$ Equations of $\overleftrightarrow{\text{AB}}$: $m_1x - y + c_2 = 0$; $\overleftrightarrow{\text{BC}} = n_1x - y + d_1 = 0$
$\overleftrightarrow{\text{CD}} = m_1x - y + c_2 = 0$ and $\overleftrightarrow{\text{DA}} = m_1x - y + d_2 = 0$

Now distance between $\overleftrightarrow{\text{AB}} \parallel \overleftrightarrow{\text{CD}}$ is $\text{P} = \dfrac{|c_1 - c_2|}{\sqrt{m_1^2 + 1}}$

A (x_1, y_1) is a point of intersection of
$\therefore$ Lines $m_1x - y + c_1 = 0$ and $n_1x - y + d_2 = 0$ is

$$(x_1, y_1) = \left(\frac{-d_2 + c_1}{-m_1 + n_1}; \frac{-(m_1d_2 - n_1c_1)}{-m_1 + n_1}\right)$$

$$\therefore \text{A } (x_1, y_1) = \left(\frac{d_2 - c_1}{m_1 - n_1}; \frac{m_1d_2 - n_1c_1}{m_1 - n_1}\right)$$

Similarly we get B $(x_2, y_2) = \left(\dfrac{d_1 - c_1}{m_1 - n_1}; \dfrac{m_1d_1 - n_1c_1}{m_1 - n_1}\right)$

$$\therefore \text{AB} = \sqrt{\left(\frac{a_2 - c_1}{m_1 + n_1} - \frac{d_1 - c_1}{m_1 - n_2}\right)^2 + \left(\frac{m_1d_2 - n_1c_1}{m_1 - n_1} - \frac{m_1d_1 - n_1c_1}{m_1 - n_1}\right)^2}$$

$$= \sqrt{\left(\frac{a_2 - d_1}{m_1 - n_1}\right)^2 + \left(\frac{m_1d_2 - m_1d_1}{m_1 - n_2}\right)^2}$$

$$= \left|\frac{d_2 - d_1}{m_1 - n_1}\right| \sqrt{1 + m_1^2}$$

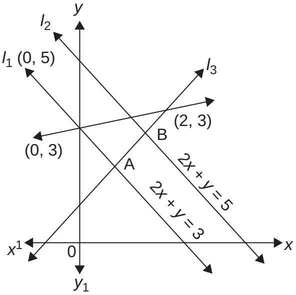
$\therefore$ Area of ▱^{m} ABCD $= \overline{\text{AB}}\,\text{P}$

$$= \left|\frac{d_2 - d_1}{m_1 - n_1}\right| \sqrt{1 + m^2}\, \frac{|c_1 - c_2|}{\sqrt{1 + m_1^2}} = \left|\frac{(d_2 - d)_1\,(c_1 - c_2)}{m_1 - n_1}\right|$$

Illustration 23 Find the equation of the line passing through $(2, 3)$ and containing a line-segment of length $\dfrac{2\sqrt{2}}{3}$ between the lines $2x + y = 3$ and $2x + y = 5$.

Solution

Let $l_1 = 2x + y - 3 = 0$ (1)
$l_2 = 2x + y - 5 = 0$ (2)
Here $l_1 \parallel l_2$



Now equation of a line l_3 passing through $(2, 3)$ and having slope m

$\therefore\ (y - y_1) = m(x - x_1)$

$\therefore\ (y - 3) = m(x - 2)$

$\therefore$ Let $l_1 \cap l_3 = A\ (x_1, y_1)$

$\therefore$ Solving eqs. (1), (3) we get

$$A\ (x_1, y_1) = \left(\frac{\begin{vmatrix} 1 & -3 \\ -1 & 3-2m \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ m & -1 \end{vmatrix}} ; \frac{\begin{vmatrix} 2 & -3 \\ m & 3-2m \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ m & -1 \end{vmatrix}} \right)$$

$$\therefore\ A\ (x_1, y_1) = A\left(\frac{2m}{m+2}; \frac{6-m}{m+2}\right)$$

Let $l_2 \cap l_3 = B\ (x_2, y_2)$

From (1) and (2)

$$B\ (x_2, y_2) = \left(\frac{\begin{vmatrix} 1 & -5 \\ -1 & 3-2m \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ m & -1 \end{vmatrix}} ; \frac{-\begin{vmatrix} 2 & -5 \\ m & 3-2m \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ m & -1 \end{vmatrix}} \right) \quad \therefore\ B = B\left(\frac{2m+2}{m+2,}\ \frac{m+6}{m+2}\right)$$

$$\therefore\ AB = \sqrt{\left(\frac{2m+2}{m+2} - \frac{2m}{m+2}\right)^2 + \left(\frac{m+6}{m+2} - \frac{6-m}{m+2}\right)^2}$$

$$= \sqrt{\left(\frac{2}{m+2}\right)^2 + \left(\frac{2m}{m+2}\right)^2} = \sqrt{\frac{4+4m^2}{(m+2)^2}}$$

Now perpendicular distance between l_1 and l_2 = $AB = \dfrac{2\sqrt{2}}{3}$

$$\therefore\ \sqrt{\frac{4+4m^2}{(m+2)^2}} = \frac{2\sqrt{2}}{3}$$

$$\therefore\ \frac{4+4m^2}{m^2+4m+4} = \frac{8}{9}$$

$\therefore\ 36 + 36m^2 = 8m^2 + 32m + 32$

$\therefore\ 7m^2 - 8m + 1 = 0$

$\therefore\ (7m - 1)\ (m - 1) = 0$

$\therefore\ m = \dfrac{1}{7}$ or $m = 1$

→Now a line passing through $(2, 3)$ having slope $\dfrac{1}{7}$ is

$(y - 3) = \dfrac{1}{7}(x - 2)$

$\therefore\ x - 7y + 19 = 0$

$\rightarrow$ and a line passing through $(2, 3)$ having slope $m = 1$

$\therefore\ (y - 3) = 1(x - 2)$

$\therefore\ x - y + 1 = 0$

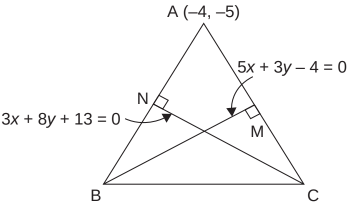
Hence required equations of lines are $x - 7y + 19 = 0$ and $x - y + 1 = 0$

$\rightarrow$ Taking line l_3 as a vertical line passing through $(2, 3)$ $l_3 = x = 2$

Intercept m $l_3 = x = 2$ by lines $l_2 = 2x + y = 3 = 0$ and $l_2 = 2x = y = 5 = 0$ is

2 units $\neq \dfrac{2\sqrt{2}}{3}$

Illustration 24 A is $(-4, -5)$ in ΔABC and the lines $5x + 3y - 4 = 0$ and $3x + 8y + 13 = 0$ contain two of the altitudes of the triangle. Find the co-ordinates of B and C.



Solution

Slope of $\overleftrightarrow{BM} = -\dfrac{5}{3}$ and

Slope of $\overleftrightarrow{CN} = -\dfrac{3}{8}$

Slope of $\overleftrightarrow{AC} = \dfrac{3}{5}$ $\left(\because \text{Slope of } \overleftrightarrow{BM} = -\dfrac{5}{3}\right)$

$\therefore$ Equation of $\overleftrightarrow{AC}$ passing through A $(-4, -5)$ and slope $\dfrac{3}{5}$ is

$(y + 5) = \dfrac{3}{5}(x + 4)$

$\therefore\ 3x - 5y - 13 = 0$

Now C (x_1, y_1) is a point of intersection of $\overleftrightarrow{CN}$ and $\overleftrightarrow{AC}$

$\therefore$ Lines $3x + 8y + 13 = 0$ and $3x - 5y - 13 = 0$

$$\therefore\ C\ (x_1, y_1) = \left(\frac{\begin{vmatrix} 8 & 13 \\ -5 & -13 \end{vmatrix}}{\begin{vmatrix} 3 & 8 \\ 3 & -5 \end{vmatrix}}; \frac{-\begin{vmatrix} 3 & 13 \\ 3 & -13 \end{vmatrix}}{\begin{vmatrix} 3 & 8 \\ 3 & -5 \end{vmatrix}}\right) = (1, -2)$$

Thus co-ordinates of B $(x_1, y_1) = (-1, 3)$ and C $(x_2, y_2) = (1, -2)$

Illustration 25 Suppose $a \neq b$, $b \neq c$, $c \neq a$, $a \neq 1$, $b \neq 1$, $c \neq 1$ and the lines $ax + y + 1 = 0$, $x + by + 1 = 0$ and $x + y + c = 0$ are concurrent prove that $\dfrac{1}{1-a} + \dfrac{1}{1-b} + \dfrac{1}{1-c} = 1$

Solution

Lines $ax + y + 1 = 0$; $x + by + 1 = 0$ and $x + y + c = 0$ are concurrent lines

$$\therefore \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} a-1 & 0 & 1 \\ 1-b & b-1 & 1 \\ 0 & 1-c & c \end{vmatrix} = 0 \quad c_{21}\left(-1\right);\ c_{32}\left(-1\right)$$

$$\therefore \left(a-1\right)\left[\left(b-1\right)c-\left(1-c\right)\right]+1\left[\left(1-b\right)\left(1-c\right)\right]=0$$

$$\therefore \left(a-1\right)\left(bc-c-1+c\right)+1\left(1-c-b+bc\right)=0$$

$$\therefore \left(a-1\right)\left(bc-1\right)+1\left(1-c-b+bc\right)=0$$

$$\therefore c\left(1-a\right)\left(1-b\right)+\left(1-a\right)\left(1-c\right)+\left(1-b\right)\left(1-c\right)=0 \text{ (By simplification)}$$

Now dividing by $\left(1-a\right)\left(1-b\right)\left(1-c\right)$

$$\therefore \frac{c}{1-c}+\frac{1}{1-b}+\frac{1}{1-a}=0$$

$$\therefore 1+\frac{c}{1-c}+\frac{1}{1-b}+\frac{1}{1-a}=1$$

$$\therefore \frac{1}{1-c}+\frac{1}{1-b}+\frac{1}{1-a}=1$$

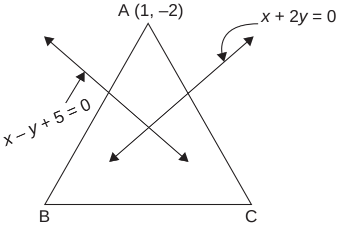
$$\therefore \frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=1$$

Illustration 26 A is $\left(1,\ 2\right)$ in ΔABC. Equations of the $\perp^{er}$ bisectors of $\overline{AB}$ and $\overline{AC}$ are $x - y + 5 = 0$ and $x + 2y = 0$ respectively. Find the co-ordinates of B and C.

Solution

Here in ΔABC A $\left(1, -2\right)$ let D and E be the mid-points of $\overline{AB}$ and $\overline{AC}$ respectively

Here let $l_1 : x - y + 5 = 0$ and $l_2 = x + 2y = 0$ be the $\perp^{er}$ bisectors of $\overline{AB}$ and $\overline{AC}$ respectively



Let the co-ordinate of B be $\left(x_1, y_1\right)$

$$\therefore \text{D} = \left(\frac{x_1+1}{2}, \frac{y_1-2}{2}\right)$$

which satisfies $l_1 = x - y + 5 = 0$

$$\therefore \left(\frac{x_1+1}{2}\right)-\left(\frac{y_1-2}{2}\right)+5=0$$

$$\therefore x_1 - y_1 + 13 = 0 \qquad (1)$$

Also slope of $\overleftrightarrow{AB}$ ∥ Slope of $l_1 = -1$

$$\therefore \frac{\left(y_1-2\right)/\left(2\right)-\left(-2\right)}{\left(x_1+1\right)/\left(2\right)-1}\times 1=-1$$

$$\therefore \frac{y_1+2}{x_1-1}=-1$$

$$\therefore y_1 + 2 = -x_1 + 1$$

$$\therefore x_1 + y_1 + 1 = 0 \qquad (2)$$

Now solving (1) and (2) we get

$\left(x_1, y_1\right) = \left(-7, 6\right)$

$\therefore$ B $= \left(x_1, y_1\right) = \left(-7, 6\right)$

Now let the co-ordinate of C be $\left(x_2, y_2\right)$

$\therefore$ E $\left(\dfrac{x_2+1}{2}, \dfrac{y_2-2}{2}\right)$

which satisfies $l_2 = x + 2y = 0$

$$\therefore \left(\frac{x_2+1}{2}\right) + 2\left(\frac{y_2-2}{2}\right) = 0$$

$$\therefore x_2 + 2y_2 - 3 = 0 \tag{3}$$

Also slope of $\overleftrightarrow{AC} \parallel$ Slope of $l_2 = -1$

$$\therefore \left(\frac{y_2+2}{x_2-1}\right)\left(-\frac{1}{2}\right) = -1$$

$$\therefore 2x_2 - y_2 - 4 = 0 \tag{4}$$

Now solving (3) and (4) we get

$$\left(x_2, y_2\right) = \left(\frac{11}{5}, \frac{2}{5}\right)$$

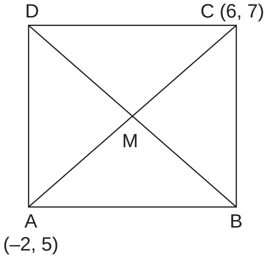
$$\therefore \text{C} = \left(x_2, y_2\right) = \left(\frac{11}{5}, \frac{2}{5}\right)$$

Thus we can say that co-ordinates of B and C are, respectively, $\left(-7, 6\right)$ and $\left(\dfrac{11}{5}, \dfrac{2}{5}\right)$

Illustration 27 Points $\left(-2, 5\right)$ and $\left(6, 7\right)$ form a pair of opposite vertices of a rhombus. Find the equations of lines that contain the diagonals of the rhombus.

Solution

Here A $\left(-2, 5\right)$ and C $\left(6, 7\right)$ are opposite vertices of a rhombus ABCD



$$\therefore \text{Equation of } \overleftrightarrow{AC} \begin{vmatrix} x & y & 1 \\ -2 & 5 & 1 \\ 6 & 7 & 1 \end{vmatrix} = 0$$

$\therefore x\left(5-7\right) - y\left(-2-6\right) + 1\left(-14-30\right) = 0$

$\therefore x - 4y + 22 = 0$

$\therefore$ Slope of $\overleftrightarrow{AC} = \dfrac{1}{4}$, slope of $\overleftrightarrow{BD} = -4$

M is the mid-point of $\overline{AC}$ $\therefore$ M $\left(2, 6\right)$

Now equation of $\overleftrightarrow{BD}$ of slope $= -4$ and it is passing through M $\left(2, 6\right)$ is

$(y-6) = -4(x-2)$

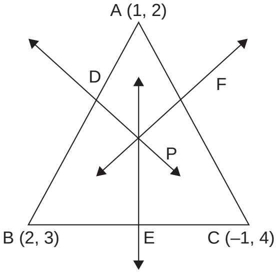
$\therefore$ $4x + y - 14 = 0$

$\therefore$ The equations of lines containing the diagonals of given rhombus are $x - 4y + 22 = 0$ and $4x + y - 14 = 0$

Illustration 28 Determine the equations of the $\perp^{\text{er}}$ bisectors of the sides of ΔABC when A is $(1, 2)$, B $(2, 3)$, C $(-1, 4)$. Use these to get the co-ordinates of the circumcentre.

Solution

Here A $(1, 2)$, B $(2, 3)$ and C $(-1, 4)$ are the vertices of ΔABC and D, E and F their respective mid-points of $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ and P is the circumcentre of ΔABC



$\therefore$ D$\left(\frac{3}{2}, \frac{5}{2}\right)$, E$\left(\frac{1}{2}, \frac{7}{2}\right)$ and F $(0, 3)$

Now slope $\overleftrightarrow{AB} = \frac{3-2}{2-1} = 1 \Rightarrow$ slope of $\overleftrightarrow{DP} = -1$

$\therefore$ Equation of $\overleftrightarrow{DP}$ having slope -1 and passing through D$\left(\frac{3}{2}, \frac{5}{2}\right)$

$\therefore$ $\left(y - \frac{5}{2}\right) = -1\left(x - \frac{3}{2}\right)$

$\therefore$ $x + y - 4 = 0$ (1)

Now slope of $\overleftrightarrow{BC} = \frac{4-3}{-1-2} = -\frac{1}{3} \Rightarrow$ slope of $\overleftrightarrow{DE} = 3$

$\therefore$ Equation of $\overleftrightarrow{PE}$ having slope = 3 and passing through E$\left(\frac{1}{2}, \frac{7}{2}\right)$

$\therefore$ $\left(y - \frac{7}{2}\right) = 3\left(x - \frac{1}{2}\right)$

$\therefore$ $3x - y + 2 = 0$ (2)

Slope of $\overleftrightarrow{AC} = \frac{4-2}{-1-1} = -1$, slope of $\overleftrightarrow{PF} = 1$

$\therefore$ Equation of $\overleftrightarrow{PF}$ having slope 1 and passing through F $(0, 3)$

$\therefore$ $(y-3) = 1(x-0)$

$\therefore$ $x - y + 3 = 0$ (3)

$\therefore$ Co-ordinates of D can be determined by solving any two of the three equations; by solving eqs. (1) and (2) we get $x = \frac{1}{2}$, $y = \frac{7}{2}$

$\therefore$ $P\ (x, y) = \left(\frac{1}{2}, \frac{7}{2}\right)$

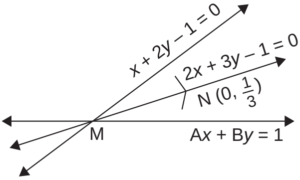
Illustration 29 The equation of a line bisecting an angle between two lines is $2x + 3y - 1 = 0$. If one of the two lines has equation $x + 2y = 1$, find the equation of the other.

Solution

Suppose the equation of other lines is $Ax + By = 1$

$l_1 = 2x + 3y = 1$

$\therefore \; y = \frac{1 - 2x}{3}$ if $x = 0 \Rightarrow \; y = \frac{1}{3}$



$\therefore \; N\left(0, \frac{1}{3}\right)$ is $2x + 3y = 1$

Now $\perp^{er}$ distance of $N\left(0, \frac{1}{3}\right)$ from $x + 2y - 1 = 0$

$$\therefore \; P = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$\therefore \; P = \frac{|1(0) + 2(1/3) + (-1)|}{\sqrt{1^2 + 2^2}} = \frac{|-1/3|}{\sqrt{5}} = \frac{1}{3\sqrt{5}}$$

$\therefore \; \perp^{er}$ distance of $N\left(0, \frac{1}{3}\right)$ from $Ax + By - 1 = 0$ is $\frac{1}{3\sqrt{5}}$

$$\therefore \; \frac{|A(0) + B(1/3) + (-1)|}{\sqrt{A^2 + B^2}} = \frac{1}{3\sqrt{5}}$$

$$\therefore \; \frac{|(B/3) - 1|}{\sqrt{A^2 + B^2}} = \frac{1}{3\sqrt{5}}, \; \frac{(B - 3)^2}{9(A^2 + B^2)} = \frac{1}{45}$$

$\therefore \; A^2 - 4B^2 + 30B - 45 = 0$ (1)

Now M is the point of intersection of $2x + 3y - 1 = 0$ and $x + 2y - 1 = 0$

Showing above equations we get M $(-1, 1)$

Now M $(-1, 1)$ is also on the line $Ax + By = 1$

$\therefore \; A(-1) + B(1) = 1$

$\therefore \; -A + B = 1$ substitute in eq. (1)

$\therefore \; (B - 1)^2 - 4B^2 + 30B - 45 = 0$

$\therefore \; -3B^2 + 28B - 44 = 0$

$\therefore \; 3B^2 - 28B - 44 = 0$

$\therefore \; (3B - 22)(B - 2) = 0$

$\therefore \; B = \frac{22}{3}$ or $B = 2$

$\therefore\ B = \frac{22}{3} \Rightarrow A = \frac{19}{3}$ and $B = 2 \Rightarrow A = 1$

$\therefore$ Substitute the value of A and B in the equation

$$\therefore\ \frac{19}{3}x + \frac{22}{3}y = 1$$

$$\therefore\ 19x + 22y - 3 = 0$$

and for A = 1, and B = 2, the equation is $x + 2y - 1 = 0$

Illustration 30 Find the points on the line $3x - 2y - 2 = 0$ which are at a distance of 3 units from $3x + 4y - 8 = 0$.

Solution

From the equation of line $3x - 2y - 2 = 0$, $y = \frac{3x-2}{2}$

$\therefore$ Any point P on this line can be taken as $P\left(x, \frac{3x-2}{2}\right)$

$\therefore$ The distance of P from line $3x + 4y - 8 = 0$ is 3 units

$$\therefore\ \left|\frac{3x + 4\left[(3x-2)/2\right] - 8}{\sqrt{3^2 + 4^2}}\right| = 3$$

$$\therefore\ \left|\frac{9x - 12}{5}\right| = 3$$

$$\therefore\ \frac{9x - 12}{5} = \pm 3$$

$$\therefore\ 9x - 12 = \pm 15$$

$$\therefore\ x = \frac{12 + 15}{9} \quad \text{or} \quad x = \frac{12 - 15}{9}$$

$$\therefore\ x = 3 \quad \text{or} \quad x = -\frac{1}{3}$$

$$\therefore\ x = 3 \Rightarrow y = \frac{3(3) - 2}{2} = \frac{7}{2}$$

$$x = -\frac{1}{3} \Rightarrow y = \frac{3(-1/3) - 2}{2} = -\frac{3}{2}$$

$\therefore$ The required points are $\left(3, \frac{7}{2}\right)$ and $\left(-\frac{1}{3}, -\frac{3}{2}\right)$

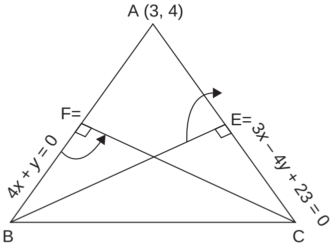
Illustration 31 In ΔABC A $(3, 4)$ and the lines containing two of the altitudes are $3x - 4y + 23 = 0$ and $4x + y = 0$. Find the co-ordinates of B and C.

Solution

Point A $(3, 4)$ does not satisfy $3x - 4y + 23 = 0$ and $4x + y = 0$

$\therefore$ These lines are not along the altitudes passing through A $(3, 4)$

$\therefore$ Take the given lines along the altitudes and point A as shown in the figure



Now $\overline{CF} \perp \overline{AB}$

$\therefore \overleftrightarrow{AB} = x - 4y + k = 0$ which passes through A $(3, 4)$

$\therefore 3 - 16 + k = 0 \Rightarrow k = 13$

$\therefore \overleftrightarrow{AB} = x - 4y + 13 = 0$ and

$\overleftrightarrow{BE} = 3x - 4y + 23 = 0$

$\therefore$ Solving these equations $x + 13 = 3x + 23$

$\Rightarrow x = -5$ and $y = 2$

$\therefore$ B (x, y) = B $(-5, 2)$

Since $\overline{BE} \perp \overline{AC}$

$\overleftrightarrow{AC} = 4x + 3y + k_1 = 0$ which is passes through A $(3, 4)$

$\therefore 12 + 12 + k_1 = 0 \Rightarrow k_1 = -24$

$\therefore \overleftrightarrow{AC} = 4x + 3y - 24 = 0$ and $\overleftrightarrow{CF} = 4x + y = 0$

Solving these equations $3y - 24 = y \Rightarrow y = 12$ and $x = -3$

$\therefore$ C (x, y) = C $(-3, 12)$

Illustration 32 Show that line $7x + 2y - 15 = 0$ is not a member of system of lines represented by the equation $(5x + 3y + 6) + \lambda(3x - 4y - 37) = 0$ $(\lambda \in R)$.

Solution

$L = 5x + 3y + 6 + \lambda(3x - 4y - 37) = 0$ $(\because \lambda \in \text{R})$

$\therefore (5 + 3\lambda)x + (3 - 4\lambda)y + (6 - 37\lambda) = 0$

If the line $7x + 2y - 15 = 0$ is not a member of L then the following equations are non-consistent for any $\lambda \in R$.

$$\begin{vmatrix} 5+3\lambda & 3-4\lambda \\ 7 & 2 \end{vmatrix} = 0 \Rightarrow -11 + 34\lambda = 0 \qquad (1)$$

$$\begin{vmatrix} 3-4\lambda & 6-37\lambda \\ 2 & -15 \end{vmatrix} = 0 \Rightarrow 134\lambda - 57 = 0 \qquad (2)$$

$$\begin{vmatrix} 5+3\lambda & 6-37\lambda \\ 7 & -15 \end{vmatrix} = 0 \Rightarrow -214\lambda - 117 = 0. \qquad (3)$$

From eq. (1) we get $\lambda = \dfrac{11}{34}$ which does not satisfy the remaining equations

$\therefore$ Eqs. (1), (2) and (3) are not consistent
$\therefore$ $7x + 2y - 15 = 0$ is not a member of the system.

Illustration 33 Show that all the points on line $2x + 11y = 5$ are equidistance from lines $24x + 7y = 20$ and $4x - 3y = 2$.

Solution

Let P (a, b) be any point on line $2x + 11y = 5$

$$\therefore\ 2a + 11b = 5\ \therefore\ b = \frac{5-2a}{11}\ \ \therefore\ \text{P} = \text{P}\left(a, \frac{5-2a}{11}\right)$$

Let P be the $\perp^{er}$ distance of P from the line $24x + 7y - 20 = 0$

$$\therefore\ \text{P}\ \frac{\left|24a + 7\left[(5-2a)/11\right] - 20\right|}{\sqrt{24^2 + 7^2}} = \frac{|264a + 35 - 14a - 220|}{11 \times 25}$$

$$\therefore\ \text{P} = \left|\frac{250a - 185}{11 \times 25}\right| = \left|\frac{50a - 37}{55}\right| \qquad (1)$$

Let P^1 be the $\perp^{er}$ distance of P from $4x - 3y = 2$ then

$$\text{P}^1 = \frac{\left|4a - 3\left[(5-2a)/11\right] - 2\right|}{\sqrt{16+9}} = \left|\frac{50a - 37}{55}\right| \qquad (2)$$

ANALYTICAL EXERCISES

1. Show that the lines $x - y - 1 = 0$; $4y + 3y - 25 = 0$; and $2x - 3y + 1 = 0$ are concurrent and also find the point of concurrence.
2. If the lines below are concurrent find the point of concurrence.
   (*a*) $x - 2y + 3 = 0$; $2x - 4y + 5 = 0$; $4x - 6y + 7 = 0$
   (*b*) ,; $x - y = 0$ $(a; b \neq 0, a \neq \text{I}b)$
3. Find k if the following lines are concurrent
   (*a*) $3x - 4y - 13 = 0$; $8x - 11y - 33 = 0$; $2x - 3y + k = 0$
   (*b*) $kx + 2y - 5 = 0$; $x - 4y - 3 = 0$; $x + 2y - 9 = 0$.
4. If a, b, c are in AP but $3a \neq 2b$; $4b \neq 2c$; and $2a \neq c$ prove that the lines $ax + 2y + 1 = 0$; $bx + 3y + 1 = 0$ and $cx + 4y + 1 = 0$ are concurrent.
5. Find the equation of the line that passes through the point of intersection of $3x - 4y + 1 = 0$ and $5x + y = 1 = 0$ and that cuts off intercepts of equal magnitude on the two axes.
6. Find the equation of lines passing through the points of intersection of $x - y - 1 = 0$ and $2x - 3y + 1 = 0$ and satisfying the following conditions (1) parallel to x-axis (2) parallel to y-axis (3) parallel to the line $3x + 4y = 14$.

7. Find the equation of the line that passes through the point of intersection of the lines $4x - 3y - 1 = 0$ and $2x - 5y + 3 = 0$ and makes congruent angles with two axes.

8. Find the equation of the line the product of whose intercepts of the two axes is -30 and that passes through the point of intersection of the lines $x - 2y - 2 = 0$ and $2x - 5y + 1 = 0$.

9. Find the equation of the line with slope $\frac{2}{3}$ and passing through the point of intersection of the lines $2x - 3y + 7 = 0$ and $x + 3y - 1 = 0$.

10. Find the equation of the line passing through the origin and also through the point of intersection of the lines $x - 3y + 1 = 0$ and $y = 2$.

11. Find the $\perp^{er}$ distance between the lines $3x - 4y + 9 = 0$ and $6x - 8y - 15 = 0$.

12. Get the $\perp^{er}$ distance from the point $(2, 3)$ to the line $y - 4 = 0$.

13. Get the $\perp^{er}$ distance from the point $(4, 2)$ to the line passing through the point $(4, 1)$ and $(2, 3)$.

14. Which of the lines $2x - y + 3 = 0$ and $x - 4y - 7 = 0$ is farther away from the origin?

15. Which of the lines $2x + 7y - 9 = 0$ and $4x - y + 11 = 0$ is farther away from the point $(2, 3)$?

16. Find the equation of the line parallel to the line $3x - 4y - 5 = 0$ and at a unit distance from it.

17. Find the equation of the lines bisecting the angle between the lines $2x + y - 3 = 0$ and $x + 2y - 1 = 0$.

18. Find the equations of the lines through $(-3, -2)$ that are parallel to the lines bisecting the angles between the lines $4x - 3y - 6 = 0$ and $3x + 4y - 12 = 0$.

19. Prove that the line $(\lambda - 1)x + (2\lambda + 1)y - 12 = 0$ passes through a fixed point for all real λ.

20. Find that co-ordinates of the orthocentre of ΔABC if A is $(2, -2)$, B is $(8, -2)$ and C is $(6, 6)$.

21. In ΔABC, C $(4, 1)$ is the line containing the attaining the median through A, $x + 2y + 7 = 0$. Find the equation of lines containing the three sides of the triangle.

22. Find the area of rectangle triangle formed with the lines $x + 4y = 9$; $x + 10 + 23 = 0$ and $7x + 2y = 11$.

23. Show that the distinct lines $(b + c)x + ay = d$; $(c + a)x + by = d$; $(a + b)x + cy = d$ are concurrent $(a + b + c \neq 0)$.

24. If the lines containing the opposite sides of a square are $5x - 12y - 65 = 0$ and $5x - 12y + 26 = 0$, find the area of the square.

25. If the lines $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$; $a_3x + b_3y = c_3 = 0$ are concurrent and none of these lines passes through the origin, prove that the points $\left(\frac{a_1}{c_1}, \frac{b_1}{c_1}\right), \left(\frac{a_2}{c_2}, \frac{b_2}{c_2}\right)$, and $\left(\frac{a_3}{c_3}, \frac{b_3}{c_3}\right)$ are collinear.

26. Prove that for all $\lambda \in R$ the line $(2 + \lambda)x + (3 - \lambda)y + 5 = 0$ passes through a fixed point. Find the coordinates of this fixed point.

27. Find the length and the foot of the $\perp^{er}$ from $(1, 3)$ to the line $5x + 6y - 1 = 0$.

28. Prove that for every real λ the line $(\lambda + 2)x + (5 - \lambda)y + \lambda - 19 = 0$ passes through a fixed point and also find the co-ordinates of this fixed point.

29. Find the equation of the line passing through the origin and containing a line-segment of length between the lines $2x - y + 1 = 0$ and $2x - y + 6 = 0$.

30. Find the points on the line $x + 5y = 13$ which are at a distance of 2 from the line $12x - 5y + 26 = 0$.

31. The lines $x - 2y + 2 = 0$; $3x - y + 6 = 0$ and $x - y = 0$ contain the three sides of a triangle. Determine the co-ordinates of the orthocentre without finding the co-ordinates of the vertices of the triangle.

32. Two of the vertices of a triangle are $(3, -1)$ and $(-2, 3)$. If the orthocentre is at the origin, find the third vertex of the triangle.

33. For ΔABC A $(-10, -13)$, B $(-2, 3)$ and C is $(2, 1)$. Obtain the $\perp^{er}$ distance of B from the lines containing the median through C.

34. Find the points of the line $x + 5y - 13 = 0$ which are at a distance of 2 units from the line $12x - 5y + 26 = 0$.

35. Two of the sides of a parallelogram are along the lines $x + 5y - 41 = 0$ and $7x - 13y + 1 = 0$. If one of the diagonals is along the line $5x + y + 11 = 0$, find the vertices of the parallelogram.

36. Find the equation of the lines passing through the intersection of lines $x + y - 7 = 0$ and $4x - 3y = 0$ and the $\perp^{er}$ distance of which from origin is maximum.

37. In ΔABC A $(2, 3)$ and B $(1, 2)$ are given. Find co-ordinates of C if orthocentre of ΔABC is $(-1, 2)$.

38. Find the orthocentre of the triangle whose sides are along the lines $y = a(x - b - c)$; $y = b(x - c - a)$; $y = c(x - a - b)$, $(a \neq b \neq c)$.

39. If $ax + by + c = 0$; $ax + by + c^1 = 0$; $a^1x + b^1y = c = 0$ and $a^1x + b^1y = c^1 = 0$ are the lines forming a rhombus, show that $a^2 + b^2 = a^{1^2} + b^{1^2}$.

40. If p is the length of the $\perp^{er}$ on a line whose intercepts on x-axis and y-axis are respectively a and b, prove that $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2}$.

ANSWERS

(1) $(4, 3)$

(2) **(a)** not consistent **(b)** $\left(\dfrac{ab}{a+b}, \dfrac{ab}{a+b}\right)$

(3) **(a)** $k = -7$ **(b)** $k = -1$

(5) $23x + 23y - 11 = 0$, $23x - 23y + 5 = 0$, $8x - 3y = 0$

(6) $y = 3$, $x = 4$, $3x + 4y = 24$

(7) $x + y - 2 = 0$, or $x - y = 0$

(8) $5x - 24y + 60 = 0$ or $5x - 6y - 30 = 0$

(9) $2x - 3y + 7 = 0$

(10) $2x - 5y = 0$,

(11) $\dfrac{33}{10}$

(12) $\dfrac{1}{\sqrt{2}}$

(14) $x + 4y - 7 = 0$, further away from $(0, 0)$

(15) $4x - y + 1 = 0$ is further away from $(2, 3)$
(16) $3x - 4y = 0$ or $3x - 4y - 10 = 0$
(17) $x - y - 2 = 0$ or $3x + 3y - 4 = 0$
(18) $x - 7y - 11 = 0,\ 7x + y + 23 = 0$
(19) $(-8, 4)$
(20) $(6, -1)$
(21) $13x - 19y + 1 = 0,\ x + 3y - 7 = 0,$
$x - 7y - 11 = 0$
(22) 26
(24) 49
(27) $\left(-\frac{49}{61}, \frac{51}{61}\right), \frac{22}{\sqrt{61}}$
(29) $(2, 3)$
(30) $\left(-3, \frac{16}{5}\right), \left(1, \frac{12}{5}\right)$
(31) $(-7, 5)$
(32) $\left(-\frac{36}{7}, -\frac{45}{7}\right)$
(33) 4
(34) $\left(1, \frac{12}{5}\right), \left(-3, \frac{16}{5}\right)$
(35) A $(-4, 9)$, B $(-17, 2)$, C $(-2, -1)$
(36) $3x + 4y - 25 = 0$
(37) C $(-2, 1)$
(38) $(-abc, 1)$

17

Circle

LEARNING OBJECTIVES

After studying this chapter, the student will be able to understand:

- The standard equation of a circle
- General equation and parametric equation of a circle
- The centre and radius of the general equation of a circle
- The equation of a circle touching the axes
- The equation of a circle where the extremities of a diameter are given
- The intersection of lines $ac + by + c = 0$ $(a^2 + b^2 \neq 0)$ with the circle $x^2 + y^2 = r^2$
- The position of a point with respect to the circle
- The equation of a tangent to the circle at a given point
- The condition in the tangent of lines to the circle and co-ordinates of contact
- The length of the tangent drawn to the circle from the point outside the circle
- Relation between two circles

INTRODUCTION

DEFINITION OF A CIRCLE

The set of points in a plane which are equidistance from a fixed point in the same plane is called a circle.

The fixed point is called the centre of the circle and its centre is called the radius of the circle.

Equation of a Circle

1. Circle having centre at $C\left(h, k\right)$ and radius r
   $\left(x - h\right)^2 + \left(y - k\right)^2 = r^2$
2. Circle with centre at origin with radius r is
   $x^2 + y^2 = r^2$
3. Circle with centre at origin with unit radius is
   $x^2 + y^2 = 1$

These cartesian forms of the equations of a circle are known as standard forms of equation of the circle.

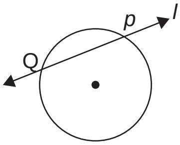
General Equations Representing a Circle

1. If $r = \sqrt{g^2 + f^2 - c}$ $fx^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle, then $g^2 + f^2 - > 0$, its centre is C $(-g, -f)$ and its radius $r = \sqrt{g^2 + f^2 - c}$.
2. Equation of the circle touching y-axis is
   $(x \pm r^2) + (y - k^2) = r^2$
3. Equation of the circle touching x-axis is
   $(x - h)^2 + (y \pm r)^2 = r^2$
4. Equation of the circle touching both axes is
   $(x \pm r)^2 + (y \pm r)^2 = r^2$

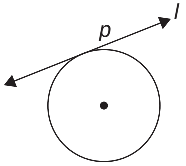
Parametric Equation of a Circle

1. The parametric equation of a circle is
   $(x - h)^2 + (y - k)^2 = r^2$
   $\therefore x = h + r \cos \theta; y = k + r \sin \theta, \theta \in (-\pi, \pi)$
2. The parametric equation of a circle = $x^2 + y^2 = r^2$
   where $x = \cos \theta, y = r \sin \theta, \theta \in (-\pi, \pi)$
3. The parametric equation of a unit circle = $x^2 + y^2 = 1$
   where $x = \cos \theta, y = \sin \theta, \theta \in (-\pi, \pi)$
4. The equation of a circle having diameter and point (x_1, y_1) is
   $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

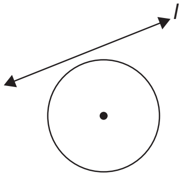
Intersection of a Circle and a Line



(1)



(2)



(3)

line $l = [(x, y) \in R^2 / ax + by + c = 0; a^2 + b^2 \neq 0]$

circle $s = [(x, y) \in R^2 / x^2 + y^2 = r^2]$

There can be three cases as shown above for the intersection of a circle with a line

1. $l \cup s = (P, Q)$ or line intersects the circle in two distinct points

   Conclusion: $r^2 > \dfrac{c^2}{a^2 + b^2}$
2. $l \cup s = (P)$ or line touches the circle at a unique point

   Conclusion: $r^2 - \dfrac{c^2}{a^2 + b^2}$
3. $l \cup s = \phi$ or line does not intersect the circle

   Conclusion: $r^2 < \dfrac{c^2}{a^2 + b^2}$

To find the intersection of a line with a circle solve their respective equations for x and y. These values of x and y are the respective co-ordinates of the point of intersection of their graphs.

This method in general can be applied to find the point of intersection of a line with any curve.

Position of a Point in the Plane with Respect to a Circle

A $\left(x_1, y_1\right)$ is a point in the plane of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

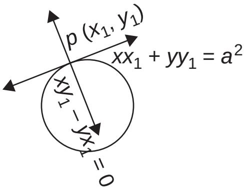
1. If $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0$ then and only then A $\left(x_1, y_1\right)$ is out side the circle.
2. If $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < 0$ then and only then A $\left(x_1, y_1\right)$ is inside the circle.
3. If $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$ then and only then A $\left(x_1, y_1\right)$ is on the circle.

Equation of Common Chord

1. If S_1: $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and S_2 : $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ are two intersecting circles, then equations
   $lS_1 + mS_2 = 0$; $l^2 + m^2 \neq 0$, $l + m \neq$
   represent a circle passing through the point of intersection of S_1 and S_2.
2. If circle $S = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$, then line
   $l = ax + by + c = 0$; $a^2 + b^2 \neq 0$
   represents a circle passing through the point of intersection of S_1 and S_2.
3. If $S_1 = 0$ and $S_2 = 0$ are two intersecting circles, then the equation
   $S_1 - S_2 = 0$
   represents a line containing the common chord of the circle.
4. If $S_1 = 0$ and $S_2 = 0$ are two intersecting circles, then the equation of any circle passing through the intersections of S_1 and S_2 can be put into the form
   $lS_1 + mS_2 = 0$; $l^2 + m^2 \neq 0$; $l + m \neq 0$.

Results (1), (2), (4) are to be assumed and (3) is to be proved but all of them are important to solve the examples.

Equation of Tangent and Normal



1. The equation of tangent of $P\left(x_1, y_1\right)$ on the circle $x^2 + y^2 = r^2$ is $xx_1 + yy_1 = a^2$
2. The equation of normal of $P\left(x_1, y_1\right)$ on the circle $x^2 + y^2 = r^2$ is $xy_1 - yx_1 = 0$

The Condition for *y* = *mx* + *c* to be Tangent to the Circle

$x^2 + y^2 = r^2$ is $c^2 = r^2\left(1 + m^2\right)$ and the co-ordinates of the point of contact are $\left(-\dfrac{a^2m}{c}, \dfrac{r^2}{c}\right)$ so the equation is $y = mx \pm r\sqrt{1 + m^2}$

Length of Tangent

The length of the tangent from $P\left(x_1, y_1\right)$ to the circle $x^2 + y^2 = r^2$ as shown in the figure is $\mathrm{PT} = \mathrm{PT}^1 = \sqrt{x_1^2 + y_1^2 - r^2}$

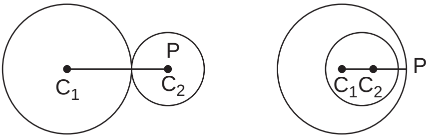
The length of the tangent from $P\left(x_1, y_1\right)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$\mathrm{PT} = \mathrm{PT}^1 = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2\,fy_1 + c}$$

Relation Between Two Circles in a Plane

When two circles are given in a plane, then there are five alternatives about their intersection as given below with their respective geometric conditions:

1. The circles may touch each other externally: $c_1c_2 = r_1 + r_2$
2. They may touch each other internally: $c_1c_2 = |\,r_1 - r_2\,|$



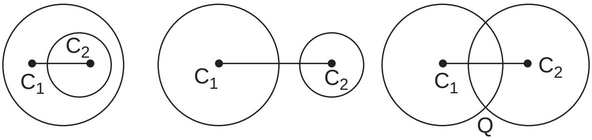
(1) (2)

In both these cases their intersection set is a singleton set (P).

3. One circle is inscribed in the circle: $c_1c_2 < |\,r_1 - r_2\,|$
4. One circle is out side the other circle: $c_1c_2 > r_1 + r_2$.

   In (3) and (4) the intersection of two circles is null set.
5. Two circles may be intersecting at two points

   $|\,r_1 - r_2\,| < c_1c_2 < r_1 + r_2$



(3) (4) (5)

ILLUSTRATIONS

Illustration 1 For each of the following circles, find the centre and radius.

(1) $x^2 + \left(y + 2\right)^2 = 0$ (2) $x^2 + y^2 + 6x - 4y + 4 = 0$

(3) $x^2 + y^2 - x + 2y - 3 = 0$

Solution

(1) $x^2 + \left(y + 2\right)^2 = 9 \Rightarrow \left(x - 0\right)^2 + \left[y - \left(-2\right)\right]^2 = \left(3\right)^2$

By comparing $\left(x - h\right)^2 + \left(y - k\right)^2 = r^2$

Centre $c\ \left(h, k\right) = \left(0, -2\right)$ and radius $r = 3$

(2) $x^2 + y^2 + 6x - 4y + 4 = 0$

The above equation compares with $x^2 + y^2 + 2gy + 2fy + c = 0$

$\therefore 2g = 6 \qquad 2f = -4 \qquad c = 4$

$\therefore g = 3 \qquad f = -2$

$\therefore g^2 + f^2 - c = 9 + 4 - 4 = 9$

$\therefore$ Centre $c\left(-g, -f\right) = \left(-\frac{1}{2}, -1\right)$ and $r = \sqrt{g^2 + f^2 - c} = \frac{\sqrt{17}}{2}$

(3) $x^2 + y^2 - x + 2y - 3 = 0$ compares with $x^2 + y^2 + 2gx + 2fy + c = 0$

$\therefore 2g = -1 \qquad 2f = 2 \qquad$ and $c = -3$

$\therefore g = \qquad \therefore f = 1$

Illustration 2 Obtain the equation of a circle given that the area 49π and that the equation of lines containing two of the diameters of the circle are $2x - 3y + 12 = 0$ and $x + 4y - 5 = 0$.

Solution

The point of intersection of the lines $2x - 23y + 12 = 0$ and $x + 4y = 5$ containing diameters of the circle is the centre of the circle.

$$\therefore \left(x, y\right) = \left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}; -\frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}\right) = \left[\frac{15 - 48}{8 + 3}; -\frac{\left(-10 - 12\right)}{8 + 3}\right]$$

$\therefore \left(x, y\right) = \left(-3, 2\right)$

$\therefore$ The centre of the circle is $\left(-3, 2\right)$

Now the area of circular region = 49π

$\therefore \pi r^2 = 49\pi$

$\therefore r = 7$

Now the equation of the circle with $c\left(-3, 2\right)$ and $r = 7$ is

$\left(x + 3\right)^2 + \left(y - 2\right)^2 = 49$

$\therefore x^2 + y^2 + 6x - 4y - 36 = 0$

Illustration 3 Prove that the radii of the circle $x^2 + y^2 = 1$; $x^2 + y^2 - 2x - 6y = 6$ and $x^2 + y^2 - 4x - 12y = 9$ are in A.P.

Solution

Here $S_1 = x^2 + y^2 = 1 \qquad \therefore$ radius $r_1 = 1$

$S_2 = x^2 + y^2 - 2x - 6y = 6 = 0$

Here $g = -1, \; f = -3, \; c = -6$

$\therefore$ Radius $r_2 = \sqrt{g^2 + f^2 - c} = \sqrt{1 + 9 + 6} = 4$

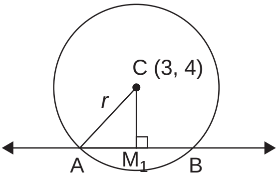
and $S_3 = x^2 + y^2 - 4x - 12y - 9 = 0$

$g = -2, \; f = -6, \; c = -9$

$\therefore$ Radius $r_3 = \sqrt{g^2 + f^2 - c} = \sqrt{4 + 36 + 9} = 7$

$\therefore$ The radii of the circle are $r_1 = 1$, $r_2 = 4$

$r_3 = 7$



$\therefore\ r_2 = \dfrac{r_1 + r_3}{2} \quad \left(4 = \dfrac{7+1}{2}\right)$

$\therefore\ r_1, r_2, r_3$ are in A.P

Illustration 4 Determine if each of the following equations represents a circle.
(1) $x^2 + y^2 + x - y = 0$ (2) $x^2 + y^2 - 12x + 3x + 3y + 10 = 0$.

Solution

(1) $x^2 + y^2 + x - y = 0$

Compare with $x^2 + y^2 + 2gx + 2fy + c = 0$

$\therefore\ g = \dfrac{1}{2},\ f = -\dfrac{1}{2},\ c = 0$

$\therefore\ g^2 + f^2 - c = \dfrac{1}{4} + \dfrac{1}{4} = \dfrac{1}{2} > 0$

Thus above equation is an equation of a circle.

(2) $x^2 + y^2 - 3x + 3y + 10 = 0$

Compare with $x^2 = y^2 + 29x + 2fy + c = 0$

$\therefore\ g = -\dfrac{3}{2},\ f = \dfrac{3}{2},\ c = 10$

$\therefore\ g^2 + f^2 - c = \dfrac{9}{4} + \dfrac{9}{4} - 10 = -\dfrac{11}{20} < 0$

Thus above equation is not an equation of a circle.

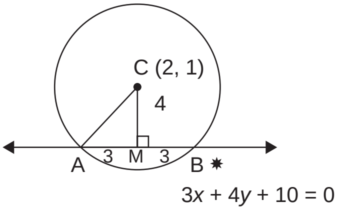
Illustration 5 Get the equation of the circle with centre $(2, 3)$ if it passes through the point of intersection of the lines $3x - 2y - 1 = 0$ and $4x + y - 27 = 0$.

Solution

Point of intersection of lines $3x - 2y - 1 = 0$ and $4x + y - 27 = 0$ are

$$(x, y) = \left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}; -\frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}\right)$$

$$= \left(\frac{54+1}{3+8}; -\frac{(-81+4)}{3+8}\right)$$



$\therefore\ (x, y) = (5, 7)$

Thus point of intersection is $(x, y) = (5, 7)$, C $(2, 3)$ is the centre of the circle and P $(5, 7)$ is on the circle.

$\therefore$ CP $= r \Rightarrow$ CP$^2 = r^2$

$\therefore\ (5 - 2)^2 + (3 - 7)^2 = r^2$

$\therefore\ 9 + 16 = r^2$

$\therefore\ r = 5$

$\therefore$ The equation of the circle with centre $(2, 3)$ with radius $r = 5$ is

$\therefore\ (x - 2)^2 + (y - 3)^2 = 5^2$

$\therefore\ x^2 - 4x + 4 + y^2 - 6y + 9 = 25$
$\therefore\ x^2 + y^2 - 4x - 6y - 12 = 0$

Illustration 6 Find the length of the chord of the circle $x^2 + y^2 - 6x - 8y - 50 = 0$ cut by the line $2x + y - 5 = 0$

Solution

Here centre of the circle
$x^2 + y^2 - 6x - 8y = 50 = 0$ is $(3, 4)$ and radius $r = \sqrt{3^2 + 4^2 + 50} = 5\sqrt{3}$
Length $\overline{\text{CM}}$ of the $\perp^{er}$ segment from $(3, 4)$ to the line $\overleftrightarrow{\text{AB}}$ is

$$\text{CM} = P = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} = \frac{|2(3) + 4 - 5|}{\sqrt{4+1}} = \sqrt{5}$$

Now in right angled ΔAMC
$\text{AM}^2 = \text{AC}^2 - \text{CM}^2 = 75 - 5 = 70$
$\therefore\ \text{AM} = \sqrt{70}$
$\therefore\ \text{AB} = 2\text{AM} = 2\sqrt{70}$

Illustration 7 Determine the equation of the circle passing through $(4, 1)$ and $(6, 5)$ and with centre on the line $4x + y = 16$.

Solution

Let the required equation of circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. This circle passes through $(4, 1)$ and $(6, 5)$; we have
$8g + 2f + c - 17 = 0$ (1) and (2)
$12g + 10f + c + 61 = 0$
Subtracting the first from second equation we get
$4g + 8f + 44 = 0$
$\therefore\ g + 2f + 11 = 0$ (3)
Centre $(-g, -f)$ lies on the line $4x + y = 16$
$\therefore\ -4g - f = 16$
$\therefore\ 4g + f + 16 = 0$ (4)
Now by solving eqs. (3) and (4) we get
$g = -3$ and $f = -4$
Substituting these values in (1)
$-24 - 8 + \text{C} + 17 = 0$
$\therefore\ \text{C} = 15$
Hence, the required equation of the circle is $x^2 + y^2 - 6x - 8y + 15 = 0$

Illustration 8 Line $3x = 4y + 10 = 0$ cuts a chord of length 6 on a circle. If the centre of the circle is $(2, 1)$, find the equation of the circle.

Solution

C $(2, 1)$ is the centre and line $3x + 4y + 10 = 0$ cuts chord $\overline{\text{AB}}$ on the circle.
M is the midpoint of $\overline{\text{AB}}$
$\therefore\ \text{AM} = \text{BM} = 3$

$$\therefore\ \text{CM} = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} = \frac{|3(2) + 1(4) + 10|}{\sqrt{3^2 + 1^2}} = 4$$

Now ΔAMC is a right angled triangle

$\therefore$ $\text{AC}^2 = \text{Am}^2 + \text{MC}^2 = 4^2 + 3^2 = 25$

$\therefore$ AC = 5

$\therefore$ $r = 5$

$\therefore$ Equation of circle is

$(x - 2)^2 + (y - 1)^2 = 5^2$

$\therefore$ $x^2 + y^2 - 4x - 2y - 20 = 0$

Illustration 9 Obtain the equation of the circle of which $(3, 4)$, $(2, -7)$ are the ends of a diameter.

Solution

A $(3, 4)$ and B $(2, -7)$ are the end-points of diameter of the circle

Now, the diametric equation of circle is

$(x - x_1)\ (x - x_2) + (y - y_1)\ (y - y_2) = 0$

$\therefore$ $(x - 3)\ (x - 2) + (y - 4)\ (9y + 7) = 0$ A $(x_1, x_1) = (3, 4)$

$\therefore$ $x^2 + y^2 - 5x + 3y - 22 = 0$ B $(x_2, y_2) = (2, -7)$

Illustration 10 Prove that points $(1, -1)$, $(-2, 2)$, $(4, 8)$ and $(7, 5)$ are cocyclic.

Solution

Suppose the equation of a circle passing through

A $(1, -1)$, B $(-2, 2)$ and C $(4, 8)$ is

$S^1 = x^2 + y^2 + 2gx + 2fy + c = 0$

A $(1, -1) \in S^1 \Rightarrow 1 + 1 + 2g - 2f + c = 0$

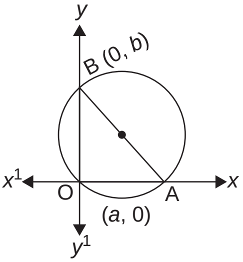
$\Rightarrow 2g - 2f + c + 2 = 0$ (1)

B $(-2, 2) \in S^1 \Rightarrow 4 + 4 - 4g = 4f + c = 0$

$\Rightarrow -4g + 4f + c + 8 = 0$ (2)

C $(4, 8) \in S^1 \Rightarrow 16 + 64 + 8g + 16f + c = 0$

$\Rightarrow 8g + 16f + c + 80 = 0$ (3)



Multiplying eqs. (1) by (2) and adding in eq. (2)

$\therefore$ $3C + 12 = 0 \Rightarrow C = -4$

But $8g + 16f + c + 80 = 0$

$\therefore$ $8g + 16f - 4 + 80 = 0$

$\therefore$ $2g + 4f = -19$ (4)

But $2g - 2f + c + 2 = 0$

$\therefore$ $2g - 2f - 4 + 2 = 0$

$\therefore$ $2g - 2f = 2$ (5)

Illustration 11 Find the equation of the circle for which the centres of the circles $x^2 + y^2 + 6x - 14y - 1 = 0$ and $x^2 + y^2 - 4x + 10y - 2 = 0$ are the ends of a diameter.

Solution

$S_1^1 = x^2 + y^2 + 6x - 14y - 1 = 0$ is $C_1^1 \left(-3, 7\right)$

$S_2^1 = x^2 + y^2 - 4x + 10y - 2 = 0$ is $C_2^1 \left(2, -5\right)$

$\therefore$ The equation of the circle with diameter $C_1^1\, C_2^1$ is

$\left(x + 3\right) \left(x - 2\right) + \left(y - 7\right) \left(y + 5\right) = 0$

$\therefore x^2 + y^2 + x - 2y - 41 = 0$

Illustration 12 Determine the position of the point $\left(7, -11\right)$ relative to the circle $x^2 + y^2 = 10x$.

Solution

If we substitute $y = 2x - 3$ in the equation of the circle we get

$x^2 + \left(2x - 3\right)^2 - 3x + 2\left(2x - 3\right) = 0$

$\therefore x^2 + 4x^2 - 12x + 9 - 3x + 4x - 6 = 0$

$\therefore 5x^2 - 11x = 0$

$\therefore x\left(5x - 11\right) = 0$

$\therefore x = 0$ or $x = \dfrac{11}{5}$

$\therefore x = 0 \Rightarrow y = 2\left(0\right) - 3 = -3$

$x = \dfrac{11}{5} \Rightarrow y = 2\left(\dfrac{11}{5}\right) - 3 = \dfrac{7}{5}$

$\therefore$ Their set of intersection is $\left[(0, -3), \left(\dfrac{1}{5}, \dfrac{7}{5}\right)\right]$

Illustration 13 Get the equation of the circle passing through the points $\left(0, 0\right)$, $\left(0, b\right)$ and $\left(a, 0\right)$. $\left(a \neq 0, b \neq 0\right)$.

Solution

Here A $\left(a, 0\right)$ and B $\left(0, b\right)$ intersect the x-axis and y-axis respectively.

$m\angle AOB = \dfrac{\pi}{2}$

$\therefore \overline{AB}$ is a diameter of the circle

$\therefore$ The equation of diametric equation of circle is

$\left(x - a\right) \left(x - 0\right) + \left(y - 0\right) \left(y - b\right) = 0$

$\therefore x^2 + y^2 - ax - by = 0$

Illustration 14 Determine the equation of the circle that passes through $\left(4, 1\right)$ and $\left(6, 5\right)$ and whose centre is on the line $4x + y - 16 = 0$.

Solution

Let $S^1 = x^2 + y^2 + 2gx + 2fy + c = 0$

$\left(4, 1\right) \in S^1 \Rightarrow 4^2 + 1^2 + 2g\left(4\right) + 2f\left(1\right) + c = 0$

$\Rightarrow 8g + 2f + c = -17 \qquad (1)$

$(6, 5) \in S^1 \Rightarrow 6^2 + 5^2 + 2g(6) + 2f(5) + c = 0$

$\Rightarrow 12g + 10f + c = -61$ (2)

But the centre $(-g, -f)$ is on the line $4x + y = 16$

$\therefore -4g - f = 16$ (3)

From eqs. (1) and (2), we get $-32 + c = -17 \Rightarrow c = 15$

Substituting the value of $c = 15$ in (1)

$12g + 10f + 15 = -61$

$\therefore 12g + 10f = -76$ (4)

Now Substituting the value of $c = 15$ in eq. (1)

$\therefore 8g + 2f + 15 = -17$

$\therefore 8g + 2f = -32$ (5)

Now solving eqs. (4) and (5) we get

$g = -3, f = -4$

$\therefore$ The equation of circle is

$x^2 + y^2 - 6x - 8y + 15 = 0$

Illustration 15 Obtain the equation of a circle with radius $\frac{5}{2}$ if it passes through $(-1, 1), (-1, -4)$.

Solution

Let the equation of circle be $S^1 : x^2 + y^2 + 2gx + 2fy + c = 0$

$(-1, 1) \in S^1 \Rightarrow 1 + 1 + 2g(-1) + 2f(1) + c = 0$

$\Rightarrow -2g + 2f + c = -2$ (1)

$(-1, -4) \in S^1 \Rightarrow 1 + 16 + 2g(-1) + 2f(-4) + c = 0$

$\Rightarrow -2g - 8f + c = -17$ (2)

Radius $r = \sqrt{g^2 + f^2 - c} \Rightarrow \frac{5}{2} = \sqrt{g^2 + f^2 - c}$

$\Rightarrow g^2 + f^2 - c = \frac{25}{4}$ (3)

Subtracting (2) from (1) we get $10f = 15$

$\therefore f = \frac{3}{2}$

Substituting the value of f in (1)

$-2g + 3 + c = -2 \Rightarrow c = 2g - 5$

Substituting value of c and f in eq. (3)

$\therefore g^2 + \frac{9}{4} - (2g - 5) = \frac{25}{4}$

$\therefore g^2 - 2g + 5 = 4$

$\therefore g^2 - 2g + 1 = 0$

$\therefore (g - 1)^2 = 0 \Rightarrow g = 1$

$\therefore c = 29 - 5 = 2 - 5 = -3$

$\therefore$ The required equation of circle is

$x^2 + y^2 + 2x + 3y - 3 = 0$

Illustration 16 If the point $(0, k)$ belongs to the circle passing through the point $(2, 3)$, $(0, 2)$, and $(4, 5)$, find k.

Solution

Suppose the equation of the circle passing through P $(2, 3)$, Q $(0, 2)$ and R $(4, 5)$ is

$S^1 = x^2 + y^2 + 2gx + 25y + c = 0$

$P\ (2, 3) \in S^1 \Rightarrow 4 + 9 + 2g(2) + 2f(3) + c = 0$

$\Rightarrow 4g + 6f + c = -13$ (1)

$Q\ (0, 2) \in S^1 \Rightarrow 0 + 4 + 2g(0) + 2f(2) + c = 0$

$\Rightarrow 4f + c = -4$ (2)

$R\ (4, 5) \in S^1 \Rightarrow 16 + 25 + 2g(4) + 2f(5) + c = 0$

$\Rightarrow 8g + 10f + c = -41$ (3)

Subtracting (2) from (1) we get

$4g + 2f = -9$ (4)

Subtracting (3) from (2) we get

$8g + 6f = -37$ (5)

Solving (4) and (5) we get

$$g = \frac{5}{2},\ f = -\frac{19}{2}$$

Now $4f + c = -4 \Rightarrow -38 + c = -4 \Rightarrow c = 34$

The equation of circle is

$x^2 + y^2 + 5x - 19y + 34 = 0$

$(0, k)$ is on circle

$\Rightarrow 0 + k^2 + 0 - 19k + 34 = 0$

$\Rightarrow k^2 - 19k + 34 = 0$

$\Rightarrow (k - 17)(k - 2) = 0$

$\Rightarrow k = 17$ or $k = 2$

Illustration 17 Find the tangents of the circle $x^2 + y^2 = 25$ that pass through the point $(7, 1)$.

Solution

As $7^2 + 1^2 = 50 > 25$ as per condition we can say that point $(7, 1)$ is outside the circle $x^2 + y^2 = 25$. Vertical tangents passe through $(5, k)$ or $(-5, k)$. Hence we can say that tangents through $(7, 1)$ are not vertical.

Let m be the slope of a tangent from $(7, 1)$ to the circle

Now a tangent with slope m has equation

$$y = mx \pm r\sqrt{1 + m^2}$$

As this tangent passes through $(7, 1)$

$$1 = 7m \pm 5\sqrt{1 + m^2}$$

$$\therefore\ (1 - 7m) = \pm 5\sqrt{1 + m^2}$$

$\therefore\ 1 - 14m + 49m^2 = 25\left(1 + m^2\right)$
$\therefore\ 24m^2 - 14m - 24 = 0$
$\therefore\ 12m^2 - 7m - 12 = 0$
$\therefore\ \left(4m + 3\right)\left(3m - 4\right) = 0$

Taking $m = -\dfrac{3}{4}$ the equation of tangent is

$\left(y - 1\right) = -\dfrac{3}{4}(x - 7)$

$\therefore\ 3x + 4y - 25 = 0$

Taking $m = \dfrac{4}{3}$ the equation of tangents is

$\left(y - 1\right) = \dfrac{4}{3}(x - 7)$

$\therefore\ 4x - 3y - 25 = 0$

Illustration 18 Get the equations of those tangents to the circle $x^2 + y^2 = 3$ that make an angle of measure $\dfrac{\pi}{3}$ with the positive direction of x-axis.

Solution

The tangent makes an angle $\dfrac{\pi}{3}$ with the +ve direction of x-axis with slope $m = \tan\dfrac{\pi}{3} = \sqrt{3}$

Radius r of circle $x^2 + y^2 = 3$ is $r = \sqrt{3}$

$\therefore$ Equation of the tangent with slope m is

$$y = mx \pm r\sqrt{1 + m^2}$$

$$y = \sqrt{3x} \pm \sqrt{3}\sqrt{(1 + 3)}$$

$$y = \sqrt{3x} \pm 2\sqrt{3} = \sqrt{3}\left(x \pm 2\right)$$

Illustration 19 If line $2x + 3y + k = 0$ touches the circle $x^2 + y^2 = 25$, find k.

Solution

The equation of tangent is $2x + 3y + k = 0$

$\therefore\ 3y = -2x - k$

$\therefore\ y = -\dfrac{2}{3}x - \dfrac{k}{3}$

Compare with $y = mx + c$

$\therefore\ m = -\dfrac{2}{3}$ and $c = -\dfrac{k}{3}$

But $x^2 + y^2 = 25 \Rightarrow r = 5$

The condition of the tangent is $c^2 = r^2\left(1 + m^2\right)$

$\therefore\ \left(-\dfrac{k}{3}\right)^2 = 25\left(1 + \dfrac{4}{9}\right) = 25\left(\dfrac{13}{9}\right)$

$$\frac{k^2}{9} = \frac{25 \times 13}{9}$$

$\therefore\ k = \pm 5\sqrt{13}$

Illustration 20 Find the tangent to the circle $4x^2 + 4y^2 = 25$ that is $\perp^{er}$ to the line $5x - 12y + 7 = 0$.

Solution

The slope of $5x - 12y + 7 = 0$ is $m = \frac{12}{5}$

Now the circle is $4x^2 + 4y^2 = 25 \Rightarrow x^2 + y^2 = \frac{25}{4} \Rightarrow r = \frac{5}{2}$

$x^2 + y^2 = r^2$ having the slope m is

$$y = mx \pm r\sqrt{1+m^2}$$

$$y = -\frac{12}{5}x \pm \frac{5}{2}\sqrt{1+\frac{144}{25}}$$

$$y = -\frac{12}{5}x \pm \frac{5}{2}\ \frac{13}{5}$$

$$y = -\frac{12}{5}x \pm \frac{13}{2}$$

$\therefore\ 24x + 10y = \pm 65$

Illustration 21 $(-3, 0)$ and $(4, 1)$ are points on a circle at which the tangents are $4x - 3y + 12 = 0$ and $3x + 4y - 10 = 0$, respectively. Find the equation of circle $3x + 4y = 16 = 0$.

Solution

The equation of the normal to the circle at $(-3, 0)$ is $3x + 4y = k$

$\therefore\ (-3, 0)$ is on the normal

$\therefore\ 3(-3) + 4(0) \neq k \Rightarrow k = -9$

$\therefore$ The equation of normal is $3x + 4y = -9$ (1)

The equation of the normal to the circle at $(4, 1)$ is

$4x - 3y = k_1$

$\therefore\ 4(4) - 3(1) = C \qquad \therefore\ C = 13$

$\therefore$ The equation of the normal is $4x - 3y = 13$ (2)

Solving (1) and (2) we get $(x, y) = (1, -3)$

$\therefore$ C $(1, -3)$ is the centre P $(4, 1)$ is on the circle

$\therefore\ r = \text{PC} = \sqrt{(4-1)^2 + (1+3)^2} = 5$

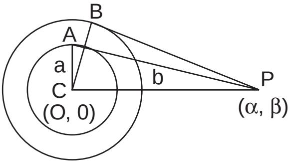
$\therefore$ The equation of the circle is

$(x - 1)^2 + (y + 3)^2 = 25$

$\therefore\ x^2 + y^2 - 2x + 6y - 15 = 0$

Illustration 22 How many tangents can be drawn from the point $(4, 3)$ to the circle $x^2 + y^2 = 26$?

Solution



$S_1 = x^2 + y^2 = 26$

For $(4, 3) = 16 + 9 - 26 = -1 < 0$

$\therefore$ $(4, 3)$ lies in the integrant of the circle

We can say that no tangent can be drawn to the circle from $(4, 3)$

Illustration 23 How many tangents can be drawn from the point $(7, 1)$ to the circle $x^2 + y^2 = 1$?

Solution

$S^1 = x^2 + y^2 = 1$

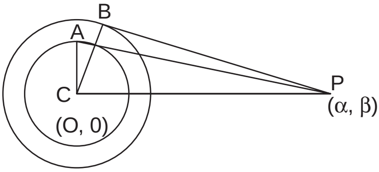
For $(7, 1) = 49 + 1 - 1 = 49 > 0$

$\therefore$ $(7, 1)$ is outside the circle $x^2 + y^2 = 1$

$\therefore$ Two tangents can be drawn to the circle from $(7, 1)$.

Illustration 24 Obtain the equation of the line containing the common chord of the circle $x^2 + y^2 + 3x + 4y + 1 = 0$ and $x^2 + y^2 + x = 8y - 1 = 0$.

Solution



$S_1 = x^2 + y^2 + 3x + 4y + 1 = 0$

$\therefore \; c_1\left(-\frac{3}{2}, -2\right)$

$$r_1 = \sqrt{\frac{9}{4} + 4 - 1} = \frac{\sqrt{69}}{2}$$

$S_2 = x^2 + y^2 + x - 8y - 1 = 0 \; c_2 = \left(-\frac{1}{2}, 4\right)$

$$r_2 = \sqrt{\frac{1}{4} + 16 + 1} = \sqrt{\frac{69}{2}}$$

Also $c_1c_2 = \sqrt{1 + 36} = \sqrt{37}$

It is clear that

$| r_1 - r_2 | < c_1c_2 < r_1 + r_2$

$\therefore$ The circles intersect in two distinct points and have a common chord

$\therefore$ The required line is $S_1 - S_2$ is

$\therefore \; (x^2 + y^2 + 2x + 4y + 1) - (x^2 + y^2 + x - 8y - 1) = 0$

$\therefore \; x + 6y + 1 = 0$

Illustration 25 Prove that the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ and $x^2 + y^2 - 18x - 16y + 120 = 0$ touch each other externally.

Solution

$S_1 = x^2 + y^2 - 2x - 4y - 20 = 0 \qquad C_1 \; (1, 2)\,;\; r_1 = \sqrt{1^2 + 2^2 + 20} = 5$

$S_2 = x^2 + y^2 - 18x - 16y + 120 = 0 \qquad C_1 \; (9, 8)\,;\; r_2 = \sqrt{9^2 + 8^2 - 120} = 5$

$\therefore\ c_1c_2 = \sqrt{(1-9)^2 + (2-8)^2} = 10$

and $r_1 + r_2 = 5 + 5 = 10$

$\therefore\ r_1 + r_2 = c_1c_2$

$\therefore$ The two circles do touch externally

Illustration 26 If the lengths of the tangents from P to two concentric circles are in inverse proportion to their radii find the locus of P.

Solution

Here $S_1 = x^2 + y^2 = a^2$

$S_2 = x^2 + y^2 = b^2$ are two concentric circles

Tangents to the circle $x^2 + y^2 = a^2$ from P (α, β) touches at A

$$\therefore\ \text{PA}\ \alpha\ \frac{1}{a} \Rightarrow \text{PA}^2\ \alpha\ \frac{1}{a^2} \qquad (1)$$

$$\text{Similarly}\ \ \text{PA}\ \alpha\ \frac{1}{a} \Rightarrow \text{PB}^2\ \alpha\ \frac{1}{b^2} \qquad (2)$$

From eqs. (1) and (2) we can say that

$$\frac{\text{PA}^2}{\text{PB}^2} = \frac{b^2}{a^2}$$

$$\therefore\ \frac{\alpha^2 + \beta^2 - a^2}{\alpha^2 + \beta^2 - b^2} = \frac{b^2}{a^2} \Rightarrow a^2(\alpha^2 + \beta^2 - a^2) = b^2(\alpha^2 + \beta^2 - b^2)$$

$$\Rightarrow (\alpha^2 + \beta^2)\ (a^2 - b^2) = (a^4 - b^4)$$

$$\Rightarrow (\alpha^2 + \beta^2) = a^2 + b^2 \qquad (\because a \neq b)$$

Illustration 27 Prove that the difference of the squares of the lengths of tangents drawn from P to two concentric circles doe not depend on P.

Solution

Let $S_1 = x^2 + y^2 = a^2$ and

$S_2 = x^2 + y^2 = b^2$ are two concentric circles. $(\because a^2 < b^2)$

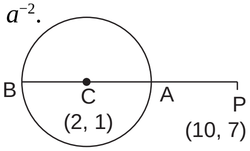
Now the length of the tangent drawn from P (α, β) to the circle $x^2 + y^2 = a^2$ is $\text{PA} = \sqrt{\alpha^2 + \beta^2 - a^2}$ and the length of the tangent from P (α, β) to the circle $x^2 + y^2 = b^2$ is

$$\text{PB} = \sqrt{\alpha^2 + \beta^2 - b^2}$$

Now $\text{PB}^2 - \text{PA}^2 = (\alpha^2 + \beta^2 - b^2) - (\alpha^2 + \beta^2 - a^2) = a^2 - b^2 =$ constant which does not depend on the position of P.

Illustration 28 If circles $x^2 + y^2 + 2gx + a^2 = 0$ and $x^2 + y^2 + 2fy + a^2 = 0$ touch each other externally, prove that P.T $g^{-2} + f^{-2} = a^{-2}$.

Solution



$S_1 = x^2 + y^2 + 2gx + a^2 = 0;\qquad \therefore\ c_1\ (-g, 0),$

and $r_1 = \sqrt{g^2 - a^2}$

and for $S_2 = x^2 + 2fy + a^2 = 0;\ c_2\ (0, -f)$

and $r_2 = \sqrt{f^2 - a^2}$

These circles touch externally

$\therefore\ c_1c_2 = r_1 + r_2$

$\therefore\ \sqrt{g^2 + f^2} = \sqrt{g^2 - a^2} + \sqrt{f^2 - a^2}$

Now Squaring both rides

$g^2 + f^2 = g^2 - a^2 + f^2 - a^2 + 2\sqrt{g^2 - a^2}\ \sqrt{f^2 - a^2}$

$\therefore\ a^2 = \sqrt{g^2 - a^2}\ \sqrt{f^2 - a^2}$

$\therefore\ a^4 = \left(g^2 - a^2\right)\ \left(f^2 - a^2\right)$

$\therefore\ a^4 = g^2f^2 - a^2f^2 - a^2g^2 + a^4$

$\therefore\ a^2f^2 + a^2g^2 = g^2f^2$

Now both sides divided by $g^2f^2a^2$, we get

$$\frac{1}{g^2} + \frac{1}{f^2} = \frac{1}{a^2}$$

$\therefore\ g^{-2} + f^{-2} = a^{-2}$

Illustration 29 Prove that the line $x + y = 2 + \sqrt{2}$ touches the circle. $x^2 + y^2 - 2x - 2y + 1 = 0$. Find the point of contact.

Solution

Here S $= x^2 + y^2 - 2x - 2y + 1 = 0$

Here $c = \left(-g, -f\right) = c\ \left(1, 1\right)$, $c = 1$

$\therefore$ Radius $r = \sqrt{g^2 + f^2 - c} = \sqrt{1 + 1 - 1} = 1$

Now the $\perp^{er}$ distance of the line $x + y = 2 + \sqrt{2}$ from the centre $\left(1, 1\right)$ is

$$\frac{|ax_1 + by_2 + c|}{\sqrt{a^2 + b^2}} = \frac{|1 + 1 - 2 - \sqrt{2}|}{\sqrt{1^2 + 1^2}} = 1 \text{ radius}$$

$\therefore$ Line $x + y = 2 + \sqrt{2}$ touches the circle.

Now a line $\perp^{er}$ to $x + y = 2 + \sqrt{2}$ is $x - y = k$

As this $\perp^{er}$ line passes through $\left(1, 1\right)$

$\therefore\ 1 - 1 = k$

$\therefore\ k = 0$

$\therefore$ The point of contact of intersection of the lines $x - y = 0$ and $x + y = 2 + \sqrt{2}$

Solving above equation, we get

$$x = \frac{2 + \sqrt{2}}{2} = y$$

$\therefore$ Point of contact is $\left(\dfrac{2 + \sqrt{2}}{2}, \dfrac{2 + \sqrt{2}}{2}\right)$

Illustration 30 Find the minimum and maximum distances of point on the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ from the point $\left(10, 7\right)$.

Solution

$S_1 = x^2 + y^2 - 4x - 2y - 20 = 0$ in above equation of circle.

Taking $x = 10$ and $y = 7$

$\therefore S_1 = 100 + 49 - 40 - 14 - 20 = 75 > 0$

$\therefore$ P $(10, 7)$ lies outside the circle

$x^2 + y^2 - 4x - 2y - 20 = 0$

$g = -2,\ f = -1,\ c = -20$

Now C $(2, 1)$ and $r = \sqrt{g^2 + f^2 - c} = \sqrt{4 + 1 + 20} = 5$

$\therefore$ CP = $\sqrt{(10 - 2)^2 + (7 - 1)^2} = \sqrt{100} = 10$

A and B are two points on the circles such that C − A − P and B − C − P. A is at minimum distance from P and B is at maximum distance from P.

$\therefore$ Minimum distance PB = CP − CA = 10 − 5 = 5 and

Maximum distance PB = CP + CB = 10 + 5 = 15

Illustration 31 If the circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$ touch each other externally, prove that $a^2, 2c, b^2$ are in H.P.

Solution

Here $S_1 = x^2 + y^2 + 2ax + c = 0$

$c_1\ (-a, 0)$ and $r_1 = \sqrt{a^2 - c}$

and for $S_2 = x^2 + y^2 + 2by + c = 0$

$c_2\ (0, -b)$ and $r_2 = \sqrt{b^2 - c}$

Now $c_1c_2 = \sqrt{a^2 + b^2}$

Both these circles touch each other externally

$\therefore c_1c_2 = r_1 + r_2$

$\therefore \sqrt{a^2 + b^2} = \sqrt{a^2 - c} + \sqrt{b^2 - c}$

Now squaring both sides

$a^2 + b^2 = a^2 + b^2 - 2c + 2\sqrt{a^2 - c}\ \sqrt{b^2 - c}$

$\therefore c = \sqrt{a^2 - c}\ \sqrt{b^2 - c}$

Again squaring both sides

$c^2 = (a^2 - c)\ (b^2 - c)$

$\therefore c^2 = a^2b^2 - a^2c - b^2c + c^2$

$\therefore c(a^2 - b^2) = a^2 + b^2$

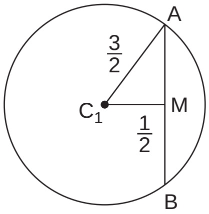
$$\therefore \frac{1}{c} = \frac{b^2 + a^2}{a^2b^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

$$\therefore \frac{2}{2c} = \frac{1}{a^2} + \frac{1}{b^2}$$

$\therefore a^2, 2c$ and b^2 are in H.P

Illustration 32 Prove that the circle drawn by taking the common chord of $(x-a)^2 + y^2 = a^2$ and $x^2 + (y-b)^2 = b^2$ as the diameter is $(a^2+b^2)(x^2+y^2) = 2ab(bx+ay)$.

Solution



Here $S_1 = (x-a)^2 + y^2 - a^2 = 0$; $c_1\ (a, 0)$; $r_1 = a$

$S_2 = x^2 + (y-b)^2 - b^2 = 0$; $c_2\ (0, b)$; $r_2 = b$

$\therefore\ c_1c_2 = \sqrt{a^2+b^2}$

$\therefore\ |a-b| < \sqrt{a^2+b^2} < a+b$

$\therefore\ |r_1 - r_2| < c_1c_2 < r_1 + r_2$

Here both the circles intersect in two distinct points and have a common chord

$\therefore$ Equation of the line containing common chord is $S_1 - S_2 = 0$

$\therefore\ (x-a)^2 + y^2 - a^2 - x^2 - (y-b)^2 + b^2 = 0$

$\therefore\ bx - ax = 0 \Rightarrow y = \dfrac{ax}{y}$

Now; substituting $y = \dfrac{ax}{b}$ in $(x-a)^2 + y^2 = a^2$

$$x^2 - 2ax + a^2 + \frac{a^2x^2}{b^2} = a^2$$

$\therefore\ b^2x^2 - 2ab^2x + a^2x^2 = 0$

$\therefore\ (a^2+b^2)x^2 - 2ab^2x = 0$

$\therefore\ x\left[(a^2+b^2)x - 2ab^2\right] = 0$

$\therefore\ x = 0$ or $x = \dfrac{2ab^2}{a^2+b^2}$

$\therefore\ x = 0 \Rightarrow y = 0$

$\therefore\ x = \dfrac{2ab^2}{a^2+b^2} \Rightarrow y = \dfrac{2a^2b}{a^2+b^2}$

$\therefore$ Diametric equation of circle is

$$(x-0)\left(x - \frac{2ab^2}{a^2+b^2}\right) + (y-0)\left(y - \frac{2a^2b}{a^2+b^2}\right) = 0$$

$\therefore\ (a^2+b^2)x^2 + (a^2+b^2)y^2 - 2ab(bx+ay) = 0$

$\therefore\ (a^2+b^2)(x^2+y^2) = 2ab(bx+ay) = 0$

Illustration 33 If the equation $3x^2 + (3-p)\ xy + qy^2 - 2px = 8pq$ represents a circle find p and q. Also determine the centre and radius of the circle.

Solution

The general condition for circle is

(1) Coefficients of x^2 and y^2 are identical $\Rightarrow q = 3$

(2) Coefficient of $xy = 0$

$\Rightarrow 3 - p = 0 \Rightarrow p = 3$

Now substituting the values of p and q in the equation of the circle we get

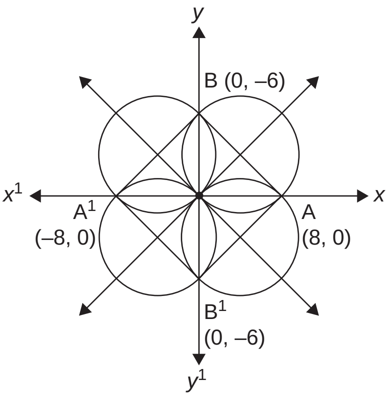
$3x^2 + 3y^2 - 6x = 72$

$\therefore\ x^2 + y^2 - 2x - 24 = 0$

$\therefore\ c\ (-g, -f\,) = (1, 0)$, and $r = \sqrt{g^2 + f^{\,2} - c} = \sqrt{1 + 0 + 24} = 5$

Illustration 34 Find the equation of the line that contains the common chord and the length of the common chord of the circle $x^2 + y^2 + 2x + 3y + 1 = 0$ and $x^2 + y^2 + 4x + 3y + 2 = 0$.

Solution



Here $S_1 = x^2 + y^2 + 2x + 3y + 1 = 0$

$\therefore\ c_1\ (-g, -f\,)\ =\ c_1\left(-1, -\frac{3}{2}\right)$

and $r_1 = \sqrt{g^2 + f^{\,2} - c} = \sqrt{1 + \frac{9}{4} - 1} = \frac{3}{2}$

and $S_2 = x^2 + y^2 + 4x + 3y + 2 = 0$

$\therefore\ c_2\ (-g, -f\,)\ =\ \left(-2, -\frac{3}{2}\right)$

$r_2 = \sqrt{g^2 + f^{\,2} - c} = \sqrt{4 + \frac{9}{4} - 2}$

$= \frac{\sqrt{17}}{2} = 2.0615$

Now the distance between their centres

$c_1c_2 = \sqrt{(-1+2)^2 + 0^2} = 1$

$|\,r_1 - r_2\,| = \left|2.0615 - \frac{3}{2}\right| = 0.5615$ and

$r_1 + r_2 = 2.0615 + 1.5 = 3.5615$

$\therefore\ 0.5615 < 1 < 3.5015$

$\therefore\ |\,r_1 - r_2\,| < c_1c_2 < r_1 + r_2$

$\therefore$ We can say that two circles intersect in two distinct points A and B.

$\therefore\ \overline{\text{AB}}$ is a common chord of circles.

The equation of common chord $\overline{\text{AB}}$ is

$\left(x^2 + y^2 + 2x + 3y + 1\right) - \left(x^2 + y^2 + 4x + 3y + 2\right) = 0$

$\therefore\ 2x + 1 = 0$

Here M is the foot of the $\perp^{er}$ from $c_1\left(-1, -\frac{3}{2}\right)$ to the common chord $\overline{\text{AB}}$

$\therefore\ \text{P} = \text{C}_1\text{M} = \frac{|\,2(-1) + 1\,|}{\sqrt{2^2 + 0}} = \frac{1}{2}\ ;\ r_1 = \text{AC}_1 = \frac{3}{2}$

$\therefore$ in $\Delta\text{C}_1\text{MA}\quad \text{AM}^2 = r_1^2 - \text{P}^2 = 2$

$\therefore\ \text{AM} = \sqrt{2}$

$\therefore\ \text{AB} = 2\ (\text{AM})\ =\ 2\sqrt{2}$

So equation of the line containing the common chord is $2x + 1 = 0$ and length of this chord = $2\sqrt{2}$

Illustration 35 Find f and k if the circle $x^2 + y^2 + 2x + fy + k = 0$ touches both axes.

Solution

Here $S = x^2 + y^2 + 2x + fy + k = 0$

Centre $c = \left(-g, -f\right) = \left(-1, -\frac{f}{2}\right)$ and

Radius $r = \sqrt{g^2 + f^2 - c} = \sqrt{1 + \frac{f^2}{4} - k}$

Here circle touches both the axes

$\therefore\ r = \left|\begin{matrix}x\text{-co-ordinate}\\ \text{of centre}\end{matrix}\right| = \left|\begin{matrix}y\text{-co-ordinate}\\ \text{of centre}\end{matrix}\right|$

$\therefore\ \sqrt{1 + \frac{f^2}{4} - k} = |-1| = \left|-\frac{f}{2}\right|$

$\therefore\ 1 + \frac{f^2}{4} - k = 1 = \frac{f^2}{4}$

$\therefore\ f^2 = 4$ and $1 - k = 1$

$\therefore\ f = \pm 2 \qquad \therefore\ k = 0$

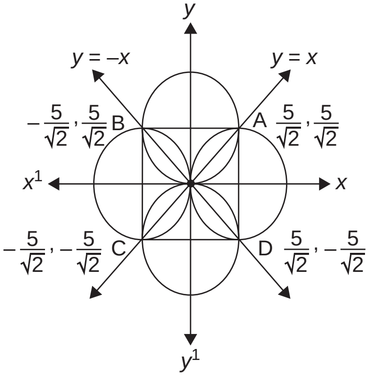


Illustration 36 Get the equation of the circle that passes through the origin and cuts chords of length 8 on x-axis and 6 on y-axis.

Solution

The given circles passe through the origin and they passe through the lengths 8 and 6 on x- and y-axis, respectively.

Here A$\left(8, 0\right)$, A′ $\left(-8, 0\right)$, B $\left(0, 6\right)$ and B′ $\left(0, -6\right)$ are the point of intersection of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ respectively.

Now the equation of circle with diameter $\overline{AB}$ is

$\left(x - 8\right)\left(x - 0\right) + \left(y - 0\right)\left(y - 6\right) = 0$

$\therefore\ x^2 + y^2 - 8x - 6y = 0$

The equation of the circle with diameter $\overline{BC}$ is

$\left(x + 8\right)\left(x - 0\right) + \left(y - 0\right)\left(y - 6\right) = 0$

$\therefore\ x^2 + y^2 + 8x - 6y = 0$

Equation of the circle with diameter $\overline{CD}$ is

$\left(x + 8\right)\left(x - 0\right) + \left(y + 6\right)\left(y - 0\right) = 0$

$\therefore\ x^2 + y^2 + 8x + 6y = 0$

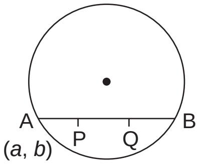
Equation of the circle with diameter $\overline{DA}$ is

$\left(x - 8\right)\left(x - 0\right) + \left(y + 6\right)\left(y - 0\right) = 0$

$\therefore\ x^2 + y^2 - 8x + 6y = 0$

Illustration 37 Get the equation of the circle that passes through the origin and cuts chord of length 5 on the line $y = \pm x \ \overline{\text{BC}}$

Solution



The circle intersects $y = x$ and A and C and $y = -x$ at B and D respectively. Here we get four circles with diameters $\overline{\text{AB}}, \overline{\text{BC}}, \overline{\text{CD}}$ and $\overline{\text{DA}}$ respectively.

Slope of $y = x$ is 1

$\therefore \ m\angle\text{AOX} = \dfrac{\pi}{4}$

Slope of $y = -x$ is -1

$\therefore \ m\angle\text{BOX} = \dfrac{3\pi}{4}$

Co-ordinates of A $= \left(5 \cos 45°, 5 \sin 45°\right) = \left(\dfrac{5}{\sqrt{2}}, \dfrac{5}{\sqrt{2}}\right)$

Co-ordinates of B $= \left(5 \cos 135°, 5 \sin 135°\right) = \left(-\dfrac{5}{\sqrt{2}}, \dfrac{5}{\sqrt{2}}\right)$

Co-ordinates of C $= \left(5 \cos 225°, 5 \sin 225°\right) = \left(-\dfrac{5}{\sqrt{2}}, -\dfrac{5}{\sqrt{2}}\right)$

Co-ordinates of D $= \left(5 \cos 315°, 5 \sin 315°\right) = \left(\dfrac{5}{\sqrt{2}}, -\dfrac{5}{\sqrt{2}}\right)$

$\rightarrow$ The equation of $\overline{\text{AB}}$:

$$\left(x - \frac{5}{\sqrt{2}}\right)\left(x + \frac{5}{\sqrt{2}}\right) + \left(y - \frac{5}{\sqrt{2}}\right)\left(y - \frac{5}{\sqrt{2}}\right) = 0$$

$\therefore \ x^2 + y^2 - 5\sqrt{2}y = 0$

$\rightarrow$ The equation of $\overline{\text{BC}}$:

$$\left(x + \frac{5}{\sqrt{2}}\right)\left(x + \frac{5}{\sqrt{2}}\right) + \left(y - \frac{5}{\sqrt{2}}\right)\left(y + \frac{5}{\sqrt{2}}\right) = 0$$

$\therefore \ x^2 + y^2 + 5\sqrt{2}x = 0$

$\rightarrow$ The equation of $\overline{\text{CD}}$:

$$\left(x + \frac{5}{\sqrt{2}}\right)\left(x - \frac{5}{\sqrt{2}}\right) + \left(y + \frac{5}{\sqrt{2}}\right)\left(y + \frac{5}{\sqrt{2}}\right) = 0$$

$\therefore \ x^2 + y^2 + 5\sqrt{2}y = 0$

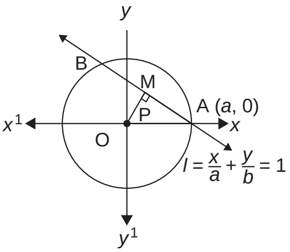
$\rightarrow$ The equation of $\overline{\text{DA}}$:

$$\left(x - \frac{5}{\sqrt{2}}\right)\left(x - \frac{5}{\sqrt{2}}\right) + \left(y - \frac{5}{\sqrt{2}}\right)\left(y + \frac{5}{\sqrt{2}}\right) = 0$$

$\therefore \ x^2 + y^2 - 5\sqrt{2}x = 0$

Illustration 38 If the points of trisection of a chord of the circle $x^2 + y^2 - 4x - 2y - c = 0$ are $\left(\frac{1}{3}, \frac{1}{3}\right)$ and $\left(\frac{8}{3}, \frac{8}{3}\right)$ find C.

Solution



Here $P\left(\frac{1}{3}, \frac{1}{3}\right)$ and $Q\left(\frac{8}{3}, \frac{8}{3}\right)$ are the points of trisection of chord $\overline{AB}$ of the circle $x^2 + y^2 - 4x - 2y - C = 0$

Let the co-ordinate of A be $A\left(a, b\right)$. $P\left(\frac{1}{3}, \frac{1}{3}\right)$ is the midpoint of $\overline{AQ}$

$$\therefore \frac{a + (8/3)}{2} = \frac{1}{3} \Rightarrow a = -2$$

$$\frac{b + (8/3)}{2} = \frac{1}{3} \Rightarrow b = -2$$

$\therefore$ A $(-2, -2)$ lies on the circle

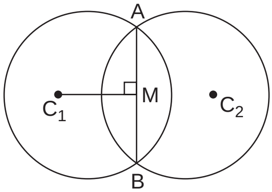
$S: x^2 + y^2 - 4x - 2y + C = 0$

$\therefore (-2)^2 + (-2)^2 - 4(-2) - 2(-2) + C = 0$

$\therefore C = 20$

Illustration 39 If the length of the chord of the circle $x^2 + y^2 = a^2$ cut off on line $y = mx + c$ is $2b$, prove that $\left(1 + m^2\right)\left(a^2 - b^2\right) = c^2$

Solution



Solving $x^2 + y^2 = a^2$ and $y = mx + c$ for (x, y) we get $x^2 + (mx + c)^2 = a^2$

$\therefore \left(1 + m^2\right)x^2 + 2mcx + c^2 - a^2 = 0$ (1)

If (x_1, y_1) and (x_2, y_2) be the intersection of $y = mx = c$ and $x^2 + y^2 = a^2$ then x_1 x_2 are the roots of eq. (1) and

$y_1 = mx_1 + c$ and $y_2 = mx_2 + c$

and from eq. (1),;

$$\alpha + \beta = x_1 + x_2 = \frac{-2mc}{1 + m^2};\ \alpha\beta = x_1x_2 = \frac{c^2 - a^2}{1 + m^2}$$

The length of the chord = $2b$

$\therefore (x_1 - x_2)^2 + (y_1 - y_2)^2 = 4b^2$

$\therefore (x_1 - x_2)^2 + m^2(x_1 - x_2)^2 = 4b^2$

$\left(1 + m^2\right)(x_1 - x_2)^2 = 4b^2$

$\therefore \left(1 + m^2\right)\left[(x_1 + x_2)^2 - 4x_1x_2\right] = 4b^2$

$$\therefore \left(1 + m^2\right)\left[\frac{4m^2c^2}{\left(1 + m^2\right)^2} - \frac{4\left(c^2 - a^2\right)}{1 + m^2}\right] = 4b^2$$

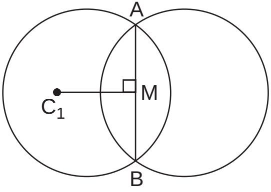
$$\therefore\ \frac{4\left(1+m^2\right)}{\left(1+m^2\right)^2}\left[m^2c^2-\left(c^2-a^2+c^2m^2-a^2m^2\right)\right]=4b^2$$

$\therefore\ a^2 - c^2 + a^2m^2 = b^2\left(1 + m^2\right)$

$\therefore\ a^2\left(1 + m^2\right) - b^2\left(1 + m^2\right) = c^2$

$\therefore\ \left(1 + m^2\right)\left(a^2 - b^2\right) = c^2$

Illustration 40 Find the length of the chord of the circle $x^2 + y^2 = a^2$ on line $\frac{x}{a}+\frac{y}{b}=1$.



Solution

Here line $l:\frac{x}{a}+\frac{y}{b}=1$ and circle $S = x^2 + y^2 = a^2$ intersect x-axis at A $(a, 0)$

$\therefore$ A is one of the end points of the chord

Length of the $\perp^{er}$ P from origin to line $l=\frac{x}{a}+\frac{y}{b}=1$ is

$$\text{OM} = \text{P} = \frac{|c|}{\sqrt{a^2+b^2}} = \frac{|-1|}{\sqrt{\left(1/a^2\right)+\left(1/b^2\right)}}$$

$$\therefore\ \text{OM} = \text{P} = \frac{|ab|}{\sqrt{a^2+b^2}}$$

$$\therefore\ \text{AB} = 2\text{AM} = 2\sqrt{\text{OA}^2-\text{P}^2} = 2\sqrt{a^2-\text{P}^2}$$

$$\therefore\ \text{AB} = 2\sqrt{a^2-\left(\frac{a^2b^2}{a^2+b^2}\right)} = \frac{2a^2}{\sqrt{a^2+b^2}}$$

Illustration 41 Show that the length of the common chord of circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$ is $2\sqrt{\left(a^2-c\right)\left(b^2+c\right)/a^2+b^2}$.

Solution

Here $S_1 = x^2 + y^2 + 2ax + c = 0$

$\therefore$ Centre $c_1\ (-a, 0)$, and $r_1=\sqrt{a^2-c}$

The equation of the line containing common chord is $S_1 - S_2 = 0$

The length of the $\perp^{er}$ from c_1 on common chord $\overline{\text{AB}}$ is

$$\text{C}_1\text{M} = \frac{|-a^2+c|}{\sqrt{a^2+b^2}}$$

Now the length of the common chord is

$$\text{AB} = 2\text{AM} = 2\sqrt{r_1^2-\text{C}_1\text{M}^2} = 2\sqrt{\left(a^2-c\right)-\left(\frac{\left(a^2-c\right)^2}{a^2+b^2}\right)}$$

$$= \sqrt{\left(a^2-c\right)\left(\frac{a^2+b^2-a^2+c}{a^2+b^2}\right)} = 2\sqrt{\frac{\left(a^2-c\right)\left(b^2+c\right)}{a^2+b^2}}$$

Illustration 42 Find the length of the common chords of the circles $(x-a)^2 + (y-b)^2 = c^2$ and $(x-b)^2 + (y-a)^2 = c^2$.

Solution

Equation of the line containing the common-chord is

$S_1 - S_2 = 0$ (i.e. $x - y = 0$)

The length of the $\perp^{er}$ from centre c_1 (a, b) of the first circle on the common chord $x - y = 0$

$$\therefore \; C_1M = \frac{|a-b|}{\sqrt{1+1}} = \frac{|a-b|}{\sqrt{2}}$$

$$\text{Now AB} = 2\text{AM} = 2\sqrt{r_1^2 - C_1M^2} = 2\sqrt{c^2 - \frac{(a-b)^2}{2}}$$

$$= \sqrt{4c^2 - 2(a-b)^2}$$

ANALYTICAL EXERCISES

1. Get the equation of the circle with radius 5 and touching x-axis at the origin.

2. Obtain the equation of the circle with centre $(2, -1)$ and passing through the point $(3, 6)$.

3. Find the equation of the circle given that its radius is 5, its centre is on the x-axis and that it passes through $(2, 3)$.

4. Prove that the centre of the circles $x^2 + y^2 = 1$; $x^2 + y^2 + 6x - 2y = 1$ and $x^2 + y^2 - 12x + 4y = 1$ are collinear points.

5. Obtain the equations of circles that touch both the axes and pass through $(-4, -2)$.

6. Find the equation of the line containing a chord of the circle $x^2 + y^2 = 25$ given that the midpoint of the chord is $(2, -3)$.

7. Find the equation of the circle passing through the point $(1, 2)$, $(3, -4)$ and $(5, -6)$.

8. A circle passes through the origin and cuts off chords of length 3 and 4 on x and y-axis respectively. Find the equation of circle.

9. Find the equation of a circle if two of its diameters are the diagonals of the square formed by the lines $x = 1$, $x = 3$, $y = 2$, $y = 4$.

10. Find the length of the chord of the circle $x^2 + y^2 = 9$ cut off by the line $x - y + 2 = 0$.

11. Find the length of the chord of the circle $x^2 + y^2 = 9$ cut off by the line $x - y + 2 = 0$.

12. Find the intersection set of line $y = 2x - 3$ and the circle $x^2 + y^2 - 3x - 2y - 3 = 0$.

13. Determine the position of the points $(0, 1)$ $(3, 1)$, $(1, 3)$ relative to the circle $x^2 + y^2 - 2x - 4y + 3 = 0$.

14. Get the equation of the circle passing through the points $(1, 0)$, $(-1, 0)$ and $(0, 1)$.

15. Get the equation of the circle passing through $(5, 5)$, $(6, 4)$ and $(-2, 4)$.

16. If one end of a diameter of the circle $x^2 + y^2 - 4x - 6y + 11 = 0$ is $(3, 4)$ find the other end.

17. Show that the points $(4, 3)$, $(8, -3)$ and $(0, 9)$ cannot be all on the same circle.

18. Find the equation of the tangent to the circle $x^2 + y^2 = 169$ at the point A $(12, -5)$

19. Find the equations of the tangents to the circle $x^2 + y^2 = 9$ from the tangents to the circle $x^2 + y^2 = 9$ from the point $\left(3, 2\right)$.

20. $y = 2x + c$ is a tangent to the circle $x^2 + y^2 = 5$ find C.

21. Find the tangent to the circle $x^2 + y^2 = 9$ that is parallel to the line $2x + y = 3$.

22. For the circle $x^2 + y^2 = 64$, find the tangent that is parallel to the y-axis.

23. Find the radius of the circle of which $12x + 5y + 16 = 0$ and $12x = 5y - 10 = 0$ are tangents.

24. Obtain the equation of the tangents to the circle $x^2 + y^2 = 16$ drawn from the point $\left(4, 7\right)$.

25. Obtain the equation of the tangent to the circle $x^2 + y^2 = 20$ drawn from the point $\left(4, 2\right)$.

26. How many tangents to the circle $x^2 + y^2 = 29$ can pass through the point $\left(5, 2\right)$?

27. Find the length of the tangent from $\left(6, -5\right)$ to $x^2 + y^2 = 49$.

28. If the circle $x^2 + y^2 + 2g_1x + 2f_1y = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y = 0$ touch each other, then prove that $f_1g_2 = f_2g_1$.

29. Find the length of the tangent from $\left(-2, 4\right)$ to the circle. $2x^2 + 2y^2 = 3$.

30. Find the equation and the length of the tangent from $\left(4, 0\right)$ to the circle. $x^2 + y^2 = 4$.

31. If the length from P $\left(3, 4\right)$ to the circles $x^2 + y^2 = 9$ touch the circle at A and B, find the area of ΔAPB.

32. If the length of tangents drawn from P to the circle $x^2 + y^2 = a^2$; $x^2 + y^2 = b^2$; $x^2 + y^2 = c^2$ are in A.P, show that a^2, b^2, c^2 must be in A.P.

33. Show that the circles $x^2 + y^2 + 6x + 2y - 90 = 0$ and $x^2 + y^2 - 34x - 28y + 260 = 0$ touch each other externally.

34. Determine whether the circles $x^2 + y^2 - 4x + 6y + 8 = 0$ and $x^2 + y^2 - 10x - 6y + 14 = 0$ touch each other or intersect in distinct points.

35. If $a > 0$, find the equation of circle that touches each of the lines $x = 0$; $x = a$ and $3x + 4y + 5a = 0$.

36. Find the maximum and minimum distances of the point $\left(-7, 2\right)$ from point on circle $x^2 + y^2 - 10x - 14y - 151 = 0$.

37. Get the equation of the circle touching both the axes and also touching the line $3x + 4y - 6 = 0$ in the first quadrant.

38. Obtain the equation of the circle with centre on the line $2x + y = 0$ and touching the lines $4x - 3y + 10 = 0$ and $4x - 3y - 30 = 0$.

39. Prove that if the circles $x^2 + y^2 = r^2$ and $x^2 + y^2 - 10x + 16 = 0$ intersect in two distinct points, then $2 < r < 8$.

40. Get the equation of the circle passing through $\left(1, 0\right)$, $\left(0, -6\right)$ and $\left(3, 4\right)$.

41. Get the equation of the line circle that touches the lines $2x + y + 2 = 0$ and $2x + y - 18 = 0$ and passes through $\left(1, 0\right)$.

42. Find the equation of the circle that touches the x-axis and passes through $\left(1, -2\right)$ and $\left(3, -4\right)$.

43. Find the equation of the incircle of the triangle formed by the line $x = 2$; $4x + 3y = 5$ and $4x - 3y + 13 = 0$.

44. Prove that the area of equilateral triangle inscribed in the circle $2x^2 + 2y^2 - 12x + ky + 18 = 0$ touches the x-axis.

45. M $\left(x_1, y_1\right)$ is the midpoint of a chord of the circle $x^2 + y^2 = a^2$. Find the equation of the line containing this chord.

46. The midpoint of a chord of the circle $x^2 + y^2 = 81$ is $\left(-6, 3\right)$. Get the equation of the line containing this chord.

47. Find the condition for line $lx + my = n = 0$; $l^2 + m^2 \neq 0$ to be a tangent to $x^2 + y^2 = a^2$. Also find the co-ordinates of the point of contact.

48. Find the equation of the line containing the common chord of circles $S_1 = x^2 + y^2 + bx + ay + c = 0$ and $S_2 = x^2 + y^2 + ax + by + c = 0$. Hence find the condition for the circles to touch each other.

ANSWERS

(1) $x^2 + y^2 \pm 10y = 0$
(2) $x^2 + y^2 - 4x + 2y - 45 = 0$
(3) $x^2 + y^2 - 12x + 11 = 0,$
$x^2 + y^2 - 4x - 21 = 0$
(5) $x^2 + y^2 + 20x + 20y + 100 = 0,$
$x^2 + y^2 - 4x - 4y = 0$
(6) $2x - 3y - 13 = 0$
(7) $x^2 + y^2 - 22x - 4y - 15 = 0,$
(8) $x^2 + y^2 + 3x \pm 4y = 0$
(9) $x^2 + y^2 - 4x + 6y - 11 = 0$
(10) $6, 2\sqrt{4}$
(11) $2\sqrt{7}$
(12) $\left(0, -3\right), \left(\frac{11}{5}, -\frac{7}{5}\right)$
(13) one outside, inside the circle
(14) $x^2 + y^2 = 1$
(15) $x^2 + y^2 - 4x - 2y - 20 = 0$
(16) $\left(1, 2\right)$
(17) $\left(17, 2\right)$
(18) $12x - 5y = 169$
(19) $5x + 12y - 39 = 0$
(20) $y = \pm 5$
(21) $2x + y \pm 3\sqrt{5} = 0$
(22) ± 8
(23) 1
(24) $33x - 53y + 260 = 0, x = 4$
(25) $2x + y = 10$
(26) One
(27) $2\sqrt{3}$
(29) $\sqrt{\frac{37}{2}}$
(30) $\sqrt{3}\, y = \pm\left(x - 4\right), 2\sqrt{3}$
(31) 7.68
(34) Touch externally
(35) $16x^2 + 16y^2 - 16ax + 72ay + 81a^2 = 0$
(36) 28 and 2
(37) $x^2 + y^2 - x - y + \frac{1}{4} = 0$
(38) $x^2 + y^2 - 2x + 4y - 11 = 0$
(40) $4x^2 + 4y^2 + 142x + 47y + 138 = 0$
(41) $5x^2 + 5y^2 - 18x - 44y + 13 = 0$
(42) $x^2 + y^2 + 10x + 20y + 25 = 0,$
$x^2 + y^2 - 6x + 4y + 9 = 0$
(43) $3x^2 + 3y^2 - 4x - 8y + 23 = 0$
(45) $xx_1 + yy_1 = x_1^2 + y_1^2$
(46) $2x + y + 5 = 0, 12$
(47) $a^2\left(l^2 + m^2\right)$
(48) $\sqrt{4c^2 - 2\left(a - b\right)^2}$

18

Parabola

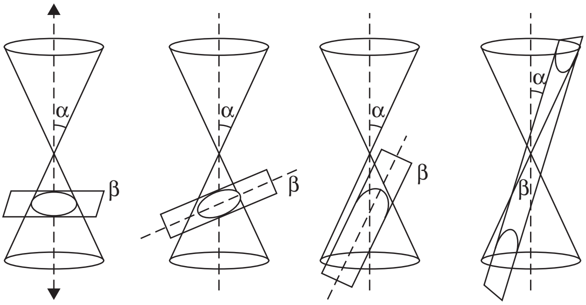
LEARNING OBJECTIVES

After studying this chapter, the student will be able to understand:

- Section of double cone by a plane
- The standard equation and parametric equation of a parabola
- Focus, diametric, latus rectum of a parabola
- Equation of a tangent at point (x_1, y_1) and t-point of a parabola
- Necessary and sufficient condition for a line $y = mx + c$ $(c \neq 0)$ to be a tangent to the parabola and the co-ordination of point of contact
- Properties of a parabola

INTRODUCTION

Conic section (Section of double cone by plane)



$\beta = \frac{\pi}{2}$ (Circle) $\quad$ $\alpha < \beta < \frac{\pi}{2}$ Ellipse $\quad$ $\alpha = \beta$ Parabola $\quad$ $\alpha\ 0 < \beta < \alpha$ Hyperbola

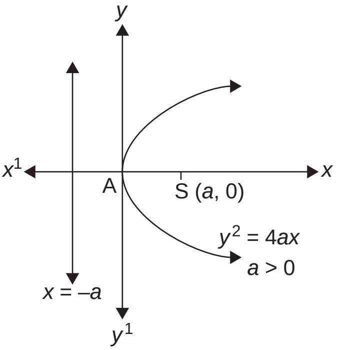
CONIC SECTION

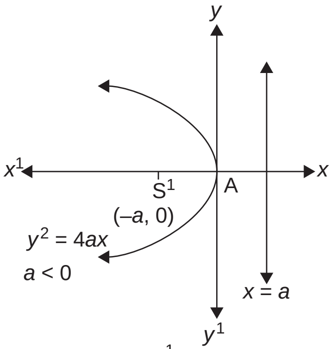
Suppose *l* be *a* fixed line in a plane and *s* be a fixed point in the plane not on the line *l*. The set of points in the plane, the ratio of whose distance from *s* and $\perp^{er}$ distance from *l* is constant is called a conic section. Here the fixed line is called directrix and the fixed point is called focus of the conic section and the constant ratio is called eccentricity (*e*) of the conic section.

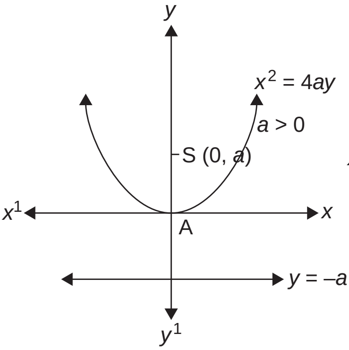
$e = 1$ then conic section is said to be a parabola
$0 < e < 1$ then conic section is said to be an ellipse
$e > 1$ then conic section is said to be a hyperbola

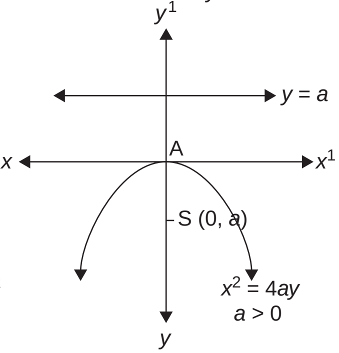
Definitions

1. Set of all points in the plane which are equidistant from a fixed line and a fixed point not on the line is called a parabola. The fixed point is called the focus of the parabola.
2. The axis and the vertex of a parabola
   The line $\perp^{er}$ to the directrix of the parabola and passing through its focus is called the axis of the parabola. The point of intersection of a parabola with its axis is



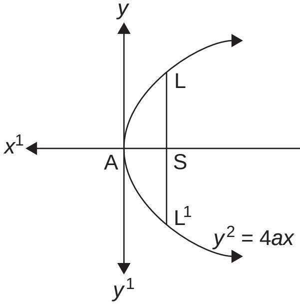


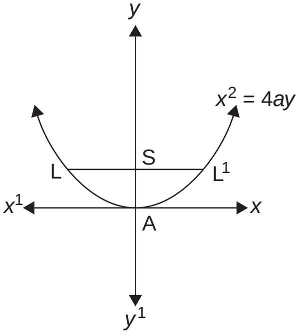




Standard Equation of a Parabola

1. If the origin is taken at the vertex and x-axis along the axis of the parabola, then the equation of the parabola is $y^2 = 4ax$, $a \in R - (0)$. Its focus is S $(a, 0)$ and its directrix is $x = -a$.
2. If the origin is taken at the vertex of the parabola then the equation of parabola is $x^2 = 4ay$; $a \in R - (0)$. Its focus is S $(0, a)$ and its directrix is $y = -a$.





Graphs of a Parabola

The symmetry of a parabola about its *x*-axis: It point (x, y) is on parabola $y^2 = 4ax$ then point $(x_1, -y)$ is also on parabola. This property is known as symmetry of the curve with respect to *x*-axis which is also axis of the parabola.

Chord: The line segment joining any two distinct points on a parabola is known as a chord of the parabola.

Focal chord: If a chord of a parabola passes through its focus, then that chord is called the focal chord of the parabola.

Π **Latus rectum:** A focal chord of the parabola, $\perp^{er}$ to the axis of the parabola is called the latus rectum of the parabola.

Π **Length of latus rectum:** For $y^2 = 4ax$; $a > 0$ length of latus rectum is $4a$. The co-ordinates of the end-points of the latus rectum are $(2a, a)$ and $(-2a, a)$.

Parametric equations: $(x, y) = (at^2, 2at)$; $t \in$ R are the parametric equations of a parabola. The point P $(at^2, 2at)$, is called parametric point on the parabola and it is written as point P(t) on the parabola. *t* is said to be the parameter.

Equation of tangent to the parabola:

(i) Equation of tangent to the parabola $y^2 = 4ax$ at P(x_1, y_1) is $yy_1 = 2a(x + x_1)$

(ii) The equation of tangent to the parabola $y^2 = 4ax$ at point P (t) on the parabola is $y = \frac{1}{t} x + at \ (t \neq 0)$.

3. If line $y = mx + c$ is a tangent to the parabola $y^2 = 4ax$ then $c = \frac{a}{m}$ and the co-ordinates of the point of contact are $\left(\frac{a}{m^2}; \frac{2a}{m}\right)$.
4. Equation of tangent, having slope *m*, to the parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$; $m \neq 0$.

Properties of Parabola

1. If the tangent at point P other than the vertex, intersects *x*-axis at T then SP = ST. Here S is the focus of parabola.
2. The foot of the $\perp^{er}$ from the focus to tangent at any point on the parabola is on the tangent to the parabola at the vertex.

3. The tangents at the end-points of a focal chord of the parabola intersect each other orthogonally on directrix.
4. If $\overline{PQ}$ is a focal chord of the parabola $y^2 = 4ax$, then $\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a}$ $(a > 0)$.
5. If $\overleftrightarrow{PQ}$ is $\perp^{er}$ to the tangent at P to the parabola $y^2 = 4ax$ $\overrightarrow{PX^1}$ and is parallel to the axis of the parabola, then $m\angle SPG = m\angle X^1PG$.

ILLUSTRATIONS

Illustration 1 For each parabola below obtain co-ordinates of the focus; equation of directrix, length of the latus rectum and the end point of the latus rectum.

(a) $y^2 = 16x$ (b) $x^2 = 24y$ (c) $y^2 = -8x$ (d) $x^2 = -6y$.

Solution

(a) $y^2 = 16x \Rightarrow y^2 = 4(4)x$

$\therefore a = 4$

$\therefore$ Focus S $(4, 0)$

The equation of directrix is $x = -a$ $\therefore x = -4$

Length of latus rectum $= 4a = 16$

Co-ordinates of latus rectum $= (a, \pm 2a) = (4, \pm 8)$

(b) $x^2 = 24y \Rightarrow x^2 = 4(6)\,y$

$\therefore a = 6$

$\therefore$ Focus S $(a, 0) = (6, 0)$

Equation of directrix: $y = -a \Rightarrow y = -6$

Length of latus rectum $= 4a = 24$

End point of latus rectum $(\pm 2a, a) = (\pm 12, 6)$

(c) $y^2 = -8x \Rightarrow y^2 = 4(-2)x$

$\therefore a = -2$

$\therefore$ Focus S $(a, 0) = (-2, 0)$

Equation of directrix: $x = -a = -(-2) = 2$

Length of latus rectum $= |4a| = 8$

End point of Latus rectum: $(-2, 4)$ and $(-2, -4)$

(d) $x^2 = -6y = 4\left(-\frac{3}{2}\right)y$

$\therefore\ a = -\frac{3}{2}$

$\therefore$ Focus S $(0, a) = \left(0, -\frac{3}{2}\right)$

$\therefore$ Equation of directrix: $y = \frac{3}{2}$

$\therefore$ Length of latus rectum $= |4a| = \left|4\left(-\frac{3}{2}\right)\right| = 6$

$\therefore$ End point of latus rectum: $\left(3, -\frac{3}{2}\right)\left(-3, -\frac{3}{2}\right)$

Illustration 2 Find the equation of parabola given the following and given that origin at the vertex of the parabola.

(1) Focus $(4, 0)$, directrix $x = -4$

(2) Focus $(0, -2)$ directrix $y = 2$

(3) Passes through $(2, 3)$ and is symmetric about x-axis.

(4) Passes through $(2, -3)$ and is symmetric about y-axis.

Solution

(1) $S\ (a, 0) = S\ (4, 0);\ x = -4$

$\therefore y^2 = 4ax = 4(4)x = 16x$

(2) $S\ (0, -2)$, direcrix $= y = 2$

$a = -2$

$\therefore x^2 = 4ay = -8y$

(3) Equation of parabola: $y^2 = 4ax$ $(2, 3)$ is on the parabola

$\therefore (3)^2 = 4a(2) \Rightarrow a = \frac{9}{8}$

$\therefore$ The equation of the parabola is $y^2 = 4\left(\frac{9}{8}\right)x$

$\therefore y^2 = \frac{9}{2}x$

$\therefore 2y^2 = 9x$

(4) The equation of parabola with respect to y-axis is $x^2 = 4ay$; But $(2, 3)$ is on the parabola

$\therefore 2^2 = 4a(-3) \Rightarrow a = -\frac{1}{3}$

$\therefore$ The equation of parabola is $x^2 = -\frac{4}{3}y$

Illustration 3 For the parabola $x^2 = 12y$, find the area of the triangle whose vertices are the vertex of the parabola and the two end-points of its latus rectum.

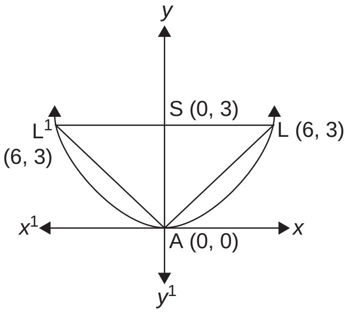
Solution

For the parabola $x^2 = 12y = 4(3)y$

$\therefore$ The end point of latus rectum L $(6, 3)$ and L^1 $(-6, 3)$ and A $(0, 0)$ is the vertex.

$\therefore$ For ΔALL^1

$$D = \begin{vmatrix} 0 & 0 & 1 \\ 6 & 3 & 1 \\ -6 & 3 & 1 \end{vmatrix} = 36$$



$$\therefore \text{ Area of } \Delta \text{ALL}^1 = \frac{1}{2}|D|$$
$$= \frac{1}{2}|36| = 18$$

Illustration 4 For the parabola $y^2 = 4ax$, one of the end-points of a focal chord is $\left(at_1^2, 2at_1\right)$. Find the other end and show that the length of this focal chord is $a\left(t_1 + \frac{1}{t_1}\right)^2$.

Solution

Let one end point of focal chord is $\mathrm{P}\left(t_1\right) = \mathrm{P}\left(at_1^2, 2at_1\right)$ then the other end point is $\mathrm{Q}\left(t_2\right) = \mathrm{Q}\left(-\frac{1}{t_1}\right) = \mathrm{Q} = \left(\frac{a}{t_1^2}; -\frac{2a}{t_1}\right) \qquad \left(\because t_1 t_2 = -1\right)$

$$\therefore \text{PQ}^2 = \left(at_1^2 - \frac{a}{t_1^2}\right)^2 + \left(2at_1 - \frac{2a}{t_1}\right)^2$$
$$= a^2\left(t_1^2 - \frac{1}{t_1^2}\right)^2 + 4a^2\left(t_1 + \frac{1}{t_1}\right)^2$$
$$= a^2\left(t_1 + \frac{1}{t_1}\right)^2\left[\left(t_1 - \frac{1}{t_1}\right)^2 + 4\right]$$
$$= a^2\left(t_1 + \frac{1}{t_1}\right)^2\left(t_1 + \frac{1}{t_1}\right)^2 = a^2\left(t_1 + \frac{1}{t_1}\right)^4$$
$$\therefore \text{PQ} = a\left(t_1 + \frac{1}{t_1}\right)^2$$

Illustration 5 If the line $3x + 4y + 16 = 0$ is tangent to the parabola $y^2 = kx$, find k and the point of contact.

Solution

$3x + 4y + 16 = 0 \Rightarrow 4y = -3x - 16$

$\Rightarrow y = -\frac{3}{4}x - 4$

From above equation, we can say that $m = -\frac{3}{4}$; $c = -4$

For parabola $y^2 = kx \qquad \therefore a = \frac{k}{4}$

The condition that $y = mx + c$ becomes tangent to the parabola $y^2 = 4ax$ is $c = \frac{a}{m}$

$\therefore -4 = \frac{\left(k/4\right)}{\left(-3/4\right)} \qquad \therefore k = 12$

$\therefore a = \frac{k}{4} = \frac{12}{4} = 3$

$$\therefore \text{ Point of contact } \left(\frac{a}{m^2}, \frac{2a}{m}\right)$$

$$= \left(\frac{3}{(9/16)}; \frac{6}{(-3/4)}\right) = \left(\frac{16}{3}, -8\right)$$

Illustration 6 For the parabola $y^2 = 6x$ find the equation of the tangents from the point $\left(\frac{3}{2}, 5\right)$ and find also the point of contact.

Solution

Here $\left(\frac{3}{2}, 5\right)$ does not satisfy the equation $y^2 = 6x$

$\therefore \left(\frac{3}{2}, 5\right)$ is not on the parabola for $y^2 = 6x = 4\left(\frac{3}{2}\right)x$

$\therefore\ a = \frac{3}{2}$

Suppose the equation of the tangent is $y = mx + \frac{a}{m}$

But the tangent passes through $\left(\frac{3}{2}, 5\right)$

$$5 = m\left(\frac{3}{2}\right) = \frac{(3/2)}{m}$$

$\therefore\ 10m = 3m^2 + 3$

$\therefore\ 3m^2 - 10m + 3 = 0$

$\therefore\ (3m - 1)\ (m - 3) = 0$

$\therefore\ m = \frac{1}{3}$ or $m = 3$

Now taking $m = 3$ equation of tangent is

$$y = 3x + \frac{3}{2}\left(\frac{1}{3}\right)$$

$$y = 3x + \frac{1}{2}$$

$\therefore\ 6x - 2y + 1 = 0$

Now the point of contact of $6x - 2y + 1 = 0$ is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

$$= \left(\frac{(3/2)}{9}, \frac{3}{3}\right) = \left(\frac{1}{6}, 1\right)$$

Now taking $m = \frac{1}{3}$ equation of tangent is

$$y = \frac{x}{3} + \frac{3}{2(1/3)} = \frac{x}{3} + \frac{9}{2}$$

$\therefore\ 2x - 6y + 27 = 0$

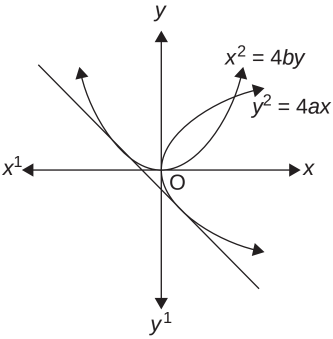
Now the point of contact of $2x - 6y + 27 = 0$ is

$$\left(\frac{a}{m^2}; \frac{2a}{m}\right) = \left(\frac{(3/2)}{(1/9)}; \frac{3}{(1/3)}\right)$$

$$= \left(\frac{27}{2}, 9\right)$$

Illustration 7 Prove that the equation of the line that is a tangent to both $y^2 = 4ax$ and $x^2 = 4by$ is $a^{1/3}x + b^{1/3}y + (ab)^{2/3} = 0$.

Solution



Here $y = mx + \frac{a}{m}$ is a tangent to the parabola

$y^2 = 4ax$; for all $m \in$ R; $m \neq 0$

Let us find the value of m such that the line also becomes a tangent to $x^2 = 4by$

Now for unique solution of

$$y = mx + \frac{a}{m} \text{ and } x^2 = 4by$$

$$\therefore\ x^2 = 4b\left(mx + \frac{a}{m}\right)$$

$$\therefore\ x^2 - 4bmx - \frac{4ab}{m} = 0$$

If this equation has one and only one root

$$\therefore\ \Delta = b^2 - 4ac = (-4bm)^2 - 4(1)\left(-\frac{4ab}{m}\right) = 0$$

$$\therefore\ m^3 = -\frac{a}{b} \Rightarrow m = -\frac{a^{1/3}}{b^{1/3}}0$$

Now substituting $y = mx + \frac{a}{m}$, the required equation of tangent is

$$y = mx + \frac{a}{m}$$

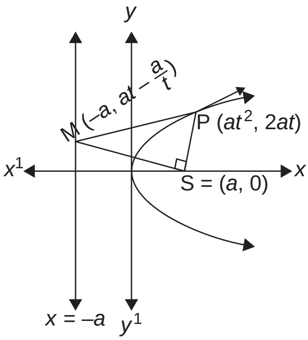
$$= -\frac{a^{1/3}}{b^{1/3}}x + \frac{a}{-a^{1/3}}b^{1/3}$$

$$\therefore\ b^{1/3}y = -a^{1/3}x - a^{2/3}b^{2/3}$$

$$\therefore\ a^{1/3}x + b^{1/3}y + (ab)^{2/3} = 0$$

Illustration 8 Prove that the line-segment on any tangent to the directrix subtends a right angle at the focus.

Solution



The tangent to the parabola at P $(at^2, 2at)$ intersects at directrix at M

Now the equation of the tangent to the parabola at point t is $y = \frac{x}{t} + at$ (1)

The equation of the directrix is $x = -a$ (2)

Now solving (1) and (2) the co-ordinates at

$$M\left(-a, at - \frac{a}{t}\right)$$

Co-ordinates of S and P are S $(a, 0)$, P $(at^2, 2at)$, respectively

$$\therefore \text{ Slope of } \overline{\text{SP}} = m = \frac{2at - 0}{at^2 - a} = \frac{2t}{t^2 - 1}$$

$$\text{Slope of } \overline{\text{SM}} = m^1 = \frac{at - (a/t) - 0}{-a - a} = \frac{t^2 - 1}{-2t}$$

$$\therefore m\,m^1 = \left(\frac{2t}{t^2 - 1}\right)\left(\frac{t^2 - 1}{-2t}\right) = -1$$

$\therefore \overline{\text{SP}} \perp \overline{\text{SM}}$

$\therefore m\angle\text{PSM} = 90°$

$\therefore \overline{\text{PM}}$ subtends right angle at S.

Illustration 9 Find the equation of the tangent at the end-point of the latus rectum of the parabola $y^2 = 4ax$.

Solution

End points of latus rectum are L = L $(a, 2a)$, $\text{L}^1 = \text{L}^1$ $(a - 2a)$

The equation of tangent at L $(a, -2a)$ is

$yy_1 = 2a(x + x_1)$

$\therefore y(2a) = 2a(x + a)$

$\therefore x - y + a = 0$

Illustration 10 For the parabola $y^2 = 16x$ (1) obtain the parametric eq. (2) If $x + y + k = 0$ is the equation of the line containing a focal chord, find k.

Solution

$y^2 = 16x \Rightarrow y^2 = 4(4)x \Rightarrow a = 4$

(1) Parametric equations are

$x = at^2 = 4t^2$ and $y = 2at = 8at$

(2) $x + y + k = 0$ is the line containing a focal chord and hence it passes through S $(4, 0)$

$\therefore 4 + 0 + k = 0 \Rightarrow k = -4$

Illustration 11 Prove that the y-co-ordinate of the point of intersection of any two tangents to a parabola is the arithmetic mean of the y-co-ordinates of their point of contact.

Solution

Let P (t_1) and Q (t_2) are intersect tangents of the parabola at T.

$\therefore$ P = P $(at_1^2, 2at_1)$; Q = Q $(at_2^2, 2at_2)$

$\therefore$ The equation of tangent at P is

$$y = \frac{1}{t_1} x + at_1 \qquad (1)$$

The equation of tangent at Q is

$$y = \frac{t}{t_2} x + at_2 \qquad (2)$$

Solving (1) and (2) we get

$x = at_1t_2;\ y = a\left(t_1 + t_2\right)$

$\therefore$ Co-ordinates of T = $\left[at_1t_2;\ a\left(t_1 + t_2\right)\right]$

Now A.M of y-co-ordinates of P and Q

$$= \frac{2at_1 + 2at_2}{2} = a\left(t_1 + t_2\right) = y\text{-co-ordinates of T}$$

Illustration 12 If a chord of $y^2 = 4ax$ subtends a right angle at the vertex of the parabola, prove that the tangents at the end-point of the chord intersect on the line $x + 4a = 0$.

Solution

Here let

$P\left(t_1\right) = \left(at_1^2, 2at_1\right)$ and

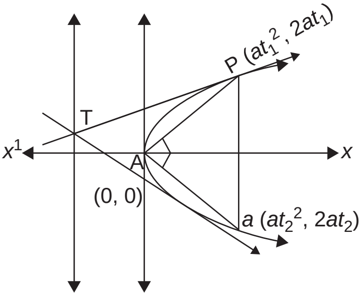
$a\left(t_2\right) = \left(at_2^2, 2at_2\right)$ are end-points of chord

$\overline{PQ}$ and $m\angle PAQ = 90°$

$\therefore\ \overline{PA} \perp \overline{AQ}$

$\therefore\ \left(\text{Slope of } \overline{PA}\right)\left(\text{Slope of } \overline{AQ}\right) = -1$

$$\therefore\ \left(\frac{2at_1 - 0}{at_1^2 - 0}\right)\left(\frac{2at_2 - 0}{at_2^2 - 0}\right) = -1$$



$\therefore\ t_1t_2 = -4$

Now the equations of tangents to the parabola at P and Q are

$$y = \frac{1}{t_1} x + at_1 \qquad (1)$$

$$y = \frac{1}{t_2} x + at_1 \qquad (2)$$

By solving eqs. (1) and (2) we get the point of intersection T are

$T\left[at_1t_2,\ a\left(t_1 + t_2\right)\right] = T\left(x, y\right)$

$\therefore\ x = at_1t_2 = -4a \qquad \left(\because t_1t_2 = -4\right)$

$\therefore\ x + 4a = 0$

Illustration 13 One end-point of a focal-chord of the parabola $y^2 = 16x$ is $\left(4, 8\right)$. Find the other end-point and the length of the focal-chord.

Solution

For parabola $y^2 = 16x = 4\left(4\right) x;\ a = 4$

Now co-ordinates of P and Q are

$P\left(at_1^2, 2at_1\right) = \left(4t_1^2, 8t_1\right)$; $Q\left(at_2^2, 2at_2\right) = \left(4t_2^2, 8t_2\right)$; co-ordinates of P = $\left(4, 8\right)$

$\therefore\ \left(4t_1^2, 8t_1\right) = \left(4, 8\right)$

$\therefore\ 4t_1^2 = 4$ and $8t_1 = 8$

$\therefore\ t_1 = 1$

$\therefore\ t_2 = -1 \qquad (\because t_1t_2 = -1)$

$\therefore$ Co-ordinates of $Q = \left[4(-1)^2, 8(-1)\right] = (4, -8)$

$\therefore$ The length of focal chord $PQ = \sqrt{(4-4)^2 + (8+8)^2} = 16$

Illustration 14 Show if a pair of tangents to the parabola $y^2 = 4ax$ cuts intercepts of equal length on the two axes, they are $\perp^{er}$ to each other and that the two tangents the directrix and the axis of the parabola $y^2 = 12x$.

Solution

The tangents at P and Q intersect each other at Z. As they make equal intercepts slope $m = \pm\ 1$

$\therefore$ The slope of the tangent at P is 1

$\therefore$ Equation of tangent at P is

$$y = mx + \frac{a}{m}$$

$$\therefore\ y = x + a \qquad (1)$$

The slope of the tangent at Q is -1

$\therefore$ Equation of tangent at Q is

$$y = mx + \frac{a}{m} = -x - a \qquad (2)$$

Now by solving (1) and (2) we get

$Z\ (x, y) = (-a, 0)$

Z lies on the directrix $x = -a$ and also x-axis.

Further product of the slopes of two tangents $(1)\ (-1) = -1$

$\therefore$ They are $\perp^{er}$ to each other.

$\therefore$ The tangents intersect orthogonally on the directrix and x-axis.

Illustration 15 Find the equation of the line containing a chord of the parabola $y^2 = 8x$ given that $(3, 1)$ is the midpoint of the chord.

Solution

Parabola $y^2 = 8x = 4(2)x$

$\therefore\ a = 2$

$(3, 1)$ is the midpoint of the chord $\overline{PQ}$

and $P\ (at_1^2, 2at_1) = (2t_1^2, 4t_1)$

$Q\ (at_2^2, 2at_2) = (2t_2^2, 4t_2)$

$$\therefore\ 3 = \frac{2t_2^2 + 2t_2^2}{2} \text{ and } 1 = \frac{4t_1 + 4t_2}{2}$$

Now, $t_1^2 + t_1^2 = 3$ and $t_1 + t_1 = \frac{1}{2}$

$$\text{Slope of } \overline{PQ} = \left(\frac{4t_1 - 4t_2}{2t_1^2 - 2t_2^2}\right)$$

$$= \frac{4(t_1 - t_2)}{2(t_1 - t_2)(t_1 + t_2)}$$

$$= \frac{2}{t_1 + t_2} = \frac{2}{(1/2)} = 4$$

$\therefore$ $\overline{PQ}$ passes through $(3, 1)$

$\therefore$ The equation of $\overline{PQ}$ is

$y - 1 = 4(x - 3)$

$\therefore y - 1 = 4x - 12$

$\therefore 4x - y - 11 = 0$

Illustration 16 Get the equation of the common tangent of the parabolas $y^2 = 32x$ and $x^2 = 108y$.

Solution

In $y^2 = 32x$ $\quad a = 8$

and $x^2 = 108y$ $\quad a = 27$

The equation of tangent to the parabola is

$$y = mx + \frac{a}{m} = mx + \frac{8}{m}$$

and also for $x^2 = 108y$

$$\therefore x^2 = 108y = 108\left(mx + \frac{8}{m}\right)$$

$$\therefore x^2 = 108mx + \frac{864}{m}$$

$\therefore mx^2 - 108m^2x - 864 = 0$

Now $\Delta = 0$

$\therefore b^2 - 4ac = 0$

$\therefore (108m^2)^2 - 4(m)(-864) = 0$

$\therefore 27m^3 + 8 = 0$

$$\therefore m = -\frac{2}{3}$$

Now substituting $m = -\frac{2}{3}$ in $y = mx + \frac{a}{m}$

$$\therefore y = -\frac{2}{3}x + \frac{8}{(-2/3)}$$

$\therefore 2x + 3y + 36 = 0$

Illustration 17 Prove that for all focal chords, the product of x-co-ordinates of the two end-points is constant.

Solution

In parabola $y^2 = 4ax$ $\overline{PQ}$ is a focal chord and co-ordinates of focal chord are P (t_1) = P $(at_1^2, 2at_1)$, Q (t_2) = Q $(at_2^2, 2at_2)$ and S $(a, 0)$.

∴ S, P, Q are collinear

$$\therefore \begin{vmatrix} a & 0 & 1 \\ at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \end{vmatrix} = 0$$

$$\therefore \left(t_1 - t_2\right)\left(1 + t_1t_2\right) = 0 \qquad \left(\because a \neq 0\right)$$

But $t_1 \neq t_2 \therefore t_1t_2 = -1 \Rightarrow t_2 = -\frac{1}{t^1}$

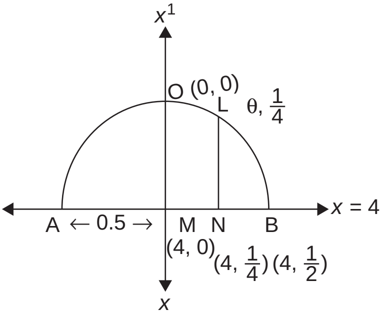
∴ The co-ordinates of P and Q are $\left(at_1^2, 2at_1\right)$ and $\left(\frac{a}{t_1^2}, \frac{-2a}{t_1}\right)$

∴ The product of *x*-co-ordinates of P and Q is

$$\left(at_1^2\right)\left(\frac{a}{t_1^2}\right) = a^2 = \text{constant}$$

Illustration 18 At a water fountain, water attains a maximum height of 4 *m* at horizontal distance of 0.5 *m* from its origin. If the path of water is a parabola, find the height of water at a horizontal distance of 0.75 *m* from the point of origin.

Solution



Here A is the origin of water fountain. $\overline{\text{AB}}$ is the path of water where B is the terminal point. O is the highest point and M is the midpoint.

OM = 4*m* AM = $\frac{1}{2}m$

If LN is the highest of water at a distance of 0.75 *m* from A on $\overline{\text{AB}}$.

AM = $\frac{3}{4}m_1$ MN = $\frac{1}{4}m$

Let O be the origin and OM be the *x*-axis

∴ M (4, 0), N $\left(4, \frac{1}{4}\right)$ and B $\left(4, \frac{1}{2}\right)$ lie on the parabola $y^2 = 4ax$

$$\therefore \left(\frac{1}{2}\right)^2 = 4a(4)$$

$$\therefore a = \frac{1}{64}$$

∴ The equation of the parabola is $y^2 = \frac{1}{64}x$

The equation of $\overleftrightarrow{\text{LN}}$ is $y = \frac{1}{4}$

Let the co-ordinates of L be $\left(\theta, \frac{1}{4}\right)$

Since L $\left(\theta, \frac{1}{4}\right)$ lies on parabola

$\therefore \left(\frac{1}{4}\right)^2 = \frac{1}{16}\theta$

$\therefore \theta = 1$

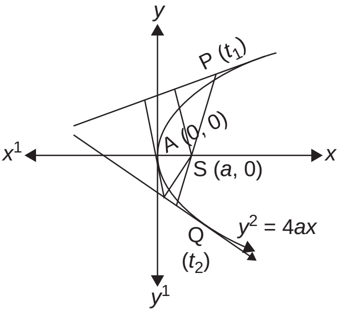
$\therefore$ L = L $\left(1, \frac{1}{4}\right)$ N $\left(4, \frac{1}{4}\right)$

$\therefore$ LN = $\sqrt{(4-1)^2 + 0} = 3$

$\therefore$ Height of water at a distance of 0.75 *m* from A is 3 *m*.

Illustration 19 $\overline{PQ}$ is a focal chord of the parabola $y^2 = 4ax$. If the length of the $\perp^{er}$ line segments from the vertex and the focus of the tangents at P and Q are P_1, P_2, P_3 and P_4 respectively, then show that $P_1, P_2, P_3, P_4 = a^4$.

Solution



Let P $(t_1) = (at_1^2, 2at_1)$ and Q $(t_2) = (at_2^2, 2at_2)$ be two points on parabola $y^2 = 4ax$; $\overline{PQ}$ is a focal chord $t_1t_2 = -1$

The equation of tangent is $y = \frac{x}{t_1} + at_1$

$\therefore x - t_1y + at_1^2 = 0$ (1)

Similarly the tangent at Q is

$x - t_2y + at_2^2 = 0$ (2)

Let P_1 and P_2 be the length of the $\perp^{er}$ from vertex and focus S $(a, 0)$ of the parabola on line (1), then

$$P_1 = \frac{|at_1^2|}{\sqrt{1+t_1^2}};\ P_2 = \frac{|a + at_1^2|}{\sqrt{1+t_1^2}}$$

Similarly if P_3 and P_4 be the length of the $\perp^{er}$ from vertex and focus on line (2) then

$$P_3 = \frac{|at_2^2|}{\sqrt{1+t_2^2}};\ P_4 = \frac{|a + at_2^2|}{\sqrt{1+t_2^2}}$$

$$\therefore P_1\, P_2\, P_3\, P_4 = \left|\frac{at_1^2(a+at_1^2)\, at_2^2(a+at_2^2}{(1+t_1^2)(1+t_2^2)}\right|$$

$$= a^4\left(t_1t_2\right)^2$$

$$= a^4$$

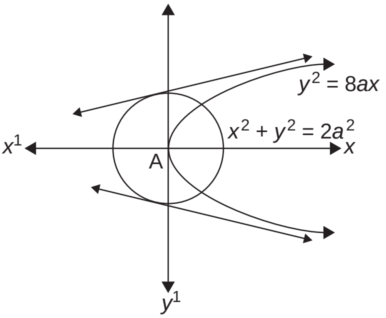
$\therefore P_1\, P_2\, P_3\, P_4 = a^4$

Illustration 20 Show that the equation of common tangents to the parabola $y^2 = 8ax$ and circle $x^2 + y^2 = 2a^2$ is $y = \pm\ (x + 2a)$.

Solution

Let m be the slope of the common tangent to both the curves.

$\therefore$ Its equation can be written in two ways



$$y = mx + \frac{2a}{m} \qquad (1)$$

$$\text{and } y = mx + \sqrt{2a}\ \sqrt{1 + m^2} \qquad (2)$$

But (1) and (2) show the same line

$$\frac{2a}{m} = \sqrt{2a}\ \sqrt{1 + m^2}$$

$$\therefore \frac{4a^2}{m^2} = 2a\left(1 + m^2\right)$$

$$\therefore m^4 + m^2 - 2 = 0$$

$$\therefore \left(m^2 + 2\right)\left(m^2 - 1\right) = 0$$

$$\therefore m = \pm 1 \qquad \left(\because m^2 + 2 \neq 0\right)$$

There are two lines touching both the curves

(1) $y = x + 2a$ $\quad\left[m = 1 \text{ from } (1)\right]$

(2) $y = -x - 2a$ $\quad\left[m = -1 \text{ from } (2)\right]$

Illustration 21 P (x_1, y_2), Q (x_2, y_2), R (x_3, y_3) are any three points on parabola $y^2 = 4ax$. Show that the area of ΔPQR is $\frac{1}{8a}\ (y_1 - y_2)\ (y_2 - y_3)\ (y_3 - y_1)\ |$.

Solution

Let P (t_1) = P $(at_1^2, 2at_1)$ = P (x_1, y_1)

Q (t_2) = Q $(at_2^2, 2at_2)$ = Q (x_2, y_2)

R (t_3) = R $(at_3^2, 2at_3)$ = R (x_3, y_3)

As shown in $\Sigma\ x$

$$\text{The area of } \Delta PQR = a^2|(t_1 - t_2)\ (t_2 - t_3)\ (t_3 - t_1)|$$

$$= \frac{8a^3}{8a}\left|(t_1 - t_2)(t_2 - t_3)(t_3 - t)_1\right|$$

$$= \frac{1}{8a}\left|(2at_1 - 2at_2)(2at_2 - 2at_3)(2at_3 - 2at_1)\right|$$

$$= \frac{1}{8a}\left|(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)\right| \qquad (\because y_1 = 2at_1 \text{ etc.})$$

Illustration 22 The tangents at A and B intersect each other at P. Prove that SP is the geometric mean of focal distances between A and B.

Solution

Let A $\left(at_1^2, 2at_1\right)$ and B $\left(at_2^2, 2at_2\right)$ be two points on $y^2 = 4ax$

The tangent at A $\left(at_1^2, 2at_1\right)$ is $y = \dfrac{x}{t_1} + at_1$ (1)

The tangent at B $\left(at_2^2, 2at_2\right)$ is $y = \dfrac{x}{t_2} + at_2$ (2)

The point of intersection of (1) and (2) is

P $\left(at_1t_2; a\left(t_1 + t_2\right)\right)$

$$\therefore \text{SP}^2 = \left(at_1t_2 - a^2\right) + a^2\left(t_1 + t_2\right)^2$$

$$= a^2\left[\left(t_1t_2 - 1\right)^2 + \left(t_1 + t_2\right)^2\right]$$

$$= a^2\left(1 + t_1^2\right)\left(1 + t_2^2\right)$$

$$\text{AS}^2 = \left(at_1^2 - a^2\right) + \left(2at_1 - 0\right)^2$$

$$= a^2\left[\left(t_1^2 - 1\right)^2 + 4t_1^2\right]$$

$$= a^2\left(1 + t_1^2\right)$$

$$\therefore \text{AS} = a\left(1 + t_1^2\right)$$

Similarly $\text{BS}^2 = a^2\left(1 + t_2^2\right)^2$

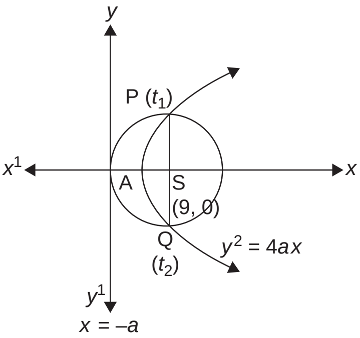
$$\therefore \text{BS} = a\left(1 + t_2^2\right)$$

$$\therefore \text{ASBS} = a^2\left(1 + t_1^2\right)\left(1 + t_2^2\right) = \text{SP}^2$$

$\therefore$ SP is the geometric mean of the focal distance between A and B.

Illustration 23 Show that the circle described on a focal chord of a parabola, as a diameter touches the directrix.

Solution



Let P $\left(at_1^2; 2at_1\right)$ and Q $\left(at_2^2, 2at_2\right)$ be end points of a focal chord of the parabola $y^2 = 4ax$

But $t_1t_2 = -1 \Rightarrow t_2 = -\dfrac{1}{t_1}$

$$\therefore \text{ Q}\left(\frac{a}{t_1^2}; -\frac{2a}{t_1}\right)$$

The equation of the circle of $\overline{\text{PQ}}$ as a diameter

$$\therefore \left(x - at_1^2\right)\left(x - \frac{a}{t_1^2}\right) + \left(y - 2at_1\right)\left(y + \frac{2a}{t_1}\right) = 0$$

Substitute $x = -a$ to get the intersection of the circle with directrix

$$\left(-a - at_1^2\right)\left(-a - \frac{a}{t_1^2}\right) + \left(y - 2at_1\right)\left(y + \frac{2a}{t_1}\right) = 0$$

$$\therefore y^2 - 2ay\left(t_1 - \frac{1}{t_1}\right) + a^2\left(t_1 - \frac{1}{t_1}\right)^2 = 0$$

For this equation $\Delta = 0 \Rightarrow b^2 - 4ac = 0$

$$\therefore\ 4a^2\left(t_1 - \frac{1}{t_1}\right)^2 - 4a^2\left(t_1 - \frac{1}{t_1}\right)^2 = 0$$

$\therefore$ The circle with $\overline{\text{PQ}}$ as a diameter touches the directrix

ANALYTICAL EXERCISES

1. There is a point on the parabola $y^2 = 2x$, whose x-co-ordinate is two times the y-co-ordinate. If this point is not a vertex of the parabola, find the point.
2. If the focus of the parabola $y^2 = 8x$ divides a focal chord in the ratio 2 : 3, find the equation of the line containing the focal chord.
3. Find the tangents to the parabola $y^2 = 8x$ that are parallel to and $\perp^{er}$ to the line $x + 2y + 5 = 0$.
4. Find the equation of tangent to $y^2 = 4ax$ that make an angle of measure $\frac{\pi}{3}$ with x-axis.
5. Find the equation of the tangent at $(7, 7)$ to the parabola $y^2 = 7x$.
6. Find the tangent to $y^2 = 8x$ for which the x-intercept is -2.
7. Find the tangent to $y^2 = 12x$ at point $t = 2$.
8. Find the equations of common tangent to the parabola $y^2 = 16x$ and the circle $x^2 + y^2 = 8$.
9. A line $\perp^{er}$ to the tangent at a point P on the parabola intersects x-axis in G. Show that PG is the geometric mean of the length of latus rectum SP.
10. Show that the set of the midpoints of all the chords through the vertex of the parabola $y^2 = 4ax$ is the parabola $y^2 = 2ax$.
11. Prove that any circle drawn with a focal chord as diameter touches the diractrix.
12. For the parabola $y^2 = 12x$, S is its focus. If SP = 6, find the co-ordinates of P.
13. M is the foot of the $\perp^{er}$ from a point P on a parabola to its axis. A is the vertex and LL^1 the latus rectum of the parabola. Prove that $PM^2 = AMLL^1$.
14. Prove that the line $y = x + 3$ is a tangent to the parabola $y^2 = 12x$.
15. Find the locus of the midpoints of the focal-chords of the parabola. $y^2 = 4ax$.
16. Get the midpoints of the focal chords of the parabola $y^2 = 8x$ lying on the line $2x + y - 8 = 0$.
17. Prove that the tangent to a parabola at its vertex is $\perp^{er}$ to the axis of the parabola.
18. Show that the line $2y = x + 4a$ is a tangent to the parabola $y^2 = 4ax$ and also get the co-ordinates of the point of contact.
19. Find the locus of point P such that the slope of the tangents drawn from P to a parabola have (1) constant sum (2) constant non-zero parabola is on the directrix.
20. Show that the orthocentre of the triangle formed by any three tangents to a parabola is on the directrix.

21. If the tangent at P of the parabola $y^2 = 4ax$ intersects the line $x = a$ in y and the directrix in z then prove that $Sy^1 = Sz$.

22. A Line passing through vertex A of parabola $y^2 = 4x$ and $\perp^{er}$ to the tangent at B ($\neq$A) intersects tangent in P and the parabola in Q. Prove that APAQ = Constant.

23. Show that the measure of angle between the parabolas $x^2 = 27y$ and $y^2 = 8x$ at the point of their intersection other than origin is $\tan^{-1}\left(\frac{9}{13}\right)$.

ANSWERS

(1) $y^2 = \frac{9}{2}x$

(2) $\pm 2\sqrt{6}x + y = \pm 4\sqrt{6}$

(3) $x + 2y + 8 = 0,\ 2x - y + 1 = 0$

(4) $3x - \sqrt{3}y + a = 0,\ 3x + \sqrt{3}y + a = 0$

(5) $x - 2y + 7 = 0$

(6) $x - y + 2 = 0,\ x + y + 2 = 0$

(7) $x - 2y + 12 = 0$

(8) $y = \pm(x + 4)$

(12) $(3, \pm 6)$

(15) $y^2 = 2a(x - a)$

(16) $(5, -2)$

(19) $p = [(x, y) \mid kx - a = 0,\ k \in \mathrm{R}]$

19

Ellipse

LEARNING OBJECTIVES

After studying this chapter, the student will be able to understand:

- The standard equation and parametric equation of ellipse, the foci, directrix, eccentricity, latus rectum
- Equation of auxilliary and directries circle of the ellipse
- The equation of tangent at point (x_1, y_1) and the necessary and sufficient condition of $y = mx + c$ to be a tangent to the ellipse and the co-ordinates of point *t*.
- Properties of the ellipse
- The position of a point with respect to the ellipse.

INTRODUCTION

Definition

Set of all points in a plane whose distance from a fixed line is said to be directrix of the ellipse and the constant ratio $e\ (e < 1)$ is said to be eccentricity of the ellipse.

The fixed point is said to be the focus of the ellipse and the fixed line is said to be the directrix of the ellipse and the constant ratio $e\ (e < 1)$ is said to be eccentricity of the ellipse.

Standard Equation of an Ellipse

The equation of an ellipse whose major and minor axes are along x and y-axis respectively is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a > b$

If the major axes are along y-axis and major axes along x-axis then the equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ b > a$

The equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is called the standard equation of ellipse.

Some Important Results Derived from the Standard Equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a>b$

(A) Curve is symmetric about both the axes. If P (x, y) is a point on the ellipse then we have the points Q $(x, -y)$, R $(-x, -y)$ and S $(-x, -y)$. The curve is symmetric about the origin O and curve passing through O is bisected at O and 0 is said to be centre of the curve.

(B) Curve intersects x-axis at A $(a, 0)$ and A′ $(-a, 0)$ and y-axis at B $(0, b)$ and B′ $(0, -b)$. AA′ = $2a$ is said to be the length of major axes and BB′ = $2b$ is said to be the length of minor axes. Here A, A′, B, B′ are the vertices of the ellipse.

(C) The relationship between a, b and e is $b^2 = a^2(1 - e^2)$

$$\therefore e^2 = \frac{a^2-b^2}{a^2} \qquad (a>b)$$

(D) Ellipse is a bifocal curve. It has two foci, two corresponding directrix as given below:

(1) Focus S $(ae, 0)$, $d = x = \frac{a}{e}$

(2) Focus S′ $(ae, 0)$ $d' = x = \frac{-a}{e}$

(E) For every point p (x, y) on ellipse $-a \le x \le a$ and $-b < y \le b$. All the points on ellipse are inside the rectangle formed by $x = \pm b$ converse not true all the points inside the said rectangle are not the points on or inside the ellipse.

(F) p (x_1, y_1) is a point in the co-ordinate plane if

(1) $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}<1$ and only then p is inside the ellipse

(2) $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$ then and only then p is on the ellipse

(3) $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}>1$ then and only then p is outside the ellipse

(G) For the standard ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\, b>a$, the major axis is along y-axis and minor axis is along x-axis. The length of the major axis is $2b$ and the length of the minor axis is $2a$.

$a^2 = b^2(1 - e^2) \therefore e^2 = \frac{b^2-a^2}{b^2}$

Foci S $(0, be)$ and S′ $(0, -be)$

Corresponding directrices $d = y = \frac{b}{e}$; $y = \frac{-b}{e}$

(H) For ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; $a>b$ the length of the latus rectum: $\frac{2b^2}{a}$ co-ordinates of the end points of the latus rectum.

$$\left(ae, \pm\frac{b^2}{a}\right); \left(-ae, \pm\frac{b^2}{a}\right)$$

The parametric equation of the ellipse is $x = a \cos \theta, y = b \sin \theta, -\pi < \theta \leq \pi$ or $0 \leq \theta < 2\pi$

The point p whose co-ordinates are $(a \cos \theta, b \cos \theta)$ is called a parametric point $p\ (\theta)$. θ is said to be parameter and θ is also an eccentric angle of point p.

Focal Distances of a Point on Ellipse

If $p\ (x, y) - \text{P}\ (a \cos \theta, b \sin \theta)$ is a point on ellipse whose foci are $\text{S}\ (ae, 0)$ and $\text{S}'\ (-ae, 0)$ then SP and S′P are called the focal distances of point p.

$$\text{SP} = a(1 - e \cos \theta) = a - e(a \cos \theta) = a - ex$$

$$\text{S}'\text{P} = a(1 + e \cos \theta) = a + e(a \cos \theta) = a + ex$$

Auxillary Circle of the Ellipse

The circle $x^2 + y^2 = a^2\ (a > 0)$ is said to be auxillary circle. If $p\ (\theta) = (a \cos \theta, b \sin \theta)$ is a point on the ellipse then its coherent point on auxillary circle is $\text{Q}\ (\theta) = (a \cos \theta, \sin \theta)$ and $m\angle\text{QOX} = \theta$ where $\text{C}\ (0, 0)$ is the centre of the ellipse.

Tangent to the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(1) The equation of tangent at $p\ (x_1, y_1)$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

(2) The tangent at $p\ (\theta) = p\ (a \cos \theta, \sin \theta)$ is $\frac{x\cos\theta}{a^2} + \frac{y\sin\theta}{b^2} = 1$

(3) The condition for line $y = mx + c$ to be a tangent:

$c^2 = a^2m^2 + b^2$ and the point of contact

$$p = p\left(\frac{-a^2m}{c}; \frac{b^2}{c}\right)$$

(4) Equation of tangent having slope mv:

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

(5) The number of tangents passing through $p\ (x, y_1)$:

(i) If $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} < 1$ then p is inside the ellipse and no tangent can be drawn through point to the curve.

(ii) If $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} < 1$ then p is on the ellipse and one and only one tangent can be drawn from point p to the curve.

(iii) $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} > 1$ p is outside the curve and two tangents can be drawn through p to the curve.

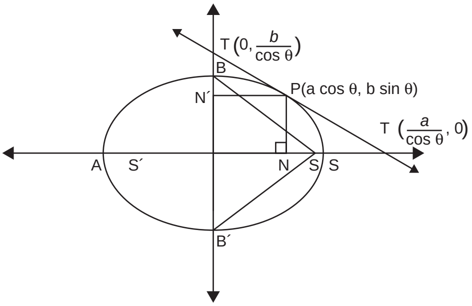
The slopes of both these tangents are the roots of equations $(y - ms_1)^2 = a^2m^2 + b^2$ assuming that no tangent passing through $p\ (x^1, y^1)$ is parallel to y-axis.

(6) Director circle of the ellipse $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$

The set of points in the plane of ellipse through which two $\perp^{er}$ tangents can be drawn to the ellipse is a circle, known as director circle of the ellipse. The equation of director circle is $x^2 + y^2 = a^2 + b^2$. Clearly this circle is centred at $(0, 0)$ and its radius is $\sqrt{a^2 + b^2}$ = AB where A and B are the vertices of the ellipse.

PROPERTIES OF AN ELLIPSE

Here C is the centre of ellipse, S and S' are foci, A, B, A′, B′ are the vertices. The tangent at the point p intersects the point p intersects the major axis at T and minor axis at T′. N and N′ are the foot of $\perp^{er}$ p to the x and y axis respectively.



- SP + S′P = 2a
- SB = S′B = a
- ASA′S = b^2
- CTCN = a^2; CT′CN′ = b^2; $\dfrac{a^2}{\text{CT}^2} + \dfrac{b^2}{CT'^2} = 1$

ILLUSTRATIONS

Illustration 1 For each ellipse below, find eccentricity, co-ordinates of foci, equation of directrix x; length of latus rectum and length of major and minor axes.

(a) $16x^2 + 25y^2 = 400$ (b) $3x^2 + 2y^2 = 6$

(c) $\dfrac{x^2}{169} + \dfrac{y^2}{144} = 1$ (d) $4x^2 + y^2 = 16$

Solution

(a) $16x^2 + 25y^2 = 400$

$\therefore \dfrac{16x^2}{400} + \dfrac{25y^2}{400} = 1$

$\therefore \dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$

$\therefore a^2 = 25 \Rightarrow a = 5$

and $b^2 = 16 \Rightarrow b = 4$

$\therefore a > b$

$\therefore 16 = 25(1 - e^2)$

$\therefore \dfrac{16}{25} = 1 - e^2$

$\therefore e^2 = 1 - \dfrac{16}{25} = \dfrac{9}{25}$

$\therefore e = \dfrac{3}{5}$, $a = 5$, $b = 4$

→ focus S $(\pm ae, 0)$

$\therefore$ S $(\pm 3, 0)$

→ The equations of directrices are

→ $x = \pm\dfrac{a}{e}$ $\therefore 3x = \pm 25$

→ Length of latus rectum

$= \dfrac{262}{a} = \dfrac{32}{5}$

→ Length of major axis

$= 2a = 10$

→ Length of minor axis

$= 2b = 8$

(b) $3x^2 + 2y^2 = 6$

$\therefore \dfrac{3x^2}{6} + \dfrac{2y^2}{6} = 1$

$\therefore \dfrac{3x^2}{6} + \dfrac{2y^2}{6} = 1$

$\Rightarrow\ a = \sqrt{2}$ and $b = \sqrt{3}$

$\therefore\ a < b$

but $a^2 = b^2\left(1 - e^2\right)$

$\therefore\ 2 = 3\left(1 - e^2\right)$

$\therefore \left(1 - e^2\right) = \dfrac{2}{3}$

$\therefore \left(1 - e^2\right) = \dfrac{2}{3}$

$\therefore e = \dfrac{1}{\sqrt{3}}$

→ Now focus S $\left(0, \pm be\right)$

= S $\left(0, \pm 1\right)$d

→ The equations of directrices are

$y = \pm \dfrac{b}{e}$

$\therefore y = \pm \dfrac{\sqrt{3}}{1/\sqrt{3}} = \pm 3$

→ Length of latus rectum

$\therefore y = \pm \dfrac{\sqrt{3}}{1/\sqrt{3}} = \pm 3$

→ Length of major axis

$= 2b = 2\sqrt{3}$

→ Length of minor axis

$= 2a = 2\sqrt{2}$

(c) $\dfrac{x^2}{169} + \dfrac{y^2}{144} = 1$

$\therefore\ a^2 = 169 \Rightarrow a = 13$

$b^2 = 144 \Rightarrow b = 12$

$\therefore\ a > b$

$\therefore\ b^2 = a^2\left(1 - e^2\right)$

$\therefore\ 144 = 169\left(1 - e^2\right)$

$\therefore\ 1 - e^2 = \dfrac{144}{169}$

$\therefore\ e^2 = 1 - \dfrac{144}{169} = \dfrac{25}{169}$

But $a = 13$

$\therefore$ Focus S $\left(\pm ae, 0\right)$

= S $\left(\pm 5, 0\right)$

→ The equations of directrices are

$x = \pm \dfrac{a}{e} = \pm \dfrac{13}{5/13} = \pm \dfrac{169}{5}$

$\therefore\ 5x = \pm 169$

→ Length of major axis

$= 2a = 26$

→ Length of minor axis

$= 2b = 24$

→ Length of latus rectum

$= \dfrac{2b^2}{a} = \dfrac{288}{13}$

$\therefore$ Eccentricity $e = \dfrac{\sqrt{3}}{2}$ and $b = 4$

(d) $4x^2 + y^2 = 16$

$\therefore \dfrac{4x^2}{16} + \dfrac{y^2}{16} = 1$

$\therefore \dfrac{x^2}{4} + \dfrac{y^2}{16} = 1$

$\therefore\ a = 2;\ b = 4$

$b > a$

→ Focus S $\left(0, \pm be\right)$

$= \mathrm{S}\left(0, \pm 2\sqrt{3}\right)$

→ The equations of directrices are

$y = \pm \dfrac{b}{e} = \pm \dfrac{8}{\sqrt{3}}$

$\Rightarrow a^2 = b^2\left(1 - e^2\right)$

$\Rightarrow 4 = 16\left(1 - e^2\right)$

$\Rightarrow \dfrac{4}{16} = 1 - e^2$

$\Rightarrow e^2 = 1 - \dfrac{4}{16} = \dfrac{12}{16} = \dfrac{3}{4}$

$e = \dfrac{\sqrt{3}}{2}$

→ Length of latus rectum

$= \dfrac{2a^2}{b} = \dfrac{2(4)}{4} = 2$

→ Length of major axis

$= 2b = 8$

→ Length of minor axis

$= 2a = 4$

Illustration 2 Get the equation of ellipse in each case below.

(1) Foci $(\pm 2, 0)$, $e = \dfrac{1}{2}$

(2) End points of major axis : $(\pm 5, 0)$, foci $(\pm 4, 0)$

(3) Length of latus rectum 5, $e = \dfrac{2}{5}$, major axis on x-axis

(4) $(4, 1)$ is a point on the ellipse; foci $(-6, 1)$

(5) Ellipse passing through $(1, 4)$ and $(-6, 1)$

(6) Length of latus rectum 4 and distance between foci $4\sqrt{2}$

Solution

(1) foci $(\pm 2, 0)$; $e = \dfrac{1}{2}$

Here we can say that

S $(2, 0)$, S′ $(-2, 0)$ and $e = \dfrac{1}{2}$

S $(ae, 0) \Rightarrow ae = 2 \Rightarrow a\left(\dfrac{1}{2}\right) = 2 \Rightarrow a = 4$

$\therefore a^2 = 16$

But $b^2 = a^2\left(1 - e^2\right) = 16\left(1 - \dfrac{1}{4}\right) = 12$

$\therefore$ Standard equation of ellipse is $\dfrac{x^2}{16} + \dfrac{y^2}{12} = 1$

(2) Vertices $(\pm 5, 0)$; foci $(\pm 4, 0)$

$\therefore$ A $(a, 0) =$ A $(5, 0)$

$\therefore a = 5$

and S $(ae, 0) =$ S $(4, 0) \Rightarrow ae = 4$

$\Rightarrow 5e = 4 \Rightarrow e = \dfrac{4}{5}$

$\therefore b^2 = a^2\left(1 - e^2\right)$

$\therefore b^2 = 25\left(1 - \dfrac{16}{25}\right) = 9$ and $a^2 = 25$

$\therefore$ Standard equation of ellipse is $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

(3) Here $e = \frac{2}{3}$

Length of latus rectum = 5 $\Rightarrow \frac{2b^2}{a} = 5 \Rightarrow 2b^2 = 5a$ (1)

But $b^2 = a^2\left(1 - e^2\right)$ (2)

By eqs. (1) and (2) we can say that

$\frac{5a}{2} = a^2\left(1\frac{4}{9}\right) = a^2\left(\frac{5}{9}\right)$

$\Rightarrow a = \frac{9}{2} \Rightarrow a^2 = \frac{81}{4}$

$\Rightarrow b^2 = \frac{5}{4}\left(\frac{9}{2}\right) = \frac{45}{4}$

$\therefore$ Equation of an ellipse $\frac{x^2}{81/4} + \frac{y^2}{45/4} = 1$

$\therefore \frac{4x^2}{81} + \frac{4y^2}{45} = 1$

(4) Focus S $\left(ae, 0\right) =$ S $\left(3, 0\right) \Rightarrow ea = 3 \Rightarrow a^2e^2 = 9$

Let the equation of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Here $b^2 = a^2\left(1 - e^2\right) = a^2 - a^2e^2$

$\therefore b^2 = a^2 - 9$

$\therefore \frac{16}{a^2} + \frac{1}{b^2} = 1$ [passes through $\left(4, 1\right)$]

$\therefore \frac{16}{a^2} + \frac{1}{a^2 - 9} = 1$

$\therefore 16\left(a^2 - 9\right) + a^2 = a^2\left(a^2 - 9\right)$

$\therefore a^4 - 26a^2 + 144 = 0$

$\therefore \left(a^2 - 18\right)\left(a^2 - 8\right) = 0$

$\therefore a^2 = 18$ or $a^2 = 8$

$a^2 = 18 \Rightarrow b^2 = 18 - 9 = 9$

$a^2 = 8 \Rightarrow b^2 = 8 - 9 = -1 < 0$

$a^2 = 8$ is not possible

$\therefore$ Equation of ellipse is $\frac{x^2}{18} + \frac{y^2}{9} = 1$

(5) Let equation of an ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which passes through $\left(1, 4\right)$ and $\left(-6, 1\right)$ if $\left(1, 4\right)$ is on the ellipse.

$\therefore \frac{1}{a^2} + \frac{16}{b^2} = 1$ (1)

$\left(-6, 1\right)$ is on the ellipse

$\therefore\ \frac{36}{a^2}+\frac{1}{b^2}=1$ (2)

From previous equation we can say that

$b^2=\frac{115\times5}{7\times5}=\frac{115}{7}$

From (2) $\frac{36}{a^2}+\frac{7}{115}=1$

$\therefore \frac{36}{a^2}=1-\frac{7}{115}=\frac{108}{115}\Rightarrow a^2=\frac{36\times115}{108}=\frac{115}{3}$

$\therefore$ Required equation is $\frac{3x^2}{115}+\frac{7y^2}{115}=1\Rightarrow 3x^2+7y^2=115$

(6) Length of latus rectum $\frac{2b^2}{a}=4\ \Rightarrow b^2=2a$ (1)

Now distance between two foci $=2ae=4\sqrt{2}\ \Rightarrow ae=2\sqrt{2}$

But $b^2=a^2\left(1-e^2\right)=a^2-a^2e^2=a^2-8$

$\therefore\ b^2=a^2-8$

$2a=a^2-8$ [From eq. (1)]

$\therefore\ a^2-2a-8=0$

$\therefore\ \left(a-4\right)\left(a+2\right)=0$

$\therefore\ a=4$ or $a=2$

But $a=-2$ is not possible

But $b^2=2a=8$

$\therefore$ Required equation of an ellipse is

$\frac{x^2}{16}+\frac{y^2}{8}=1$

Illustration 3 Find the eccentricity of the ellipse the length of whose latus rectum is

(1) Half the length of major (2) Half the length of minor axis

Solution

(1) Length of latus rectum = $\frac{2b^2}{a}=a$

$\therefore\ 2b^2=a^2$

But $b^2=a^2\left(1-e^2\right)\Rightarrow b^2=2b^2\left(1-e^2\right)$

$\therefore 1-e^2=\frac{1}{2}\Rightarrow e^2=\frac{1}{2}\Rightarrow e=\frac{1}{\sqrt{2}}$

(2) Here length of latus rectum $\frac{2b^2}{a}=b\Rightarrow 2b=a$

$\Rightarrow b=\frac{a}{2}$

But $b^2=a^2\left(1-e^2\right)$

$\therefore \frac{a^2}{4}=a^2\left(1-e^2\right)\Rightarrow 1-e^2=\frac{1}{4}\Rightarrow e=\frac{\sqrt{3}}{2}$

Illustration 4 Obtain the distance of the point $p\left(5, 4\sqrt{2}\right)$ on the ellipse $16x^2 + 25y^2 = 1600$ from the foci.

Solution

$$16x^2 + 25y^2 = 1600$$

$$\therefore \frac{x^2}{100} + \frac{y^2}{64} = 1 \Rightarrow a^2 = 100 \qquad \text{and } b^2 = 64$$

$$\Rightarrow a = 10 \qquad \text{and } b = 8$$

$$\Rightarrow a > b$$

$$\therefore b^2 = a^2\left(1 - e^2\right)$$

$$\therefore 64 = 100\left(1 - e^2\right)$$

$$\therefore 1 - e^2 = \frac{64}{100} \Rightarrow e^2 = 1 - \frac{64}{100} = \frac{36}{100} \Rightarrow e = \frac{6}{10} = \frac{3}{5}$$

$$\mathrm{S}\left(ae, 0\right) = \mathrm{S}\left(6, 0\right), \mathrm{P}\left(5, 4\sqrt{3}\right)$$

$$\therefore \mathrm{SP} = \sqrt{\left(6 - 5\right)^2 + \left(0 - 4\sqrt{3}\right)^2} = \sqrt{1 + 48} = 7$$

$$\text{and } \mathrm{S}'\left(-ae, 0\right) = \mathrm{S}'\left(-6, 0\right), \mathrm{P}\left(5, 4\sqrt{3}\right)$$

$$\Rightarrow \mathrm{S}'\mathrm{P} = \sqrt{\left(5 - 6\right)^2 + \left(4\sqrt{3} - 0\right)^2} = \sqrt{169} = 13$$

Illustration 5 Show that for every real value of t the point $\left[\frac{a\left(1-t^2\right)}{1+t^2}, \frac{2bt}{1+t^2}\right]$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Solution

$$\text{L.H.S.} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$= \frac{a^2\left(1-t^2\right)^2}{a^2\left(1+t^2\right)^2} + \frac{4b^2t^2}{b^2\left(1+t^2\right)}$$

$$= \frac{\left(1-t^2\right)4t^2}{\left(1+t^2\right)^2}$$

$$= \frac{\left(1+t^2\right)^2}{\left(1+t^2\right)^2} = 1$$

$$\therefore \text{The point } \left[\frac{a\left(1-t^2\right)}{1+t^2}, \frac{2bt}{1+t^2}\right] \text{ lies on the ellipse.}$$

Illustration 6 Obtain the equation of the tangent at the point of the ellipse $4x^2 + 9y^2 = 20.$

Solution

$4x^2 + 9y^2 = 20$

$\therefore \dfrac{4x^2}{20} + \dfrac{9y^2}{20} = 1$

$\therefore \dfrac{x^2}{5} + \dfrac{y^2}{20/9} = 1$

Now the equation of the tangent to the ellipse at the point (x_1, y_1) is

$\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$

Here $(x_1, y_2) = \left(1, \dfrac{4}{3}\right)$ (given)

and $a^2 = 5$ $\quad b^2 = \dfrac{20}{9}$

$\therefore$ The required equation of tangent is

$\dfrac{x(1)}{5} + \dfrac{y(4/3)}{20/9} = 1$

$\therefore \dfrac{x}{5} + \dfrac{3y}{5} = 1$

$\therefore x + 3y = 5$

Illustration 7 Find the tangent to the ellipse $3x^2 + 4y^2 = 12$ that are parallel to the line $3x + y - 2 = 0$.

Solution

Here equation of ellipse is $3x^2 + 4y^2 = 12 \Rightarrow \dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$

$\therefore a^2 = 4 \Rightarrow a = 2$ and $b^2 = 3 \Rightarrow b = \sqrt{3}$

But the line $3x + y = 12$ is parallel to the tangent to the ellipse.

$\therefore$ Slope (tangent) $m = -\dfrac{a}{b} = -3$

Now the equation of tangent to the ellipse having slope m is

$y = mx \pm \sqrt{a^2m^2 + b^2}$

$y = -3x \pm \sqrt{4(9) + 3}$

$3x + y = \pm\sqrt{39}$

Illustration 8 Find the equation of line that is common tangent to the parabola $y^2 = 4x$ and ellipse $2x^2 + 3y^2 = 6$.

Solution

The equation of ellipse is $2x^2 + 3y^2 = 6 \Rightarrow \dfrac{x^2}{3} + \dfrac{y^2}{2} = 1$

$\therefore a^2 = 3,\ b^2 = 2$

Let the tangent to ellipse be $y = mx \pm \sqrt{a^2m^2 + b^2}$

$\therefore$ Equation of tangent is

$$y = mx \pm \sqrt{3x^2 + 2} \tag{1}$$

But for the parabola $y^2 = 4x$, $a = 1$

Let the tangent to the parabola be

$$y = mx + \frac{a}{m} = mx + \frac{1}{m} \qquad (\because a = 1) \tag{2}$$

By comparing (1) and (2) we can say that $\pm\sqrt{3m^2+2}=\frac{1}{m}$

$\therefore 3m^2+2=\frac{1}{m^2}$

$\therefore\ 3m^4 + 2m^2 - 1 = 0$

$\therefore\ \left(3m^2 - 1\right) = 0$ but $m^2 + 1 \neq 0$

$\therefore m=\pm\frac{1}{\sqrt{3}}$

$\therefore$ Required equations of tangent are

$y=\frac{1}{\sqrt{3}}x+\sqrt{3}$ $\qquad$ $y=-\frac{x}{\sqrt{3}}-\sqrt{3}$

$\therefore \sqrt{3}y=x+3$ $\qquad$ $\therefore \sqrt{3}\,y=-x-\sqrt{3}$

$\therefore x-\sqrt{3}y+3=0$ $\qquad$ $\therefore x+\sqrt{3}\,y+3=0$

Illustration 9 Show that line $x + y = \sqrt{a^2+b^2}$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and get the point of contact.

Solution

Here $x+y=\sqrt{a^2+b^2}$

$\therefore y=-x+\sqrt{a^2+b^2}$

$\therefore m=-1$, and $c=\sqrt{a^2+b^2}$

But the condition for $y = mx + c$ to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $c^2=a^2m^2+b^2$

But R.H.S. $= a^2m^2 + b^2 = a^2 + b^2 = c^2$ $\qquad$ $\left(\because\ m=-1\right)$

$\therefore$ Condition $c^2 = a^2m^2 + b^2$ is satisfied

$$\text{Now tangent point} = \left(-\frac{am^2}{c},\frac{b^2}{c}\right)$$

$$=\left(-\frac{a^2(-1)}{\sqrt{a^2+b^2}},\frac{b^2}{\sqrt{a^2+b^2}}\right)$$

$$=\left(\frac{a^2}{\sqrt{a^2+b^2}},\frac{b^2}{\sqrt{a^2+b^2}}\right)$$

Illustration 10 Get the equation of the tangent at the points on the ellipse $2x^2 + 3y^2 = 6$ whose y-co-ordinate is $\frac{1}{\sqrt{3}}$.

Solution

Let point of contact $=\left(x_1,\frac{2}{\sqrt{3}}\right)$ and equation of ellipse is

$2x^2+3y^2=6$ $\qquad$ $\frac{x^2}{3}+\frac{y^2}{2}=1$

Point $\left(x_1, \frac{2}{\sqrt{3}}\right)$ satisfies the equation of ellipse

$\therefore \frac{x_1^{\ 2}}{3} + \frac{4}{3(2)} = 1$

$\therefore \frac{x_1^2}{3} = \frac{1}{3} \Rightarrow x_1 = \pm 1$

$\therefore$ Point of contact $= \left(\pm 1, \frac{2}{\sqrt{3}}\right)$

But the equation of the tangent to the ellipse at point (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

$\therefore$ Equation of tangent at point $\left(1, \frac{2}{\sqrt{3}}\right)$ is $\frac{x(1)}{3} + \frac{y}{2}\left(\frac{2}{\sqrt{3}}\right) = 1$

$\therefore x + \sqrt{3}\,y = 3$

Equation of tangent at point $\left(-1, \frac{2}{\sqrt{3}}\right)$ is $\frac{x(-1)}{3} + \frac{y}{2}\left(\frac{2}{\sqrt{3}}\right) = 1$

$\frac{x(-1)}{3} + \frac{y}{2}\left(\frac{2}{\sqrt{3}}\right) = 1$

$\therefore\ -x + \sqrt{3}y = 3$

$\therefore\ x - \sqrt{3}y + 3 = 0$

Illustration 11 Obtain the equation of tangents at $(3, 1)$ and $(3, -1)$ to the ellipse $x^2 + 2x^2 = 11$. so that these tangents intersect on the major axis.

Solution

The equation of ellipse is $x^2 + 2y^2 = 11 \Rightarrow \frac{x^2}{11} + \frac{y^2}{11/2} = 1$

$\therefore a^2 = 11, b^2 = \frac{11}{2}$

Now the equation of tangent at $(3, 1)$ is

$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \Rightarrow \frac{x(3)}{11} + \frac{y(1)}{11/2} = 1$

$\therefore 3x - 2y = 11$ (1)

and the equation of tangent at $(3, -1)$ is

$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} \Rightarrow \frac{x(3)}{11} + \frac{y(-1)}{11/2} = 1$

$\therefore 3x - 2y = 11$ (2)

Now, by solving (1) and (2) we get interesection of tangents is on the major axis.

Illustration 12 Show that the tangents drawn from $(3, 2)$ to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ are $\perp^{er}$ to each other.

Solution

Equation of ellipse is $\frac{x^2}{9}+\frac{y^2}{4}=1$

$\therefore$ $a^2 = 9$ and $b^2 = 4$

Now equation of director circle of ellipse is

$x^2 + y^2 = a^2 + b^2 = 9 + 4$

$\therefore$ $x^2 + y^2 = 13$

Now $(3, 2)$ is on the director circle of ellipse which is on the director circle.

$\therefore$ They are mutually perpendicular.

ANALYTICAL EXERCISES

1. Find the eccentricity of the ellipse in which the distance between the two directrices is three times the distance between the two foci.
2. Show that the line $x + 2y + 5 = 0$ touches the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$. Also find the point of contact.

ANSWERS

(1) $\left(e=\frac{1}{\sqrt{3}}\right)$

(2) $\left(-\frac{9}{5},-\frac{8}{5}\right)$

20

Hyperbola

LEARNING OBJECTIVES

After studying this chapter, the student will be able to understand:

- Standard equation
- Parametric equation
- Foci; directrix; eccentricity, latus rectum
- Equation of auxillary and direct circle of the hyperbola
- Equation of tangent at point (x_1, y_1) and at θ point and the necessary and sufficient condition for $y = mx + c$ to be a tangent to the hyperbola and the co-ordinates of point of contact
- Properties of Hyperbola
- Asymptoter and rectangular hyperbola—its equation and foci

INTRODUCTION

Definition

The set of all points in the plane whose distance from a fixed point is in a constant ratio e to its distance from a fixed line is called a hyperbola and $e > 1$.

The fixed point is said to be focus, the fixed line is called the directrix and the constant ratio e is called the eccentricity of the hyperbola.

Standard Equation of Hyperbola

If C $(0, 0)$ be the centre and the transverse axis is along x-axis then the equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

The above equation is said to be equations of hyperbola and these are the standard equations of hyperbola.

Properties of Hyperbola

(1) The length of the transverse axis is $2a$, the length of the conjugate axis is $2b$ and the relation between a, b, e is $b^2 = a^2(e^2 - 1)$.

(2) Hyperbola intersects the transverse axis in two points, A $(a, 0)$ and A^1 $(-a, 0)$, but it does not intersect the conjugate axis.

(3) Hyperbola is symmetric about both its axis and is also symmetric about the centre.

(4) Hyperbola is also a bifocal curve. The two foci and their corresponding directrices are as under:

Focus: S $(ae, 0)$, directrix = $x = \frac{a}{e}$

Focus: S^1 $(-ae, 0)$, directrix = $x = -\frac{a}{e}$

(5) For hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ the relation between a, b and e is $a^2 = b^2(e^2 - 1)$; foci S $(0, \pm be)$, directrix = $y = \pm\frac{b}{e}$.

(6) For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ the length of the latus rectum is $\frac{2b^2}{a}$ and end points of the latus rectum are $\left(ae, \pm\frac{b^2}{a}\right)$; $\left(-ae, \pm\frac{b^2}{a}\right)$

(7) The parametric equation of hyperbola is $(x, y) = (a \sec\theta; b \tan\theta)$, where $\theta \in (-\neq, \neq]$, $\theta \neq \pm\frac{\pi}{2}$. Any point on the hyperbola can be taken as P (θ) = P $(a \sec\theta, b \tan\theta)$.

Auxilliary Circle of the Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } x^2 + y^2 = a^2$$

Tangent to the Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

(1) At point P (x_1, y_1) on the hyperbola:

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

(2) At point P (θ) on the hyperbola:

$$\frac{x \sec\theta}{a} - \frac{y \tan\theta}{b} = 1$$

(3) Condition for line $y = mx + c$ to be a tangent to the hyperbola: $c^2 = a^2m^2 - b^2$

(4) Equation of the tangent having slope m:

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

Number of Tangents from a Point P (x_1, y_1) in the Plane of the Hyperbola

If the quadratic equation is m,

$$(x_1^2 - a^2)m^2 - 2x_1y_1m + y^2 + b^2 = 0$$

- has two distinct real roots, then two tangents can be drawn from P to the hyperbola.

- has only one real root, then one tangent through A (x_1, y_1) can be drawn to the hyperbola.
- has no real root, then no tangent can be drawn through A (x_1, y_1) to the hyperbola.

DIRECTOR CIRCLE

The set of points in the plane through which two $\perp^{er}$ tangents can be drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $(a > b)$ is a circle and known equation of director circle is $x^2 + y^2 = a^2 - b^2$.

If $a^2 < b^2$, no such circle exists and if $a = b$ then it becomes a point $(0, 0)$ which is the circle of the hyperbola.

Asymptotes of the Curve

Let $y = f(x)$ be a curve and $y = mx + c$ be a line such that

$$|x| \underset{\to\infty}{\lim} \; |f(x) - (xm + c)| = 0$$

then $y = xm + c$ is said to be an asymptoter of the curve $y = f(x)$ and the vertical asymptote of the curve $y = f(x)$ so that the line $y = \pm\frac{b}{a}x$ are the asymptotes of the hyperbola. The combined equation of asymptotes is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

The angle between two asymptotes is $2\tan^{-1}\left(\frac{b}{a}\right) = 1$ where $a \geq b$.

RECTANGULAR HYPERBOLA

If the angle between two asymptotes of a hyperbola is right angle then the hyperbola is said to be a rectangular hyperbola and $e = \sqrt{2}$ and also the parametric co-ordinates of point of rectangular hyperbola are $(a \sec\theta; b\tan\theta)$

Characteristics of Hyperbola

If P is a point on the hyperbola then its distances from S and S^1 are known as focal distances of P.

If P (x, y) = P $(a\sec\theta; b\tan\theta)$ then SP = $a(e\sec\theta - 1) = ex - a$ and $S^1P = a(e\sec\theta + 1) = ex + a$, then

(1) $|SP - S^1P| = 2a$ = constant

(2) If the tangent at P intersects the transverse axis at T then $\left|\frac{ST}{S^1T}\right| = \left|\frac{SP}{S^1P}\right|$.

(3) If L and L^1 are the feet of the $\perp^{er}$ from S and S^1 to the tangent at P to the hyperbola then $SLS^1L^1 = b^2$.

Another form of a hyperbola = $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

(1) The foci are S $(0, be)$, S^1 $(0, -be)$ and their directrices are $y = \pm\frac{b}{e}$.

(2) $a^2 = b^2(e^2 - 1)$ using this we can get the value of e.

(3) $AA^1 = 2a$ and $BB^1 = 2b$ where $A^1\left(-a, 0\right)$ and $A\left(a, 0\right)$. Here BB^1 is called transverse axis and AA^1 is called conjugate axis.

(4) Length of latus rectum is $\dfrac{2a^2}{b}$ and end points of latus rectum are $\left(\pm\dfrac{a^2}{b}, \pm be\right)$.

ILLUSTRATIONS

Illustration 1 For the following hyperbola find the co-ordinates of foci, equation of directrices, eccentricity, length of latus rectum and the length of the axes.

(a) $3x^2 - 12y^2 = 36$ (b) $3x^2 - 2y^2 = 1$ (c) $y^2 - 16y^2 = 16$

Solution

(a) $3x^2 - 12y^2 = 36$

$$\Rightarrow \frac{3x^2}{36} - \frac{12y^2}{36} = 1$$

$$\Rightarrow \frac{x^2}{12} - \frac{y^2}{3} = 0 \Rightarrow a^2 = 12 \text{ and } b^2 = 3$$

but $b^2 = a^2\left(e^2 - 1\right) \Rightarrow a = 2\sqrt{3}$ and $b = \sqrt{3}$

$\therefore 3 = 12\left(e^2 - 1\right)$

$\therefore e^2 = \dfrac{5}{4}$

$\therefore e = \dfrac{\sqrt{5}}{2}$

and foci S $\left(\pm ae, 0\right) = \left(\pm\sqrt{15}, 0\right)$

directrix; $x = \pm\dfrac{a}{e} = \pm\dfrac{2\sqrt{3}}{\sqrt{5}/2} = \pm 4\sqrt{\dfrac{3}{5}}$

Length of latus rectum $= \dfrac{2b^2}{a} = \dfrac{2(3)}{2\sqrt{3}} = \sqrt{3}$

Length of transverse axis $= 2a = 2\left(2\sqrt{3}\right) = 4\sqrt{3}$

Length of conjugate axis $= 2b = 2\sqrt{3}$

(b) $3x^2 - 2y^2 = 1 \Rightarrow \dfrac{x^2}{\left(1/\sqrt{3}\right)^2} - \dfrac{y^2}{\left(1/\sqrt{2}\right)^2} = 1$

$$\Rightarrow a = \frac{1}{\sqrt{3}}, \text{ and } b = \frac{1}{\sqrt{2}}$$

Now $b^2 = a^2\left(e^2 - 1\right)$

$\therefore\ \frac{1}{2} = \frac{1}{3}\left(e^2 - 1\right)$

$\Rightarrow e^2 - 1 = \frac{3}{2}$

$\Rightarrow e = \sqrt{\frac{5}{2}}$

foci $\left(\pm ae,\ 0\right) = \left(\pm\sqrt{\frac{5}{6}}, 0\right)$

Equation of directrix $x = \pm\frac{a}{e} = \pm\frac{1/\sqrt{3}}{\sqrt{5/2}} = \pm\sqrt{\frac{2}{15}}$

$\therefore\ x = \pm\sqrt{\frac{2}{15}}$

Length of latus rectum = $\frac{2b^2}{a} = \frac{2\left(1/2\right)}{1/\sqrt{3}} = \sqrt{3}$

Length of transverse axis = $2a = \frac{2}{\sqrt{3}}$

Length of conjugate axis = $2b = \frac{2}{\sqrt{2}} = \sqrt{2}$

(c) $y^2 - 16x^2 = 16 \Rightarrow \frac{y^2}{16} - \frac{x^2}{1} = 1 \Rightarrow x^2 - \frac{y^2}{16} = -1$

$\therefore\ a^2 = 1$ and $b^2 = 16$; $a = 1$ and $b = 4$

$\therefore\ a^2 = b^2\left(e^2 - 1\right)$

$\therefore\ 1 = 16\left(e^2 - 1\right);\ e^2 - 1 = \frac{1}{16} \Rightarrow e^2 = \frac{17}{16}$

$\Rightarrow e = \frac{\sqrt{17}}{4}$

Foci = S $\left(0, \pm be\right) = \left(0, \pm 4\frac{\sqrt{17}}{4}\right) = \left(0, \pm\sqrt{17}\right)$

Equation of directrices = $y = \pm\frac{b}{e} = \pm\frac{4}{\sqrt{17}/4} = \pm\frac{16}{7}$

Length of latus rectum = $\frac{2b^2}{a} = \frac{2\left(1\right)}{4} = \frac{1}{2}$

Length of transverse axis = $2b = 8$
Length of conjugate axis = $2a = 2$

Illustration 2 Given the following conditions obtain the standard equation of the hyperbola:

(a) Foci $(\pm 2,\ 0)$, $e = \dfrac{3}{2}$

(b) Passing through $(2,\ 1)$ and distance between the directrices is $\dfrac{4\sqrt{3}}{5}$

(c) Passing through $(5,\ -2)$ and length of the transverse axis is 7

Solution

(a) Foci: S $(\pm 2,\ 0)$, $= (\pm ae,\ 0)$ and

$\therefore\ ae = 2$

$$\therefore\ a\left(\frac{3}{2}\right) = 2$$

$$\therefore\ a = \frac{4}{3}$$

$$\therefore\ a^2 = \frac{16}{9}$$

$$\text{Now } b^2 = a^2\left(e^2 - 1\right) = \frac{16}{9}\left(\frac{9}{1} - 1\right) = \frac{16}{9}\cdot\frac{5}{4} = \frac{20}{9}$$

$$\therefore\ b^2 = \frac{20}{9}$$

$\therefore$ The required equation of hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\therefore\ \frac{x^2}{(16/9)} - \frac{y^2}{(20/9)} = 1$$

$$\therefore\ \frac{9x^2}{16} - \frac{9y^2}{20} = 1$$

$\therefore\ 45x^2 - 36y^2 = 80$

(b) Let the equation of hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\therefore \text{ Distance between the directrices } = \frac{4\sqrt{3}}{3}$$

$\therefore\ \dfrac{2a}{e} = \dfrac{4\sqrt{3}}{3}$

$\therefore\ a = \dfrac{2\sqrt{3}}{3}e$

$\therefore\ a^2 = \dfrac{4}{3}e^2$

But point $(2, 1)$ is on the hyperbola $(\because \text{given})$

$\therefore\ \dfrac{4}{a^2} - \dfrac{1}{b^2} = 1$

$\therefore\ \dfrac{4}{a^2} - \dfrac{1}{a^2\left(e^2 - 1\right)} = 1$

$\therefore\ \dfrac{4}{(4/3)e^2} - \dfrac{1}{(4/3)e^2\left(e^2 - 1\right)} = 1$

$\therefore\ > \dfrac{3}{e^2} - \dfrac{3}{4e^2\left(e^2 - 1\right)} = 1$

$\therefore\ 12\left(e^2 - 1\right) - 3 = 4e^2\left(e^2 - 1\right)$

$\therefore\ 12e^2 - 12 - 3 = 4e^4 - 4e^2$

$\therefore\ 4e^4 - 16e^2 + 15 = 0$

$\therefore\ 4e^4 - 10e^2 - 6a^2 + 15 = 0$

$\therefore\ 2e^2\left(2e^2 - 5\right) - 3\left(2e^2 - 5\right) = 0$

$\therefore\ 2e^2 - 3 = 0$ or $2e^2 - 5 = 0$

$\therefore\ e^2 = \dfrac{3}{2}$ or $e^2 = \dfrac{5}{2}$

$\therefore\ e = \sqrt{\dfrac{3}{2}}$ or $e = \sqrt{\dfrac{5}{2}}$

If $e = \sqrt{\dfrac{3}{2}}$ or If $e = \sqrt{\dfrac{5}{2}}$

$\therefore\ a = \dfrac{2\sqrt{3}}{3}e = \dfrac{2\sqrt{3}}{3} \cdot \dfrac{\sqrt{3}}{\sqrt{2}} = \sqrt{2}$

$\therefore\ a^2 = 2$

But $b^2 = a^2\left(e^2 - 1\right) = 1$

$= 2\left(\dfrac{3}{2} - 1\right) = 1$

$\therefore$ The required equation of hyperbola is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$\therefore\ a = \dfrac{2\sqrt{3}}{3}\dfrac{\sqrt{5}}{\sqrt{2}} = \sqrt{\dfrac{10}{3}}$

$\therefore\ a^2 = \dfrac{10}{3}$

But $b^2 = a^2\left(e^2 - 1\right)$

$= \dfrac{10}{3}\left(\dfrac{5}{2} - 1\right)$

$= \dfrac{10}{3} \times \dfrac{3}{2} = 5$

$\therefore$ The required equation of hyperbola is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$$\therefore \frac{x^2}{2} - \frac{y^2}{1} = 1$$

$\therefore x^2 - 2y^2 = 2$

$$\therefore \frac{x^2}{(10/3)} - \frac{y^2}{5} = 1$$

$$\therefore \frac{3x^2}{10} - \frac{y^2}{5} = 1$$

$\therefore 3x^2 - 2y^2 = 10$

(c) Here length of transverse axis = 7

Let the equation of hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

But it passes through $(5, -2)$

$$\therefore \frac{25}{a^2} - \frac{4}{b^2} = 1$$

$$\therefore \frac{25}{49/4} - \frac{4}{b^2} = 1$$

$$\therefore \frac{100}{49} - 1 = \frac{4}{b^2}$$

$$\therefore \frac{4}{b^2} = \frac{51}{49}$$

$$\therefore b^2 = \frac{196}{51}$$

$\therefore$ The required equation of hyperbola is

$$\frac{x^2}{(49/4)} - \frac{y^2}{(196/51)} = 1 \Rightarrow 16x^2 - 51y^2 = 196$$

Illustration 3 If the eccentricities of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ are e_1 and e_2 respectively then prove that $e_1^2 + e_2^2 = e_1^2\, e_2^2$.

Solution

If e_1 and e_2 are the eccentricities of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, respectively, then

$$e_1^2 + e_2^2 = \frac{a^2 + b^2}{a^2} + \frac{a^2 + b^2}{b^2}$$

$$= (a^2 + b^2)\left(\frac{1}{a^2} + \frac{1}{b^2}\right)$$

$$= (a^2 + b^2)\left(\frac{a^2 + b^2}{a^2 b^2}\right)$$

$$= \left(\frac{a^2+b^2}{a^2}\right)+\left(\frac{a^2+b^2}{b^2}\right)$$

$$= e_1^2\, e_2^2$$

$$\therefore\ e_1^2 + e_2^2 = e_1^2\, e_2^2$$

Illustration 4 If e_1 the eccentricity of $\frac{x^2}{25}+\frac{y^2}{9}=1$ and e_2 is the eccentricity of $9x^2 - 16y^2 = 144$, prove that $e_1e_2 = 1$.

Solution

From the equation of ellipse

$$\frac{x^2}{25}-\frac{y^2}{9}=1$$

$a^2 = 25$ and $b^2 = 9$ and eccentricity = e_1

But $b^2 = a^2\left(1 - e_1^2\right)$

$$\therefore\ 9 = 25\left(1 - e^2\right)$$

$$\therefore\ 1 - e_1^2 = \frac{9}{25}$$

$$\therefore\ e_1^2 = 1-\frac{9}{25} = \frac{16}{25}$$

$$\therefore\ e_1 = \frac{4}{5} \qquad (1)$$

For hyperbola:

$9x^2 - 16y^2 = 144$

$$\therefore\ \frac{9x^2}{144}-\frac{16y^2}{144} = 1$$

$$\therefore\ \frac{9x^2}{16}-\frac{16y^2}{19} = 1$$

$\therefore\ a^2 = 16$ and $b^2 = 9$ and eccentricity = e_2

$$\therefore\ b^2 = a^2\left(e_2^2 - 1\right)$$

$$\therefore\ 9 = 16\left(e_2^2 - 1\right)$$

$$\therefore\ e_2^2 - 1 = \frac{9}{16}$$

$$\therefore\ e_2^2 = \frac{9}{16}+1=\frac{25}{16}$$

$$\therefore\ e_2 = \frac{5}{4} \qquad (2)$$

Now from eqs. (1) and (2) we can say that

$$e_1e_2 = \frac{4}{5}\times\frac{5}{4}$$

$$\therefore\ e_1e_2 = 1$$

Illustration 5 Get the equation of the tangents to the hyperbola $\frac{x^2}{4}-\frac{y^2}{3}=1$ that are (1) parallel to (2) $\perp^{er}$ to the line $x - y + 2 = 0$.

Solution

In the equation of hyperbola

$$\frac{x^2}{4}-\frac{y^2}{3}=1$$

$a^2 = 4$ and $b^2 = 3$

and the slope of the parallel line to $x - y + 2 = 0$ is $m = 1$.

∴ The equation of the tangent to the hyperbola having slope m

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

$$= x \pm \sqrt{4(-1)^2 - 3}$$

$\therefore y = x \pm 1$

Illustration 6 If the line $lx + my + n = 0$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, prove that $a^2l^2 - b^2m^2 = n^2$.

Solution

Here $lx + my + n = 0 \Rightarrow y = \left(-\frac{l}{m}\right)x + \left(\frac{n}{m}\right)$

Compare with $y = \mathrm{M}_1x + \mathrm{C}_1$

$\therefore \mathrm{M}_1 = -\frac{l}{m}$ and $C_1 = -\frac{n}{m}$

The condition that $y = \mathrm{M}_1x + \mathrm{C}_1$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\mathrm{C}_1^2 = a^2\mathrm{M}_1^2 - b^2$

$$\therefore \left(-\frac{n}{m}\right)^2 = a^2\left(-\frac{l}{m}\right)^2 - b^2$$

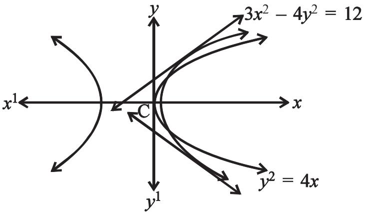
$$\therefore \frac{n^2}{m^2} = \frac{a^2l^2}{m^2} - b^2$$

$\therefore n^2 = a^2l^2 - b^2m^2$

∴ If line $lx + my + n = 0$ is the tangent to the hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ then $n^2 = a^2l^2 - b^2m^2$

Illustration 7 Obtain the equation of common tangents to the hyperbola $3x^2 - 4y^2 = 12$ and the parabola $y^2 = 4x$.



Solution

No tangent of parabola and hyperbola is a horizontal line.

The equation of tangent to the parabola $y^2 = 4x$ having slope m is

$$y = mx + \frac{1}{m} \quad (m \neq 0) \tag{1}$$

and the equation of tangent to the hyperbola $\frac{x^2}{4} - \frac{y^2}{3} = 1$ having slope m is

$$mx \pm \sqrt{a^2m^2 - b^2}$$

$$\therefore \; x \pm mx\sqrt{4m^2 - 3} \tag{2}$$

from (1) and (2) we can say that

$$\frac{1}{m} = \pm\sqrt{4m^2 - 3}$$

$$\therefore \; \frac{1}{m^2} = 4m^2 - 3 \qquad (\because \text{Squaring})$$

$$\therefore \; 4m^2 - 3m^2 - 1 = 0$$

$$\therefore \; (m^2 - 1)(4m^2 + 1) = 0$$

$$\therefore \; m^2 - 1 = 0 \qquad (4m^2 + 1 \notin \text{R})$$

$$\therefore \; m = \pm 1$$

If $m = 1$ then the equation of common tangent is

$y = x + 1$ [From eq. (1)]

If $m = -1$ then the equation of common tangent is

$y = -(x + 1)$ [From eq. (1)]

Illustration 8 Find the equation of a curve from every point of which the tangents to the hyperbola $\frac{x^2}{144} - \frac{y^2}{36} = 1$ intersect at right angles.

Solution

In equation of hyperbola

$$\frac{x^2}{144} - \frac{y^2}{36} = 1$$

$$\therefore \; a^2 = 144 \text{ and } b^2 = 36$$

The point of intersection of orthogonal tangents of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ lie on the director circle.

$$x^2 + y^2 = a^2 - b^2 \qquad (\because a^2 > b^2)$$

$$\therefore \; x^2 + y^2 = 144 - 36 = 108$$

$$\therefore \; x^2 + y^2 = 108$$

Illustration 9 Get the equation of a hyperbola if the foci are $(\pm 6, 0)$ and the distance between the directrices is 6.

Solution

Here foci are S $(\pm 6, 0)$ = S $(ae, 0)$

$$\therefore \; ae = 6 \tag{1}$$

Distance between two directrices = 6

$$\therefore \ \frac{2a}{e} = 6 \ \Rightarrow \ \frac{a}{e} = 3 \ \Rightarrow a = 3e \qquad (2)$$

From eqs. (1) and (2) we can say that

$$ae\frac{a}{e} = 6 \times 3 \ \Rightarrow a^2 = 18$$

But $b^2 = a^2\left(e^2 - 1\right) = a^2e^2 - a^2 = 36 - 18 = 18$

$\therefore$ Equation of hyperbola is

$x^2 - y^2 = 18$

$\therefore$ It is a rectangular hyperbola.

Illustration 10 Prove that there is no point in the plane of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ from which a pair of mutually $\perp^{er}$ tangents can be drawn to the hyperbola.

Solution

In given hyperbola $a^2 = 9$ and $b^2 = 16$

The point from which the tangents to the hyperbola are $\perp^{er}$ to each other lies on $x^2 + y^2 = a^2 - b^2$

$\therefore x^2 + y^2 = 9 - 16 = -7 < 0$

$\therefore$ There is no point in the plane of hyperbola from which two $\perp^{er}$ tangent can be drawn.

ANALYTICAL EXERCISES

1. Find the equation of the hyperbola for which the distance from one vertex to the two foci are 9 and 1.
2. Find the equation of the hyperbola which has the same foci as the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ and whose eccentricity is 2.
3. Get the equation of the tangents to the hyperbola $3x^2 - 2y^2 = 10$.
4. Obtain the equation of the tangents to the hyperbola $5x^2 - y^2 = 5$ from the point $\left(0, 2\right)$.
5. If $\overline{\text{SK}}$ is the $\perp^{er}$ line-segment drawn from the focus S of the hyperbola to the tangent at a point P of the hyperbola, show that k is on the auxillary circle.
6. Find the measure of angle between the asymptotes of $3x^2 - 2y^2 = 1$.
7. Show that the point of intersects of the asymptotes and the directrices are on the auxillary circle.
8. Find the length of the $\perp^{er}$ from a focus to an asymptote of $x^2 - 4y^2 = 20$.
9. For the rectangular hyperbola $x^2 - y^2 = 9$, consider the tangent at $\left(5, 4\right)$. Find the area of the triangle which this tangent makes with the two asymptotes.
10. Get the equations of the tangents to $4x^2 - y^2 = 64$ that are parallel to $15x - 6y + 11 = 0$ if possible.
11. Find c if $5x + 12y + c = 0$ touches $\frac{x^2}{9} - \frac{y^2}{1} = 1$. Also find the point of contact.

ANSWERS

(1) $9x^2 - 16y^2 = 144$

(2) $3x^2 - y^2 = 12$

(3) $3x - y = 5$

(4) $3x - y + 2 = 0, 3x + y - 2 = 0,$

(6) $2\tan^{-1}\left(\sqrt{\frac{3}{2}}\right)$

(8) $\sqrt{5}$

(9) 9

(10) $5x - 2y = \pm 12$

(11) $c = \pm 9, \left(-5, \frac{4}{3}\right), \left(5, -\frac{4}{3}\right)$

21

Determinant

LEARNING OBJECTIVES

After studying this chapter, the student will be able to understand:

- Understand the meaning of determinants of different orders
- Know basic terminology and rules of determinants
- Know how to multiply two determinants
- Know Cramer's method for evaluation of simultanious linear equation
- Know the important application of determinants

INTRODUCTION

A square arrangement of an expression or number of elements into rows and columns, enclosed within $|\;|$ is called a *determinant.*

The representation of an expression $ad - bc$ in the form of $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is called a determinant, say $|A|$ order two and $ad - bc$ is called the *value of the determinant.*

$$\therefore |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Thus the value of the determinant of order two can be obtained as the product of the diagonal elements minus product of antidiagonal elements.

$$\text{e.g. } |A| = \begin{vmatrix} 2 & 2 \\ 5 & 9 \end{vmatrix} = 2 \times 9 - 3 \times 5 = 18 - 15 = 3$$

$$\therefore |B| = \begin{vmatrix} 4 & -3 \\ 2 & -5 \end{vmatrix} = 4 \times 5 - (-3 \times 2) = 20 + 6 = 26$$

The representation of an algebraic expression

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{31}a_{21} - a_{11}a_{23}a_{32}$$

into three rows and three columns is called *determinant of order three* and is denoted as

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{12} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

A determinant obtained by leaving the row and column in which a particular element will lie is called *minor* of that element.

If we prefix the sign $(-1)^{1+j}$ to the minor of the element, then it is called *co–factor of the element* (ii).

∴ Co-factor of element (ii) = $(-1)^{i+j}$ the minor of (ii)

In order to expand $|A|$ about a row or a column we multiply each element a_{ij} in its row with $(-1)^{itj}$ times its minor.

$$\text{e.g. } |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{12} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

We can expand this determinant about first row as under

$$|A| = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

We can expand this determinant about second row as under

$$|A| = (-1)^{2+1} a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{2+2} a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{2+3} a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= -a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{22}(a_{11}a_{33} - a_{13}a_{31}) + a_{23}(a_{11}a_{32} - a_{12}a_{31})$$

$$= a_{11} + a_{22} + a_{33} + a_{12} + a_{23} + a_{31} + a_{13} + a_{21} + a_{32} -$$
$$a_{11} + a_{23} + a_{32} - a_{12} + a_{21} + a_{33} - a_{13} + a_{22} + a_{31}$$

Thus we can expand the given determinant by considering any row or column, its value remains the same. For example

$$|A| = \begin{vmatrix} 2 & 3 & 4 \\ 0 & -1 & 3 \\ 2 & 1 & -1 \end{vmatrix}$$

can be evaluated as under:

$$|A| = 2\begin{vmatrix} -1 & 3 \\ 1 & -1 \end{vmatrix} - 3\begin{vmatrix} 0 & 3 \\ 2 & -1 \end{vmatrix} + 4\begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= 2(1-3) - 3(0-6) + 4[0-(-2)]$$
$$= -4 + 18 + 8$$
$$= 22$$

ILLUSTRATIONS

Illustration 1 Expand following determinant by considering third column.

$$|A| = \begin{vmatrix} -4 & 3 & -3 \\ 0 & 3 & 2 \\ -1 & 3 & 1 \end{vmatrix}$$

Solution

We can expand $|A|$ by considering C_3

$$|A| = (-1)^{1+3}(-3)\begin{vmatrix} 0 & 3 \\ -1 & 3 \end{vmatrix} + (-1)^{2+3}(2)\begin{vmatrix} -4 & 3 \\ -1 & 3 \end{vmatrix} + (-1)^{3+3}(1)\begin{vmatrix} -4 & 3 \\ 0 & 3 \end{vmatrix}$$

$$= -3\left[0 - (-3)\right] - 2\left[-12 - (-3)\right] + 1(-12 - 0)$$

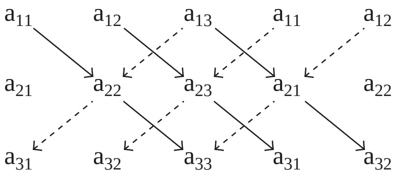
$$= -9 - 18 - 1$$

$$= -39$$

Evaluation of determinant by using **SAARUS** method:

Consider the determinant $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{12} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Enlarge the determinant by adjoining the first two columns on the right and then obtain the value of determinant as the sum of the products of the elements in the lines parallel to the diagonal minus the sum of the product of the elements parallel to the antidiagonal. This can be shown as under.



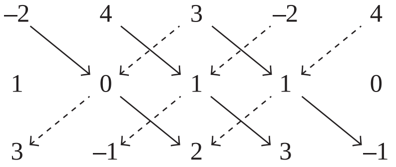
$$= (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33})$$

Here we can note that this method does not work for determinants of order more than three.

Illustration 2 Evaluate $|A| = \begin{vmatrix} -2 & 4 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{vmatrix}$ by using SARRUS method.

Solution

Repeating first two columns to right, we have



$$
\begin{aligned}
&= (-2\times 0\times 2+4\times 1\times 3+3\times 1\times -1)-(3\times 0\times 3+-2\times 1\times -1+4\times 1\times 2)\\
&= 9-10\\
&= -1
\end{aligned}
$$

PROPERTIES OF DETERMINANTS

(1) If the rows and columns of a determinant are interchanged, then the value of determinant remains unchanged.

Proof

Suppose $D=\begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & b_3 & c_3\end{vmatrix}$ and $D'=\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}$

$$
\begin{aligned}
\text{Now } D &= a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)\\
&= a_1b_2c_3-a_1b_3c_2-a_2b_1c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1\\
&= a_1b_2c_3-a_1b_3c_2-b_1a_2c_3+b_1a_3c_2+c_1a_2b_3-c_1a_3b_2\\
&= a_1(b_2c_3-b_3c_2)-b_1(a_2c_3+a_3c_2)+c_1(a_2b_3-a_3b_2)\\
&= D'
\end{aligned}
$$

(2) If we interchange the respective elements of any two rows or columns then the value of determinant changes by its sign only.

Proof

Suppose $D=\begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & b_3 & c_3\end{vmatrix}$ and $D'=\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}$

$$
\begin{aligned}
\text{Now } D &= a_1b_2c_3-a_1b_3c_2-a_2b_1c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1\\
&= -b_1a_2c_3+b_1a_3c_2+b_2a_1c_3-b_2a_3c_1+b_3a_2c_1-b_3a_1c_2\\
&= -\left[b_1(a_2c_3+a_3c_2)-b_2(a_1c_3-a_3c_1)+b_3(a_2c_1-a_1c_2)\right]\\
&= -D'
\end{aligned}
$$

(3) If there is a common factor in each element of any row or column of a determinant then it can be taken as the common factor of the determinant.

Proof

Suppose $D=\begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & b_3 & c_3\end{vmatrix}$ and $D'=\begin{vmatrix} ka_1 & ka_2 & ka_3\\ b_1 & b_2 & b_3\\ c_1 & b_3 & c_3\end{vmatrix}$

where $k\in R$

$$
\begin{aligned}
\text{Now } D' &= ka_1(b_2c_3-b_3c_2)-ka_2(b_1c_3-b_3c_1)+ka_3(b_1c_2-b_2c_1)\\
&= k\left[a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)\right]\\
&= kD
\end{aligned}
$$

(4) If the elements of any two rows or columns of a determinant are identical, then the value of determinant is zero.

Proof

$$\text{Suppose } D = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \text{ then}$$

$$\begin{aligned} D &= a_1a_2b_3 - a_1a_3b_2 - a_2a_1b_3 + a_2a_3b_1 + a_3a_1b_2 - a_3a_2b_1 \\ &= a_1a_2b_3 - a_1a_2b_3 - a_1a_3b_2 + a_1a_3b_2 + a_2a_3b_1 - a_2a_3b_1 \\ &= 0 \end{aligned}$$

(5) If each element of any row or column of a determinant is the sum of two terms then the determinant can be expressed as the sum of two determinants of the same order.

Proof

$$\text{We have to prove } \begin{vmatrix} a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

$$\text{L.H.S.} = \begin{vmatrix} a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

$$\begin{aligned} &= (a_1 + b_1)(c_2d_3 - c_3d_2) - (a_2 + b_2)(c_1d_3 - c_3d_1) + (a_3 + b_3)(c_1d_2 - c_2d_1) \\ &= a_1(c_2d_3 - c_2d_3) + b_1(c_2d_3 - c_3d_2) - a_2(c_1d_3 - c_3d_1) - b_2(c_1d_3 - c_3d_1) \\ &= \left[a_1(c_2d_3 - c_2d_3) - a_2(c_1d_3 - c_3d_1) + a_3(c_1d_2 - c_2d_1)\right] \\ &\quad \left[b_1(c_2d_3 - c_3d_2) - b_2(c_1d_3 - c_3d_1) + b_3(c_1d_2 - c_2d_1)\right] \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} \\ &= \text{R.H.S.} \end{aligned}$$

(6) If all the elements of any row or columns of a determinant are multiplied by a constant and added to or subtracted from the respective elements of any other row or column then the value of determinant remains unchanged.

Proof

$$\text{Let } D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \text{ and } D' = \begin{vmatrix} a_1 + kb_1 & a_2 + kb_2 & a_3 + kb_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\text{Now consider } D' = \begin{vmatrix} a_1 + kb_1 & a_2 + kb_2 & a_3 + kb_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} kb_1 & kb_2 & kb_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= D + k\begin{vmatrix} b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \qquad \text{using rule 3}$$

$$= D + k(0) \qquad \text{using rule 4}$$

$$= D$$

Note:

1. The sum of the products of elements of any row or column with their co-factor is always equal to the value of the determinant.
2. The sum of the products of the elements of any row or column with the co-factors of the corresponding elements of some other row or column is zero.
3. If each element in a row or column of a determinant is zero then the value of that determinant is zero.
4. If all elements above the diagonal or below the diagonal are zero then the value of that determinant is the product of the diagonal elements.
5. If A and B are two determinants of same order then $|AB| = |A|\,|B|$.

Illustration 3 Obtain the value of $\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix}$

Solution

$$\text{Here } |A| = a\begin{vmatrix} d & e \\ e & f \end{vmatrix} - b\begin{vmatrix} b & e \\ c & f \end{vmatrix} + c\begin{vmatrix} b & d \\ c & e \end{vmatrix}$$

$$= a(df - e^2) - b(bf - ce) + c(be - dc)$$

$$= adf - ae^2 - b^2f + bce + bce - dc^2$$

$$= adf + 2bce - ae^2 - b^2f - c^2d$$

Illustration 4 Without expansion solve the following

$$\text{(i)} \begin{vmatrix} 5 & 1 & 2 \\ x & 4 & 8 \\ 7 & -6 & 9 \end{vmatrix} = 0 \text{ (ii)} \begin{vmatrix} x & 1 & 2 \\ -4 & -6 & 7 \\ -2 & -3 & 8 \end{vmatrix} + \begin{vmatrix} -2 & -3 & 8 \\ -2 & 4 & 3 \\ 4 & 6 & 7 \end{vmatrix} = \begin{vmatrix} -4 & -6 & 7 \\ -4 & -5 & -5 \\ -2 & -3 & 8 \end{vmatrix}$$

Solution

(i) Here the value of determinant is zero; this means that the elements of any two rows or columns are identical. By comparing first and second row we can see that if we take 4 common from the elements of second row then they are exactly equal to the elements of first row. Then we have $\frac{x}{4} = 5$

$\therefore\ x = 20$

(ii) Here we have

$$\begin{vmatrix} x & 1 & 2 \\ -4 & -6 & 7 \\ -2 & -3 & 8 \end{vmatrix} + \begin{vmatrix} -2 & -3 & 8 \\ -2 & 4 & 3 \\ -4 & -6 & 7 \end{vmatrix} = \begin{vmatrix} -4 & -6 & 7 \\ -4 & -5 & -5 \\ -2 & -3 & 8 \end{vmatrix}$$

R_{13} = interchanging 1st and 3rd row

R_{13} = interchanging

R_{12} = 1st and 2nd row

$$\therefore \begin{vmatrix} x & 1 & 2 \\ -4 & -6 & 7 \\ -2 & -3 & 8 \end{vmatrix} + \begin{vmatrix} -4 & -6 & 7 \\ -2 & 4 & 3 \\ -2 & -3 & 8 \end{vmatrix} = \begin{vmatrix} -4 & -5 & -5 \\ -4 & -6 & 7 \\ -2 & -3 & 8 \end{vmatrix}\ R_1 \rightarrow -1 \times R_1$$

$$\therefore \begin{vmatrix} x & 1 & 2 \\ -4 & -6 & 7 \\ -2 & -3 & 8 \end{vmatrix} + \begin{vmatrix} -2 & 4 & 3 \\ -4 & -6 & 7 \\ -2 & -3 & 8 \end{vmatrix} = \begin{vmatrix} 4 & 5 & 5 \\ -4 & -6 & 7 \\ -2 & -3 & 8 \end{vmatrix}$$

$$\therefore \begin{vmatrix} x-2 & 5 & 5 \\ -4 & -6 & 7 \\ -2 & -3 & 8 \end{vmatrix} = \begin{vmatrix} 4 & 5 & 5 \\ -4 & -6 & 7 \\ -2 & -3 & 8 \end{vmatrix}$$

Comparing both determinants we have

$x - 2 = 4 \qquad x = 6$

Illustration 5 Solve the equation $\begin{vmatrix} x+1 & 2 & 3 \\ 2 & x+1 & 3 \\ 3 & 3 & x+1 \end{vmatrix} = 0$

Solution

Here we have

$$\begin{vmatrix} x+1 & 2 & 3 \\ 2 & x+1 & 3 \\ 3 & 3 & x+1 \end{vmatrix} = 0\ \ R_1 \rightarrow R_1 + R_2 + R_{23}$$

Adding second and third rows to the first row

$$\begin{vmatrix} x+6 & x+6 & x+6 \\ 2 & x+1 & 2 \\ 3 & 3 & x+1 \end{vmatrix} = 0$$

$$\therefore (x+6)\begin{vmatrix} 1 & 1 & 1 \\ 2 & x+1 & 2 \\ 3 & 3 & x+1 \end{vmatrix} = 0 \quad \begin{matrix} c_2 \to c_2 - c_1 \\ c_3 \to c_3 - c_1 \end{matrix}$$

Subtracting first column from second and third columns

$$(x+6)\begin{vmatrix} 1 & 0 & 0 \\ 2 & x-1 & 0 \\ 3 & 0 & x+1 \end{vmatrix} = 0$$

$\therefore (x+6)(x^2-1) = 0$

(Value of determinant is equal to the product of diagonals)

$x = -6$ or $x^2 = 1$

$\therefore x = \pm 1$

Illustration 6 Prove the following $\begin{vmatrix} 1 & 1 & 1 \\ xy(x+y) & yz(y+z) & zx(z+x) \\ xy & yz & zx \end{vmatrix} = 0$

Solution

Multiplying the first, second and third column by z, x and y, respectively, we have

$$\text{L.H.S.} = \frac{1}{xyz}\begin{vmatrix} z & x & y \\ xyz(x+y) & xyz(y+z) & xyz(z+x) \\ xyz & xyz & xyz \end{vmatrix}$$

$$= (xyz)\begin{vmatrix} z & x & y \\ x+y & y+z & z+x \\ 1 & 1 & 1 \end{vmatrix} \quad R_2 \to R_1 + R_2$$

Adding elements of first row to the corresponding elements of second row

$$= (xyz)\begin{vmatrix} z & x & y \\ x+y+z & x+y+z & x+y+z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (xyz)(x+y+z)\begin{vmatrix} z & x & y \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$= (xyz)(x+y+z)(0)$

(second and third row are the same)

$= 0$

Illustration 7 Without expansion find the value of $\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$

Solution

$$\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix} \quad \begin{matrix} c_1 \to c_1 - c_3 \\ c_2 \to c_2 - c_3 \end{matrix}$$

Subtracting the elements of third column from first and second column

$$\begin{vmatrix} 46 & 21 & 219 \\ 42 & 27 & 198 \\ 38 & 17 & 181 \end{vmatrix} \quad c_1 \to c_1 - 2c_2$$

Multiplying the elements of second column by 2 and subtracting these from the respective elements of column one

$$\begin{vmatrix} 4 & 21 & 219 \\ -12 & 27 & 198 \\ 4 & 17 & 181 \end{vmatrix} \quad c_3 \to c_3 - 10c_2$$

Multiplying the elements of second column by 10 and subtracting these from the respective elements of column three

$$\begin{vmatrix} 4 & 21 & 9 \\ -12 & 27 & -72 \\ 4 & 17 & 11 \end{vmatrix} \quad \begin{matrix} R_1 \to R_1 - R_3 \\ R_2 \to R_2 + 3R_3 \end{matrix}$$

Subtracting the elements of third row from second row and multiplying the elements of third row by 3 and adding these to the respective elements of second row value

$$\begin{vmatrix} 0 & 4 & -2 \\ 0 & 78 & -39 \\ 4 & 17 & 11 \end{vmatrix}$$

$$= 2(39)\begin{vmatrix} 0 & 2 & -1 \\ 0 & 2 & -1 \\ 4 & 17 & 11 \end{vmatrix}$$

$= 78(0)$

$= 0$

Illustration 8 Evaluate the following determinant $\begin{vmatrix} x & x+y & x+y+z \\ 2x & 3x+2y & 4x+3y+2z \\ 3x & 6x+3y & 10x+6y+3z \end{vmatrix}$

Solution

Second column is the sum of first two factors

$$\therefore \begin{vmatrix} x & x & x+y+z \\ 2x & 3x & 4x+3y+2z \\ 3x & 6x & 10x+6y+3z \end{vmatrix} + \begin{vmatrix} x & y & x+y+z \\ 2x & 2y & 4x+3y+2z \\ 3x & 3y & 10x+6y+3z \end{vmatrix}$$

$$= xx\begin{vmatrix} 1 & 1 & x+y+z \\ 2 & 3 & 4x+3y+2z \\ 3 & 6 & 10x+6y+3z \end{vmatrix} + xy\begin{vmatrix} 1 & 1 & x+y+z \\ 2 & 2 & 4x+3y+2z \\ 3 & 3 & 10x+6y+3z \end{vmatrix}$$

$$= xx\left[\begin{vmatrix} 1 & 1 & x \\ 2 & 3 & 4x \\ 3 & 6 & 10x \end{vmatrix} + \begin{vmatrix} 1 & 1 & y \\ 2 & 3 & 3y \\ 3 & 6 & 6y \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2z \\ 3 & 6 & 3z \end{vmatrix}\right] + xy(0)$$

$$= x^2x\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} + x^2y\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 6 & 6 \end{vmatrix} + x^2z\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 6 & 3 \end{vmatrix} + 0$$

$$= x^3\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 3 & 7 \end{vmatrix} + x^2y(0) + x^2z(0)$$

$$= x^3\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 3 & 7 \end{vmatrix}$$

$$= x^3(1)$$

$$= x^3$$

PRODUCT OF DETERMINANTS

Consider det $A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and det $B = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

then the product of two determinants can be carried out as under

$$\det A \det B = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1x_1 + b_1x_2 + c_1x_3 & a_1y_1 + b_1y_2 + c_1y_3 & a_1z_1 + b_1z_2 + c_1z_3 \\ a_2x_1 + b_2x_2 + c_2x_3 & a_2y_1 + b_2y_2 + c_2y_3 & a_2z_1 + b_2z_2 + c_2z_3 \\ a_3x_1 + b_3x_2 + c_3x_3 & a_3y_1 + b_3y_2 + c_3y_3 & a_3z_1 + b_3z_2 + c_3z_3 \end{vmatrix}$$

Thus the multiplication of two determinants is carried out by adding the multiplication of the respective row elements of the first determinant and column elements of the second determinant. Here we note that the value of the determinant remains unchanged by interchanging rows and columns. So we can also carry out multiplication by adding the multiplication of respective row–by–row elements or column–by–column elements.

Illustration 9 If $\det A = \begin{vmatrix} 2 & 3 & 1 \\ 4 & 0 & 1 \\ 3 & -1 & 2 \end{vmatrix}$ and $\det B = \begin{vmatrix} -1 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 2 \end{vmatrix}$ then find the multiplication of these two determinants.

Solution

$$\det A \det B = \begin{vmatrix} 2 & 3 & 1 \\ 4 & 0 & 1 \\ 3 & -1 & 2 \end{vmatrix}\begin{vmatrix} -1 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} -2+6+1 & 4+9+1 & 2+0+2 \\ -4+0+1 & 8+0+1 & 4+0+2 \\ -3-2+2 & 6-3+2 & 3+0+4 \end{vmatrix}$$

$$= \begin{vmatrix} 5 & 14 & 4 \\ 14 & 9 & 6 \\ 4 & 5 & 7 \end{vmatrix}$$

$$= 5(63-30) - 14(-21+18) + 4(-15+21)$$

$$= 231$$

Illustration 10 By using the multiplication of determinant prove that

$$\begin{vmatrix} 2x_1y_1 & x_1y_2 + x_2y_1 & x_1y_3 + x_3y_1 \\ x_1y_2 + x_2y_1 & 2x_2y_2 & x_2y_3 + x_3y_2 \\ x_1y_3 + x_3y_1 & x_2y_3 + x_3y_2 & 2x_3y_3 \end{vmatrix} = 0$$

Solution

$$\text{L.H.S.} = \begin{vmatrix} 2x_1y_1 & x_1y_2 + x_2y_1 & x_1y_3 + x_3y_1 \\ x_1y_2 + x_2y_1 & 2x_2y_2 & x_2y_3 + x_3y_2 \\ x_1y_3 + x_3y_1 & x_2y_3 + x_3y_2 & 2x_3y_3 \end{vmatrix}$$

$$= \begin{vmatrix} x_1y_1 + x_1y_1 & x_1y_2 + x_2y_1 & x_1y_3 + x_3y_1 \\ x_1y_2 + x_2y_1 & x_2y_2 + x_2y_2 & x_2y_3 + x_3y_2 \\ x_1y_3 + x_3y_1 & x_2y_3 + x_3y_2 & x_3y_3 + x_3y_3 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ x_3 & y_3 & 0 \end{vmatrix} + \begin{vmatrix} y_1 & x_1 & 0 \\ y_2 & x_2 & 0 \\ y_3 & x_3 & 0 \end{vmatrix}$$

$$= 0 + 0$$

$$= 0$$

APPLICATION OF DETERMINANTS

(1) Area of a triangle whose vertices are (x_1, y_1) (x_2, y_2) and (x_3, y_3) is given by the expression

$$\text{Area of triangle} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 3 \end{vmatrix}$$

$$= \frac{1}{2}\left[x_1(y_1 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]$$

Here we note that the area is always a positive quantity; therefore we always take the absolute value of the determinant for the area.

Illustration 11 Find the area of a triangle formed by the points A $(2, 3)$, B $(-1, 4)$, and C $(0, 5)$.

Solution

$$\text{Area of triangle} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ -1 & 4 & 1 \\ 0 & 5 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[2(4-5) - 3(-1-0) + 1(5-0)\right]$$

$$= \frac{1}{2}[-4]$$

$$= |-2| \qquad \text{(Area cannot be negative)}$$

$$= 2$$

(2) If three different points A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3) are collinear then area of triangle ABC is zero

$$\text{i.e. } \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Illustration 12 Show that the points $(2, 4)$ $(3, 3)$ and $(1, 5)$ are collinear.

Solution

We have area of triangle $= \frac{1}{2}\begin{vmatrix} 2 & 4 & 1 \\ 3 & 3 & 1 \\ 1 & 5 & 1 \end{vmatrix}$

Adding the elements of second and third column, we have

$$= \frac{1}{2}\begin{vmatrix} 2 & 6 & 1 \\ 3 & 6 & 1 \\ 1 & 6 & 1 \end{vmatrix}$$

$$= \frac{6}{2}\begin{vmatrix} 2 & 1 & 1 \\ 3 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

The given three points are collinear $= 3(0) = 0$

(3) Equation of a line passing through two different points A (x_1, y_1) and B (x_2, y_2)

$$\text{is } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Illustration 13 Obtain an equation of a line passing from the points A $(2, 3)$ and B $(-1, 5)$.

Solution

By using the determinant, the equation of line is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} x & y & 1 \\ 2 & 3 & 1 \\ -1 & 5 & 1 \end{vmatrix} = 0$$

$\therefore x(3 - 5) - y(2 + 1) + 1(10 + 3) = 0$

$\therefore -2x - 3y + 13 = 0$

$\therefore 2x + 3y - 13 = 0$

(4) For solving simultaneous linear equations (Cramer's rule for solving linear equations)

4.1 Consider a set of linear equations as

$$\begin{array}{l} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{array}$$

then the solution to the above simultaneous equations is given as $x = \dfrac{Dx}{D}$ and $y = \dfrac{Dy}{D}$ where

$$Dx = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix},\ Dy = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \text{ and } D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix};\ D \neq 0$$

Proof

Let us solve the given equation by the method of elimination

$$\begin{array}{l} a_1b_2x + b_1b_2y = c_1b_2 \\ a_2b_1x + b_1b_2y = c_2b_1 \\ - \qquad\ \ - \qquad\ \ - \\ \left(a_1b_2 - a_2b_1\right)x = c_1b_2 - c_2b_1 \end{array}$$

$$\therefore x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_x}{D}$$

Now by eliminating x we have

$$\begin{array}{l} a_1a_2x + a_2b_1y = a_2c_1 \\ a_1b_2x + a_1b_2y = a_2c_2 \\ - \qquad\ \ - \qquad\ \ - \\ \left(a_2b_1 - a_1b_2\right)y = a_2c_1 - a_1c_2 \\ \therefore -\left(a_1b_2 - a_2b_1\right)y = a_2c_1 - a_1c_2 \end{array}$$

$$\therefore y = \frac{-\left(a_2c_1 - a_1c_2\right)}{\left(a_1b_2 - a_2b_1\right)} = \frac{-\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{-D_y}{D}$$

Thus the solution to the given set of equations is

$$x = \frac{D_x}{x},\ y = \frac{D_y}{D}$$

$$\text{where } D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix},\ D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix},\ D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

4.2 Consider a system of simultaneous linear equations given by

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

then the solution to the above system of equations is given as

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D} \text{ and } z = \frac{D_z}{D}$$

where $D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$

$$D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \text{ and } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} ; D \neq 0$$

Proof

Consider the determinant of coefficients

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\text{Now } xD = x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix} \quad c_1 \rightarrow c_1 + yc_2 + zc_3$$

Multiplying 2nd column by y and third by z and then adding to the first column, we have

$$= \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \quad \text{by using given set of equations.}$$

$$xD = D_x$$

$$\therefore x = \frac{D_x}{D}$$

Similarly $yD = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} = D_y$ and $zD = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

$\therefore\ x = \frac{D_x}{D},\ y = \frac{D_y}{D}$, and $z = \frac{D_z}{D}$, is the solution to the given set of equations.

Note:

(i) Here if $d_1 = d_2 = d_3 = 0$ then the equations are called homogeneous equations. Otherwise they are called non–homogeneous equations.

(ii) If $D = 0$ and remaining determinants also have zero value then the given systems of equations is consistent and has infinitely many solutions.

(iii) If $D = 0$ and remaining determinants non–zero then the given sets of equations are called inconsistent.

(iv) If $D \neq 0$ then the given sets of equations is called consistent.

4.3 The above method of solving a system of three linear equations in three variables can be used exactly the same way to solve as system of n equations in n equations as under.

Consider the set of n linear equations as

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots.a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots.a_{2n}x_n = b_2$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots.a_{nn}x_n = b_n$$

Let D = the determinant of coefficients then $\begin{vmatrix} a_{11} + a_{12} + \ldots\ldots.a_{1n} \\ a_{21} + a_{22} + \ldots\ldots.a_{2n} \\ \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \\ a_{n1} + a_{n2} + \ldots\ldots.a_{nn} \end{vmatrix}$ and

the determinant Dj can be obtained by replacing jth column by

$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ then the solution is

$$x_1 = \frac{D_1}{D},\ x_2 = \frac{D_2}{D}, \ldots\ldots,\ x_n = \frac{D_n}{D}$$

Illustration 14 Solve the following set of equations by using Cramer's method: $4x - 3y - 5 = 0$ and $-2x + y + 3 = 0$.

Solution

The given equations are $4x - 3y - 5 = 0$ and $-2x + y + 3 = 0$

Now according to Cramer's rule

$$D = \begin{vmatrix} 4 & -3 \\ -2 & 1 \end{vmatrix} = 4 - 6 = -2$$

$$D_x = \begin{vmatrix} 5 & -3 \\ -3 & 1 \end{vmatrix} = 5 - 9 = -4$$

$$D_y = \begin{vmatrix} 4 & 5 \\ -2 & -3 \end{vmatrix} = -12 + 10 = -2$$

Hence the solution is

$$x = \frac{D_x}{D} = \frac{-4}{-2} = 2 \text{ and } y = \frac{D_y}{D} = \frac{-2}{-2} = 1$$

$x = 2$ and $y = 1$ is the solution to given set of equations.

Illustration 15 Solve the following set of equations by using determinants: $2x + 3y - z = 8$, $4x + z = 7$, $3y + 2z = 1$

Solution

$2x + 3y - z = 8$

$4x + z = 7$

$3y + 2z = 1$

Now by using Cramer's rule

$$D = \begin{vmatrix} 2 & 3 & -1 \\ 4 & 0 & 1 \\ 0 & 3 & 2 \end{vmatrix}$$

$$= 2(0 - 3) - 3(8 - 0) + (-1)(12 - 0)$$

$$= -2 - 12$$

$$= -42$$

$$D_x = \begin{vmatrix} 8 & 3 & -1 \\ 7 & 0 & 1 \\ 1 & 3 & 2 \end{vmatrix}$$

$$= 8(0 - 3) - 3(14 - 1) + (-1)(21 - 0)$$

$$= -24 - 39 - 21$$

$$= -84$$

$$D_y = \begin{vmatrix} 2 & 8 & -1 \\ 4 & 7 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

$$\begin{aligned} &= 2(14-1) - 8(8-0) + (-1)(4-0) \\ &= 26 - 64 - 4 \\ &= -42 \end{aligned}$$

$$D_z = \begin{vmatrix} 2 & 3 & 8 \\ 4 & 0 & 7 \\ 0 & 3 & 1 \end{vmatrix}$$

$$\begin{aligned} &= 2(0-21) - 3(4-0) + 8(12-0) \\ &= -42 - 12 + 96 \\ &= 42 \end{aligned}$$

The solution to the given system of equations

$$x = \frac{D_x}{D} = \frac{-84}{-42} = 2$$

$$y = \frac{D_y}{D} = \frac{-42}{-42} = 1$$

$$z = \frac{D_z}{D} = \frac{42}{-42} = -1$$

Hence the solution is $x = 2$, $y = 1$ and $z = -1$.

Illustration 16 Solve the system of equations $3x + 2y = 5$ and $6x + 4y = 10$ by using determinants.

Solution

Here we have $D = \begin{vmatrix} 3 & 2 \\ 6 & 4 \end{vmatrix} = 12 - 12 = 0$

and $D_x = \begin{vmatrix} 5 & 2 \\ 10 & 4 \end{vmatrix} = 20 - 20 = 0$

$$D_y = \begin{vmatrix} 3 & 5 \\ 6 & 10 \end{vmatrix} = 30 - 30 = 0$$

$D = D_x = D_y = 0$

So the given system of equations has infinite number of solutions.

Suppose $y = k$. Then $x + 2y = 3$

$x = 3 - 2k$

Hence $x = 3 - 2k$, $y = k$ is the given system of equations, where k is any arbitrage real number.

Illustration 17 Show that the system of the equations $4x - 3y = 9$ and $12x - 9y = 20$ is inconsistent.

Solution

Here $D = \begin{vmatrix} 4 & -3 \\ 12 & -9 \end{vmatrix} = -36 - (-36) = 0$

Now $D_x = \begin{vmatrix} 9 & -3 \\ 20 & -9 \end{vmatrix} = -81 - (-60) = -21 \neq 0$

Since $D = 0$, therefore the given system is consistent.

Illustration 18 Solve the following system of equations by using determinants.
$x + 3y + 5z = 7, x + y + z = 1, x + 2y + 3z = 4.$

Solution

Consider

$$D = \begin{vmatrix} 1 & 3 & 5 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= 1(3-2) - 3(3-1) + 5(2-1)$$
$$= 1 - 6 + 5$$
$$= 0$$

$$D_x = \begin{vmatrix} 7 & 3 & 5 \\ 1 & 1 & 1 \\ 4 & 2 & 3 \end{vmatrix}$$

$$= 7(3-2) - 3(3-4) + 5(2-4)$$
$$= 7 + 3 - 10$$
$$= 0$$

$$D_y = \begin{vmatrix} 1 & 7 & 5 \\ 1 & 1 & 1 \\ 1 & 4 & 3 \end{vmatrix}$$

$$= 1(3-4) - 7(3-1) + 5(4-1)$$
$$= 0$$

$$D_z = \begin{vmatrix} 1 & 3 & 7 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix}$$

$$= 1(4-2) - 3(4-1) + 7(2-1)$$
$$= 2 - 9 + 17$$
$$= 0$$

Since $D = Dx = Dy = Dz = 0$.

So the given system of equations has infinitely many solutions. Consider the first two equations as

$x + 3y = 7 - 5z$

$x + y = 1 - z$

Solving these equations by using Cramer's rule, we have

$$D = \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = 1 - 3 = -2$$

$$\text{and } D_x = \begin{vmatrix} 7-5z & 3 \\ 1-z & 1 \end{vmatrix}$$

$$= 7 - 5z - 3(1-z)$$

$$= 7 - 5z - 3 + 3z$$

$$= 4 - 2z$$

$$\text{and } D_y = \begin{vmatrix} 1 & 7-5z \\ 1 & 1-z \end{vmatrix}$$

$$= 1 - z - (7 - 5z)$$

$$= 1 - z - 7 + 5z$$

$$= 4z - 6$$

Hence

$$x = \frac{D_x}{D} = \frac{4-2z}{-2} = \frac{-2(z-2)}{-z} = z - 2$$

$$y = \frac{D_y}{D} = \frac{4z-6}{-2} = \frac{-2(3-2z)}{-z} = 3 - 2z$$

Suppose $z = k$, where k is any real number, then the set of solution of given equations is

$x = k - 2, y = 3 - 2k, z = k$

Illustration 19 Check the consistency of the following equations: $2x + y + 3z = 4$, $-x + 3y + 2z = 12$, $x + 2y + 3z = 10$

Solution

$$\text{Here } D = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 3 & 2 \\ 1 & 2 & 3 \end{vmatrix} = 0 \quad c_2\ c_1 + c_2$$

Adding the elements of first and second column we have

$$D = \begin{vmatrix} 2 & 3 & 3 \\ -1 & 2 & 2 \\ 1 & 3 & 3 \end{vmatrix} = 0$$

$$\text{and } D_x = \begin{vmatrix} 4 & 1 & 3 \\ 12 & 3 & 2 \\ 10 & 2 & 3 \end{vmatrix}$$

$$= 4(9-4) - 1(36-20) + 3(24-30)$$

$$= 20 - 16 + 18$$

Since $D = 0$, therefore the given system of equations is inconsistent.

Illustration 20 If the system of equations $x + y + z = 6$, $x + 2y + 3z = 10$ and $x + 2y + az = b$ have infinite number of solutions then find the value of a and b. Also find the solution to the given set of equations.

Solution

Since the given set of equations has infinite solutions so we have

$D_x = D_y = D_z = D = 0$

$$\text{Now } D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & a \end{vmatrix} = 0$$

$1(2a - 6) - 1(a - 3) + 1(2 - 2) = 0$

$2a - 6 - a + 3 + 1(0) = 0$

$a - 3 = 0$

$a = 3$

or

Comparing the elements of 2nd and 3rd row we have $a = 3$

$$\text{and } D_x = \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ b & 2 & 3 \end{vmatrix} = 0$$

Again comparing the elements of 2nd and 3rd rows we have $b = 10$

Hence the given equations are

$x + y + z = 6$, $x + 2y + 3z = 10$ and $x + 2y + 3z = 10$

Consider the first two equations as

$x + y = 6 - z$

$x + 2y = 10 - 3z$

Solving the two equations by using elimination method we have

$$\begin{array}{l} 2x + 2y = 12 - 2z \\ x + 2y = 10 - 3z \\ - \quad - \quad\; - \quad + \\ \hline x = 2 + z \end{array}$$

Put $x = 2 + z$ in $x + 2y = 6 - z$

$2 + z + 2y = 6 - z$

$2y = 6 - z - 2 - z$

$2y = 4 - 2z$

$y = 2 - z$

Now suppose $z = k$ then the solution to the given set of equations is $x = 2 + k$, $y = 2 - k$, $z = k$

Illustration 21 Solve the following systems of equations by using determinants. It is known that $x + y + z = 1$, $ax + by + cz = k$, $ax^2 + b^2y + c^2z = k^2$

Solution

We have

$$D = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \quad \begin{array}{l} c_1 \to c_1 - c_2 \\ c_2 \to c_2 - c_3 \end{array}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ a+b & b+c & c^2 \end{vmatrix} \quad c_1 \to c_1 - c_2$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ a-c & b+c & c^2 \end{vmatrix}$$

$D = (a-b)(b-c)(c-a)$

Similarly we find D_x, D_y and D_z as under

$$D_x = \begin{vmatrix} 1 & 1 & 1 \\ k & b & c \\ k^2 & b^2 & c^2 \end{vmatrix} \quad \begin{matrix} c_1 \to c_1 - c_2 \\ c_2 \to c_2 - c_3 \end{matrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ k-b & b-c & c \\ k^2-b^2 & b^2-c^2 & c^2 \end{vmatrix}$$

$$= (k-b)(b-c)(c-k)$$

$$D_y = \begin{vmatrix} 1 & 1 & 1 \\ a & k & c \\ a^2 & k^2 & c^2 \end{vmatrix} = (a-k)(k-c)(c-a)$$

$$D_z = \begin{vmatrix} 1 & 1 & 1 \\ a & b & k \\ a^2 & b^2 & k^2 \end{vmatrix} = (a-b)(b-k)(k-a)$$

Hence the solution to the given equations is

$$x = \frac{D_x}{D} = \frac{(k-b)(b-c)(c-k)}{(a-b)(b-c)(c-a)} = \frac{(k-b)(c-k)}{(a-b)(c-a)}$$

$$= \frac{k^2-(b+c)k+bc}{a^2-(b+c)a+bc}$$

$$y = \frac{D_y}{D} = \frac{(a-k)(k-c)(c-a)}{(a-b)(b-c)(c-a)} = \frac{(a-k)(k-c)}{(a-b)(b-c)}$$

$$= \frac{k^2-(a+1)k+ac}{b^2-(a+c)b+ac}$$

$$z = \frac{D_z}{D} = \frac{(a-b)(b-k)(k-a)}{(a-b)(b-c)(c-a)} = \frac{(b-k)(k-a)}{(b-c)(c-a)}$$

$$= \frac{k^2 - (a+b)k + ab}{c^2 - (a+b)c + ab}$$

Illustration 22 Prove the following

$$\begin{vmatrix} p-q-z & 2q & 2z \\ 2p & q-z-p & 2z \\ 2p & 2q & z-p-q \end{vmatrix} = (p+q+z)^3$$

Solution

$$\text{L.H.S.} \begin{vmatrix} p-q-z & 2q & 2z \\ 2p & q-z-p & 2z \\ 2p & 2q & z-p-q \end{vmatrix} \quad c_1 \rightarrow c_1 + c_2 + c_3$$

$$= \begin{vmatrix} p+q+z & 2q & 2z \\ p+q+z & q-z-p & 2z \\ p+q+z & 2p & z-p-q \end{vmatrix}$$

$$= (p+q+z) \begin{vmatrix} 1 & 2q & 2z \\ 1 & q-z-p & 2z \\ 1 & 2q & z-p-q \end{vmatrix} \quad \begin{matrix} R_1 \rightarrow R_1 - R_3 \\ R_2 \rightarrow R_2 - R_3 \end{matrix}$$

$$= (p+q+z) \begin{vmatrix} 0 & 0 & p+q+z \\ 0 & -p-q-z & p+q+z \\ 1 & 2q & z-p-q \end{vmatrix}$$

$$= (p+q+z)^3 \begin{vmatrix} 0 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 2q & z-p-q \end{vmatrix}$$

$$= (p+q+z)^3 (1)$$

$$= (p+q+z)^3$$

Illustration 23 Evaluate the following determinant

$$\begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix}$$

Solution

We have

$$= \begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix}$$

$$= \begin{vmatrix} a^2-2ax+x^2 & a^2-2ay+y^2 & a^2-2az+z^2 \\ b^2-2bx+x^2 & b^2-2by+y^2 & b^2-2bz+z^2 \\ c^2-2cx+x^2 & c^2-2cy+y^2 & c^2-2cz+z^2 \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & -2a & 1 \\ b^2 & -2b & 1 \\ c^2 & -2c & 1 \end{vmatrix} \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \text{ using row-by-row multiplication}$$

$$= -2\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= \pm 2(a-b)(b-c)(c-a)(x-y)(y-z)(z-x)$$

Illustration 24 Prove that $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (c+a)^2 \end{vmatrix} = 2abc(a+b+c)^3$

Solution

We have

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (c+a)^2 \end{vmatrix} \begin{matrix} c_1 \to c_1 - c_3 \\ c_2 \to c_2 - c_3 \end{matrix}$$

$$= \begin{vmatrix} (b+c)^2-a^2 & 0 & a^2 \\ 0 & (c+a)^2-b^2 & b^2 \\ c^2-(a+b)^2 & c^2-(a+b)^2 & (a+b)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (a+b+c)(b+c-a) & 0 & a^2 \\ 0 & (a+b+c)(a+c-b) & b^2 \\ (a+b+c)(c-a-b) & (a+b+c)(c-a-b) & (a+b)^2 \end{vmatrix}$$

$$= (a+b+c)(a+b+c)\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & a+c-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix}$$

$R_3 \to R_3 - R_1 - R_2$

$$= (a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & a+c-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix}$$

$$= (a+b+c)^2$$

$$\left[(b+c-2a)\begin{vmatrix} a+c-b & b^2 \\ -2a & 2ab \end{vmatrix} - 0\begin{vmatrix} 0 & b^2 \\ -2b & 2ab \end{vmatrix} + a^2\begin{vmatrix} 0 & a+c-b \\ -2b & -2a \end{vmatrix}\right]$$

$$= (a+b+c)^2\left[(b+c-a)\left(2a^2b-2ab^2+2ab^2\right)+0+a^2\left(0+2ab+2bc-2b^2\right)\right]$$

$$= (a+b+c)^2\left[(b+c-a)\left(2a^2b+2abc\right)+2a^3b+2a^2bc-2a^2b^2\right)\right]$$

$$= (a+b+c)^2\ \left(2a^2b^2+2a^2bc+2abc^2-2a^3b-2a^2bc+2a^3b+2a^2bc-2a^2b^2\right)$$

$$= (a+b+c)^2\ \left(2a^{2bc}+2ab^2c+2abc^2\right)$$

$$= (a+b+c)^2\ 2abc(a+b+c)$$

$$= 2abc(a+b+c)^3$$

Illustration 25 If det $A = \begin{vmatrix} 1 & 2 \\ 5 & 3 \end{vmatrix}$ and det $B = \begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix}$ then show that det AB = det A det B.

Solution

Here det $AB = \begin{vmatrix} 1 & 2 \\ 5 & 3 \end{vmatrix}$

$$= \begin{vmatrix} 2+6 & -2+2 \\ 10+9 & -10+3 \end{vmatrix}$$

$$= \begin{vmatrix} 8 & 0 \\ 0 & -7 \end{vmatrix}$$

$$= -56 - 0$$

$$= -56$$

Now det $A = \begin{vmatrix} 1 & 2 \\ 5 & 3 \end{vmatrix} = 3 - 10 = -7$

and det $B = \begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix} = 2 + 6 = 8$

Now det A det $B = (-7) \times 8 = -56$ = det AB.

Illustration 26 Solve the equation $\begin{vmatrix} 4x-8 & 4 & 4 \\ 4 & 4x-8 & 4 \\ 4 & 4 & 4x-8 \end{vmatrix} = 0$

Solution

Here it is given that

$$\begin{vmatrix} 4x-8 & 4 & 4 \\ 4 & 4x-8 & 4 \\ 4 & 4 & 4x-8 \end{vmatrix} = 0 \quad R_1 \to R_1 + R_2 + R_3$$

$$= \begin{vmatrix} 4x & 4x & 4x \\ 4 & 4x-8 & 4 \\ 4 & 1 & 4x-8 \end{vmatrix} = 0$$

$$= 4x\begin{vmatrix} 1 & 1 & 1 \\ 4 & 4x-8 & 4 \\ 4 & 4 & 4x-8 \end{vmatrix} = 0 \quad \begin{matrix} c_2 \to c_2 - c_1 \\ c_3 \to c_3 - c_1 \end{matrix}$$

$$= 4x\begin{vmatrix} 1 & 0 & 0 \\ 4 & 4x-12 & 0 \\ 4 & 0 & 4x-12 \end{vmatrix} = 0$$

$$= 4x(4x-12)^2 = 0$$

$$\therefore 4x = 0 \text{ or } (4x-12)^2 = 0$$

$$\therefore x = 0 \text{ or } x = 3$$

Illustration 27 Without expanding prove that the value of $\begin{vmatrix} 6 & 5 & 7 \\ 3 & 5 & 4 \\ 1 & 1 & 5 \end{vmatrix}$ is multiple of 19.

(Hint: 361, 551, 475)

Solution

Here we have

$$= \begin{vmatrix} 6 & 5 & 7 \\ 3 & 5 & 4 \\ 1 & 1 & 5 \end{vmatrix} \quad R_3 \to R_3 + 100R_2$$

$$= \begin{vmatrix} 6 & 5 & 7 \\ 3 & 5 & 4 \\ 301 & 501 & 405 \end{vmatrix} \quad R_3 \to R_3 + 10R_1$$

$$= \begin{vmatrix} 6 & 5 & 7 \\ 3 & 5 & 4 \\ 361 & 551 & 475 \end{vmatrix}$$

$$= 19\begin{vmatrix} 6 & 5 & 7 \\ 3 & 5 & 4 \\ 19 & 29 & 25 \end{vmatrix}$$

$= 19 \times$ constant.

Hence the value of given determinant is multiple of 19.

Illustration 28 Without evaluating the determinant, prove that

$$\begin{vmatrix} (a+1)^3 & a^2 & a \\ (b+1)^3 & b^2 & b \\ (c+1)^3 & c^2 & c \end{vmatrix} = \begin{vmatrix} a^3+1 & a^2 & a \\ b^3+1 & b^2 & b \\ c^3+1 & c^2 & c \end{vmatrix}$$

Solution

Here

$$\text{R.H.S.} = \begin{vmatrix} (a+1)^3 & a^2 & a \\ (b+1)^3 & b^2 & b \\ (c+1)^3 & c^2 & c \end{vmatrix}$$

$$= \begin{vmatrix} a^3+3a^2+3a+1 & a^2 & a \\ b^3+3b^2+3b+1 & b^2 & b \\ c^3+3c^2+3c+1 & c^2 & c \end{vmatrix}$$

$$= \begin{vmatrix} a^3+1 & a^2 & a \\ b^3+1 & b^2 & b \\ c^3+1 & c^2 & c \end{vmatrix} + \begin{vmatrix} 3a^2 & a^2 & a \\ 3b^2 & b^2 & b \\ 3c^2 & c^2 & c \end{vmatrix} + \begin{vmatrix} 3a & a^2 & a \\ 3b & b^2 & b \\ 3c & c^2 & c \end{vmatrix}$$

$$= \begin{vmatrix} a^3+1 & a^2 & 1 \\ b^3+1 & b^2 & 1 \\ c^3+1 & c^2 & 1 \end{vmatrix} + 3(0) + 3(0)$$

$$= \begin{vmatrix} a^3+1 & a^2 & 1 \\ b^3+1 & b^2 & 1 \\ c^3+1 & c^2 & 1 \end{vmatrix}$$

= R.H.S.

Illustration 29 If the area of a triangle formed by the vertices $(x, 8)$, $(4, 2)$ and $(5, -1)$ is $\frac{75}{2}$ then find the value of x.

Solution

$$\text{Area of triangle } = \frac{1}{2}\begin{vmatrix} x & 8 & 1 \\ -4 & 2 & 1 \\ 5 & -1 & 1 \end{vmatrix}$$

$$\therefore \frac{75}{2} = \frac{1}{2}\left(x\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} - 8\begin{vmatrix} -4 & 1 \\ 5 & 1 \end{vmatrix} + 1\begin{vmatrix} -4 & 2 \\ 5 & -1 \end{vmatrix} \right)$$

$$\therefore \frac{75}{2} = x(2+1) - 8(-4-5) + 1(4-10)$$

$$\therefore \frac{75}{2} = 3x + 72 - 6$$

$$\therefore 3x = 9$$

$$\therefore x = 3$$

Illustration 30 Prove that the points $(b, c+a)$, $(c, a+b)$ and $(a, b+c)$ are collinear points.

Solution

$$\text{Area of triangle } = \frac{1}{2}\begin{vmatrix} b & c+a & 1 \\ c & a+b & 1 \\ a & b+c & 1 \end{vmatrix} \quad c_2 \to c_2 + c_1$$

$$= \frac{1}{2}\begin{vmatrix} b & a+b+c & 1 \\ c & a+b+c & 1 \\ a & a+b+c & 1 \end{vmatrix}$$

$$= \frac{(a+b+c)}{2}\begin{vmatrix} b & 1 & 1 \\ c & 1 & 1 \\ a & 1 & 1 \end{vmatrix}$$

$$= \frac{(a+b+c)}{2}(0)$$

$$= 0$$

Since the area of the triangle formed by the given points is zero, therefore the given points are collinear.

Illustration 31 If $x = \lambda x_2 + (1-\lambda)x_1$ and $y = \lambda y_2 + (1-\lambda)y_1$ where $\lambda \in R$ then prove that $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$

Solution

We have $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$

Multiplying second row by $(1-\lambda)$ and third row by λ we have

$$= \frac{1}{\lambda(\lambda-1)} \begin{vmatrix} x & y & 1 \\ (1-\lambda)x_1 & (1-\lambda)y_1 & 1-\lambda \\ \lambda x_2 & \lambda y_2 & \lambda \end{vmatrix} \text{ Now } R_3 \to R_3 + R_2$$

$$= \frac{1}{\lambda(\lambda-1)} \begin{vmatrix} x & y & 1 \\ (1-\lambda)x_1 & (1-\lambda)y_1 & 1-\lambda \\ \lambda x_2 + (1-\lambda)x_1 & \lambda y_2 + (1-\lambda)y_1 & \lambda + 1 - \lambda \end{vmatrix}$$

$$= \frac{1}{\lambda(\lambda-1)} \begin{vmatrix} x & y & 1 \\ (1-\lambda)x_1 & (1-\lambda)y_1 & 1-\lambda \\ x & y & 1 \end{vmatrix}$$

$$= \frac{1}{\lambda(\lambda-1)}(0)$$

$$= 0$$

ANALYTICAL EXERCISES

1. Evaluate the following determinants

(i) $\begin{vmatrix} 2 & 1 \\ -3 & -1 \end{vmatrix}$ (ii) $\begin{vmatrix} x & y(2+y) \\ -x & x \end{vmatrix}$

2. Evaluate the following determinants

(i) $\begin{vmatrix} 2 & 1 & -1 \\ -3 & 0 & 3 \\ 4 & -2 & 4 \end{vmatrix}$ (ii) $\begin{vmatrix} 4 & -2 & 2 \\ -2 & -1 & 0 \\ -3 & 0 & 3 \end{vmatrix}$ (iii) $\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$

3. Obtain the value of following determinants by using SAARUS method

(i) $\begin{vmatrix} 2 & 4 & -3 \\ -1 & -1 & 0 \\ 3 & 2 & 1 \end{vmatrix}$ (ii) $\begin{vmatrix} -2 & 0 & -4 \\ 4 & 3 & 3 \\ 1 & -1 & 2 \end{vmatrix}$ (iii) $\begin{vmatrix} 1 & 2 & 3 \\ a & -a & b \\ -a & 0 & -b \end{vmatrix}$

4. I $\begin{vmatrix} a & 0 & 2 \\ a & 2 & 5 \\ 6 & 8 & 0 \end{vmatrix}$ $f = -48$ then find value of a.

5. If $\begin{vmatrix} 3 & 4 & 7 \\ 2 & k & 3 \\ -5 & k & 2 \end{vmatrix} = -30$ then find value of k.

6. If $\begin{vmatrix} 2 & 3 & p \\ 1 & 0 & 7 \\ p & 5 & 6 \end{vmatrix} = 16$ then find the value of p.

7. If $\begin{vmatrix} 2 & 4a & 9 \\ 2a & 5 & 6 \\ 1 & a & 3 \end{vmatrix} = 0$ then obtain the value of a.

8. If $\begin{vmatrix} 1 & k & 3 \\ 0 & 5 & 0 \\ k & 2k & 3 \end{vmatrix} = 105$ what is the value of k.

9. *When* $\begin{vmatrix} p & p & 0 \\ p & 1 & 1 \\ -7 & p & -3 \end{vmatrix} = -12$ then find the values of p.

10. Expand the following determinant about second row $\begin{vmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{vmatrix}$

11. Expand the following determinant about third column $\begin{vmatrix} 1 & 0 & -4 \\ -2 & 2 & 5 \\ 3 & -1 & 2 \end{vmatrix}$

12. Expand the following determinant by considering second column $\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 1 & 4 & -3 \end{vmatrix}$

13. Without expanding find the value of x from the following $\begin{vmatrix} 7 & 8 & 4 \\ 2 & 3 & 5 \\ x-4 & -6 & -10 \end{vmatrix} = 10$

14. Find the values of following determinants by using properties of determinant

(i) $\begin{vmatrix} 8 & 11 & 14 \\ 5 & 8 & 11 \\ 2 & 5 & 8 \end{vmatrix}$ (ii) $\begin{vmatrix} 12 & 14 & 16 \\ 10 & 20 & 30 \\ 6 & 7 & 8 \end{vmatrix}$ (iii) $\begin{vmatrix} 0 & a & b \\ -a & 0 & -c \\ -b & c & 0 \end{vmatrix}$

15. Without expanding show that the following is a multiple of 11.

$$\begin{vmatrix} 8 & 3 & 6 \\ 7 & 3 & 7 \\ 8 & 3 & 6 \end{vmatrix}$$

(Hint : 616, 737, 836 are multiples of 11)

16. Without expanding show that the following is divisible by 18. $\begin{vmatrix} 4 & 3 & 1 \\ 3 & 2 & 2 \\ 2 & 4 & 6 \end{vmatrix}$

(Hint : 126, 324, 732 are divisible by 18)

17. By using properties of determinant, find the value of x if $\begin{vmatrix} 12 & 6 & x \\ 24 & 2x+8 & 4 \\ 6 & 3 & 5 \end{vmatrix}$

18. Without expanding find the value of k.

$$\begin{vmatrix} 2 & -1 & 5 \\ 2k-3 & -4 & -7 \\ 1 & 2 & 5 \end{vmatrix} = \begin{vmatrix} k & 2 & 4 \\ 2 & -1 & 5 \\ 1 & 2 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 6 \\ 4 & 2 & 3 \\ 2 & -1 & 5 \end{vmatrix}$$

19. Without the help of properties of determinant prove the following

(i) $\begin{vmatrix} 7^2 & 24^2 & 25^2 \\ 5^2 & 12^2 & 13^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix} = 0$ (ii) $\begin{vmatrix} 1^2 & 5^2 & 3^2 \\ 2^2 & 25^2 & 24^2 \\ 3^2 & 41^2 & 40^2 \end{vmatrix} + \begin{vmatrix} 4^2 & 1^2 & 5^2 \\ 7^2 & 2^2 & 25^2 \\ 9^2 & 3^2 & 41^2 \end{vmatrix} = 0$

20. Prove the following. (Use properties of determinants)

(i) $\begin{vmatrix} p+q & q+r & r+p \\ q+r & r+p & p+q \\ r+p & p+q & q+r \end{vmatrix} = 2\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix}$

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix} \left(x, y, z \neq 0\right)$

(iii) $\begin{vmatrix} a & b & c \\ bq & qr & rp \\ xy & yz & zx \end{vmatrix} = \begin{vmatrix} arz & bpx & cqy \\ z & x & y \\ r & p & q \end{vmatrix}$ (iv) $\begin{vmatrix} yz & zx & xy \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^2 & y^3 & z^3 \end{vmatrix}$

21. Using the rules of determinant prove the following

(i) $\begin{vmatrix} (x+1)^3 & x^2 & x \\ (y+1)^3 & y^2 & y \\ (z+1)^3 & z^2 & z \end{vmatrix} = \begin{vmatrix} x^3+1 & x^2 & x \\ y^3+1 & y^2 & y \\ z^3+1 & z^2 & z \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & p & (p-1)^2 \\ 1 & q & (q-1)^2 \\ 1 & r & (r-1)^2 \end{vmatrix} = \begin{vmatrix} 1 & p & p^2 \\ 1 & q & q^2 \\ 1 & r & r^2 \end{vmatrix}$

22. If p, q, r are different real numbers then prove that $\begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ p^2 & q^2 & r^2 \end{vmatrix} \neq 0$

23. For any different values of p, q, r prove that

$$\begin{vmatrix} q^2 + z^2 & pq & pz \\ pq & z^2 + p^2 & qz \\ pz & qz & p^2 + q^2 \end{vmatrix} = 4p^2q^2z^2$$

24. Prove that $\begin{vmatrix} 1 + a^2 & ab & ac \\ ab & 1 + b^2 & bc \\ ac & bc & 1 + c^2 \end{vmatrix} = a^2 + b^2 + c^2 + 1$

25. Without expanding prove that $\begin{vmatrix} x + y & y + z & z + x \\ y & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0$

26. Without expanding prove that $\begin{vmatrix} (yz)^2 & yz & y + z \\ (zx)^2 & zx & z + x \\ (xy)^2 & xy & x + y \end{vmatrix} = 0$

27. Without expanding evaluate the determinant

$$\begin{vmatrix} \left(a^x + \frac{1}{a^x}\right)^2 & \left(a^x - \frac{1}{a^x}\right)^2 & 1 \\ \left(a^y + \frac{1}{a^y}\right)^2 & \left(a^y - \frac{1}{a^y}\right)^2 & 1 \\ \left(a^z + \frac{1}{a^z}\right)^2 & \left(a^z - \frac{1}{a^z}\right)^2 & 1 \end{vmatrix}$$

28. Without expanding find the value of the following

(i) $\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$ (ii) $\begin{vmatrix} 50 & 55 & 80 \\ 151 & 155 & 140 \\ 201 & 210 & 220 \end{vmatrix}$ (iii) $\begin{vmatrix} 200 & 408 & 207 \\ 100 & 205 & 105 \\ 300 & 608 & 310 \end{vmatrix}$

29. If area of a triangle formed by vertices $(5, 4)$, $(k, 4)$ and $(2, -6)$ is 35. Find the value of k.

30. Find the area of a triangle whose one vertice is origin and other two vertices are (a, b) and (b, a) respectively where $a > b$.

31. The points (x, x^2), (y, y^2), (z, z^2) are vertices of a triangle. What is the area of the triangle?

32. Show that the points $(a, b + c)$, $(b, a + c)$ and $(c, a + b)$ are collinear points.

33. Show that $\begin{vmatrix} p+q & p-q & p-q \\ p-q & p+q & p-q \\ p-q & p-q & p+q \end{vmatrix} = 4q^2(3p-q)$

34. Show that $\begin{vmatrix} x+p & q & r \\ r & x+q & p \\ p & q & x+r \end{vmatrix} = x^2(x+p+q+r)$

35. Show that $\begin{vmatrix} p & q & r \\ p^2 & q^2 & r^2 \\ q^2 & q^2 & pq \end{vmatrix} = (p-q)(q-r)(r-p)(pq+qr+pr)$

36. Show that $\begin{vmatrix} p & q & r \\ p & q & q \\ q & p & p \end{vmatrix} = p(p-q)(r-q)$

37. Show that $\begin{vmatrix} a+b+2c & a & b \\ c & a+b+2c & b \\ c & a & a+b+2c \end{vmatrix} = 2(a+b+c)$

38. Show that $\begin{vmatrix} p & q & r \\ p-q & q-r & r-p \\ q+r & r+p & p+q \end{vmatrix} = p^3+q^3+r^3-3pqr$

39. Show that $\begin{vmatrix} 1+p & 1 & 1 \\ 1 & 1+q & 1 \\ 1 & 1 & 1+r \end{vmatrix} = pqr\left(1+\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)$

40. Show that $\begin{vmatrix} 1+x^2 & xy & xz \\ xy & 1+y^2 & yz \\ xz & yz & 1+z^2 \end{vmatrix} = 1+x^2+y^2+z^2$

41. Show that $\begin{vmatrix} 1-x^2+y^2 & 2xy & -2x \\ 2xy & 1+x^2-y^2 & 2y \\ 2x & -2y & 1-x^2-y^2 \end{vmatrix} = (1+x^2+y^2)^3$

42. If w is cube root of unity then prove that $\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix} = 0$

43. Show that $\begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & b^2c & 0 \end{vmatrix} = 2a^3b^3c^3$

44. Solve the following equations by using Cramer's rule

(i) $2x + 5y = 16$ and $3x + y = 11$

(ii) $2x + 5y - 21 = 0$ and $x + 2y - 8 = 0$

(iii) $2xy - 3x - y = 0$ and $2xy - x + y = 0$

(iv) $\frac{1}{x} + \frac{2}{y} = 4$ and $\frac{3}{x} + \frac{4}{y} = 10$

(v) $-4x + 3y + xy = 0$ and $3x + 4y = 7xy$

45. If the following equations are consistent then find the value of constant k where k is any real number.

(i) $kx - y = 2$
$2x + ky = 5$
$4x - y = 3$

(ii) $x + 2y + 3 = 0$
$2x + 4y + 5 = 0$
$x + ky + 8 = 0$

(iii) $3x + 2y + 4 = 0$
$4x - y + k = 0$
$x - y + 3 = 0$

46. Prove that the followings different equations are consistent if $a + b + c = 0$

$ax + by + c = 0$
$bx - cy + a = 0$
$cx - ay + b = 0$
where $a, b, c, d \in R - (0)$

47. Using determinant solve the flowing equations

(i) $2x + y - z = 3$
$x + y + z = 1$
$x - 2y - 3z = 4$

(ii) $\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0$
$\frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0$
$\frac{2}{x} - \frac{5}{y} - \frac{2}{z} - 3 = 0$

48. Solve the equations (Use Cramer's rule)

(i) $x + 3y + 4z - 8 = 0$
$2x + y + 2z - 5 = 0$
$5x + 2z - 7 = 0$

(ii) $2x - 5y + 7z = 6$
$x - 3y + 4z = 3$
$3x - 8y + 11z = 11$

49. Solve the following equations (if possible)

(i) $x + y + z = 1$
$y + z - 1 = 0$
$x + z = 0$

(ii) $x - 3y - 8z + 10 = 0$
$3x + y - 4z = 0$
$2x + 5y + 6z - 13 = 0$

50. A company is manufacturing two types of cycles for gents and ladies separately which are assembled and finished in two workshops w_1 and w_2. Each type takes

15 hrs and 10 hrs for assembly and 5 hrs and 2 hrs for finishing in respective workshops. If the total number of hours available are 400 and 120 in workshops w_1 and w_2 respectively, by using determinant, calculate the number of units of cycles produced.

51. Persons A, B and C have Rs. 1,250, Rs. 1,700 and Rs. 2,100, respectively. They utilize the amount to purchase three types of shares x, y and z. Person A purchases 20 shares of x, 50 shares of y and 30 shares of z. Person B purchases 44 shares of x, 30 shares of y and 60 shares of z. Person C purchases 12 shares of x, 40 shares of y and 100 shares of z. Find the purchase price of three different types of shares.

ANSWERS

(1) (i) 1 (ii) $(x+y)^2$
(2) (i) 30 (ii) −30 (iii) 0
(3) (i) 1 (ii) −10 (iii) $a(b-3a)$
(4) $a = 1$
(5) $k = 1$
(6) $p = 4$
(7) $a = 2$ or $a = -5$
(8) $k = 3$ or $k = -6$
(9) $p = 2$ or $p = 3$
(10) 11
(11) 25
(12) −5
(13) $x = 0$
(14) (i) 0 (ii) 0 (iii) 0
(17) $x = 2$
(18) $k = 10$
(27) 0
(28) (i) 0 (ii) 0 (iii) 1100
(29) $k = -2$
(30) $\frac{a^2 - b^2}{2}$
(31) $\frac{1}{2}(x-y)(y-z)(z-x)$
(44) (i) $x = 3, y = 2$
  (ii) $x = -2, y = 5$
  (iii) $x = -1, y = 1$
  (iv) $x = \frac{1}{2}, y = 1$
  (v) $x = 1, y = 1$
(45) (i) $k = 3$ or $k = -2$
  (ii) Equations are inconsistent
  (iii) $k = 9$
(47) (i) $x = 2, y = 1, z = 0$
  (ii) $x = 1, y = 3, z = 3$
(48) (i) $x = \frac{7-2a}{5}, \quad y = \frac{11-6a}{5}, \quad z = a$
  (ii) No solution
(49) (i) $x = -1, y = 0, z = 1$
  (ii) $x = 2k - 1, y = -2k + 3, z = k$
(50) 20 units of gents cycles and 10 units of ladies cycles
(51) $x = 10, y = 12, z = 15$

22

Matrix Algebra

LEARNING OBJECTIVES

After studying this chapter, the student will be able to understand:

- Understand the meaning of matrix
- Know different types of matrices
- Know basic algebra of matrices
- Use matrix to solve simultaneous linear equations

INTRODUCTION

In order to represent large amount of data into a shorter form matrix is used. Matrix is an arrangement of elements into rows and columns enclosed within [], {} or (). Suppose *mn* elements are to be arranged into *m* rows and *n* columns then the general of matrix is as under.

$$A=\begin{bmatrix} a_{11}\ a_{12}\ldots\ldots\ldots\ldots a_{14} \\ a_{21}\ a_{22}\ldots\ldots\ldots\ldots a_{24} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}\ a_{m2}\ldots\ldots\ldots\ldots a_{mn} \end{bmatrix}_{m\times n}$$

Each matrix is uniquely identified by its order, which indicates the total number of rows and total number of columns and also the total number of elements.

In a matrix each element is identified by its position e.g. *aij* indicates the elements of *i*th row and *j*th column.

Different Types of Matrices

Equal Matrices

Two or more matrices are said to be equal matrices if they have same order and the corresponding elements of the matrices are equal e.g.

$$A_{2\times2}=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix},\ B_{2\times2}=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\ \text{then } A=B$$

Square Matrix

A matrix in which total number of rows and total number of columns are equal is called a square matrix, e.g.

$$A_{2\times 2} = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix},\ B_{3\times 3} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Diagonal Elements

In a square matrix the elements for which row numbers and column numbers are equal is called a diagonal element,. e.g. in the above matrix $a_{11} = -1$, $a_{22} = -4$ are diagonal elements.

Row Matrix

A matrix having only one row and any number of columns is called a row matrix.

Column Matrix

A matrix having only one column and any number of rows is called a column matrix, e.g.

$$B_{3\times 1} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

Null Matrix

A matrix in which all elements are zero is called null number or zero matrix and is denoted by O, e.g.

$$O_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ O_{2\times 3} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Diagonal Matrix

A square matrix whose all elements except the diagonal elements are zero is called a diagonal matrix.

$$A_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix},\ B_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Identity Matrix

A square matrix whose all diagonal elements are unity (one) and rest of the elements are zero is called an identity matrix or unit matrix and is denoted by I, e.g.

$$I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ I_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

Scalar Matrix

A square matrix whose all diagonal elements are identical and rest of the elements are zero is called a scalar matrix, e.g.

$$A_{2\times 2} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},\ B_{3\times 3} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Transpose Matrix

A matrix obtained by interchanging rows into columns or columns into rows is called transpose of the given matrix and is denoted by A' or A^T, e.g.

$$A_{2\times 3} = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 5 & 0 \end{bmatrix} \text{then } A'_{3\times 2} = \begin{bmatrix} 2 & 4 \\ -1 & 5 \\ 3 & 0 \end{bmatrix}$$

Triangular Matrix

A square matrix whose all elements above the diagonal elements are zero is called a lower triangular matrix, e.g.

$$\text{A} = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

A square matrix whose all elements below the diagonal elements are zero is called an upper triangular matrix, e.g.

$$B_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

Symmetric Matrix

A square matrix whose transpose is a matrix itself is called a symmetric matrix. If A is a square matrix of order n then it is called asymmetric matrix if $A = A'$, i.e. if $\left[aij\right] = \left[aji\right]$ i.e. the element of ith row and jth column is equal to the element of jth row and ith column. e.g.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 0 \end{bmatrix}$$

Here $a_{12} = 2$ $a_{13} = 3$ and $a_{23} = -5$
$a_{21} = 2$ $a_{31} = 3$ $a_{32} = -5$

Skew Symmetric Matrix

If A is a square matrix of order n then it is said to be a skew symmetric matrix if $A = -A'$ i.e. if $\left[aij\right] = \left[aji\right]$ i.e. the element of ith row and jth column and the elements of jth row and ith column are equal in magnitude and opposite in sign, e.g.

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & +4 \\ 3 & -4 & 0 \end{bmatrix}$$

Here $a_{12} = 2$ $a_{13} = -3$ and $a_{23} = 4$
$a_{21} = 2$ $a_{31} = 3$ $a_{32} = -4$

Submatrix

A matrix obtained by ignoring some rows or columns or both of a given matrix is called submatrix of a given matrix, e.g.

$$A_{2\times 3} = \begin{bmatrix} -2 & 3 & 0 \\ 4 & -1 & 5 \end{bmatrix} \text{ then } \begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix} \text{etc.}$$

are called submatrices of given matrix A.

Orthogonal Matrix

If A is a square matrix of order n then it is said to be an orthogonal matrix if $AA' = A'A = \text{I}$.

Idempotant Matrix

If A is a square matrix of order n then it is said to be an idempotant matrix if $A^2 = A$.

Nilpotent Matrix

A square matrix A of order n is said to be a nilpotent matrix if for some positive integer value of n, $A^n = 0$.

MATRIX ALGEBRA

By using matrix algebra we can solve many real life problems efficiently. The main operations in matrix algebra are addition and multiplication while subtraction can be performed by using addition and the division can be performed in the form of multiplication.

Addition and Subtraction of Two or More Matrices

The necessary condition for adding or subtracting two or more matrices is that they must have same order and the addition or subtraction is carried out with the corresponding elements.

Properties

(1) Addition of two matrices possesses the order (commutative) property, i.e. $A + B = B + A$

Proof

Let A be a matrix of order $m \times n$ and B is also a matrix of same order then $m \times n$

$$\begin{aligned} A + B &= (aij) + (bij) = (aij + bij) \\ &= (bij + aij) \\ &= (bij) + (aij) \\ &= B + A \end{aligned}$$

Hence $A + B = B + A$

(2) Addition of the matrices possesses the property of associationship, i.e.

$$A + (B + C) = (A + B) + C$$

Proof

Let A, B and C are three matrices of same order then

$$(B + C) = [\mathrm{b}ij + \mathrm{c}ij]$$

$$\begin{aligned} \therefore A + (B + C) &= [aij] + ([bij] + [cij]) \\ &= [aij + bij] + [cij] \\ &= (A + B) + C \end{aligned}$$

(3) Addition of matrices possesses the property of distributive with respect to scalar, i.e. $\in R$.

Proof

Let A and B are any two matrices of same order then

$$\begin{aligned} k(A + B) &= k(aij + bij) \\ &= (k\,aij + k\,bij) \\ &= (k\,aij) + (k\,bij) \\ &= kA + kB, \; k \in R \end{aligned}$$

(4) If A is matrix of order $m \times n$ then there exists a matrix O of same order such that $A + O = O + A = A$, i.e. there exists additive identity.

Proof

Let A is a matrix of order $m \times n$ and O is a null matrix of same order then

$$A+O=\left[aij+Oij\right]=\left[aij\right]=A$$

and

$$O+A=\left[Oij+aij\right]=\left[aij\right]=A$$

Hence $A + O = O + A = A$.

(5) If A is any matrix of order $m \times n$ than the r exists another B matrix of same order such that $A + B = O$ then B is called an additive inverse of A and is denoted by $-A$.

Hence $A + \left(-A\right) = O$.

Proof

Let $A = \left[aij\right]$ is a matrix of order $m \times n$ and let there exist another matrix $B = \left[-aij\right]$ of same order then

$$\begin{aligned} A+B &= \left[aij\right]+\left[-aij\right] \\ &= \left[aij-aij\right] \\ &= O \end{aligned}$$

Hence B is the additive inverse of A.

(6) If A, B and C are the three matrices of same order and if

$A + C = B + C$ then $A = B$.

Proof

We have A, B and C three matrices of same order and

$$\begin{aligned} & A + C = B + C \\ & \therefore \left(A+C\right) + \left(-C\right) = \left(B+C\right) + \left(-C\right) \\ & \therefore A + \left(C + \left(-C\right) = B + \left(C + \left(-C\right)\right) \\ & \therefore A + O = B + O \\ & \therefore A = B \end{aligned}$$

ILLUSTRATIONS

Illustration 1 If $A=\begin{bmatrix} 2 & 3 & -1 & 4 \\ 2 & 3 & 3 & -1 \end{bmatrix}$, $B=\begin{bmatrix} -2 & 4 \\ -3 & 2 \\ -4 & -1 \\ 1 & 3 \end{bmatrix}$ and $C=\begin{bmatrix} 0 & -4 \\ 1 & -3 \\ -2 & 1 \\ 3 & 2 \end{bmatrix}$ then find the following (if possible)

(i) $A + 2B + C$, (ii) $2B - C$, (iii) $A + 2B'$, (iv) $A' - 2B + C$

Solution

Here $A = 2 \times 4, B = 4 \times 2$ and $C = 4 \times 2$.

(i) Since the order of the matrices is not same so A + 2B + C is not possible.

(ii) Since the order of matrices B and C is same so $2B - C$ is possible.

$$2B - C = 2\begin{bmatrix} -2 & 4 \\ -3 & 2 \\ -4 & -1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 0 & -4 \\ 1 & -3 \\ -2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 8 \\ -6 & 4 \\ -8 & -2 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 0 & -4 \\ 1 & -3 \\ -2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 8 & 5 & -4 \\ -6 & 4 & 8 & (-3) \\ -8 & (-2) & -2 & -1 \\ 2 & -3 & 6 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 4 \\ -7 & 11 \\ -6 & -3 \\ -1 & 4 \end{bmatrix}$$

(iii) Since the order of matrices $A = 2 \times 4$ and $B' = 2 \times 4$ are same so $A + 2B^1$ is possible.

$$A + 2B' = \begin{bmatrix} 2 & 3 & -1 & 4 \\ 1 & 2 & 3 & -1 \end{bmatrix} + 2\begin{bmatrix} -2 & -3 & -4 & 1 \\ 4 & 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & -1 & 4 \\ 1 & 2 & 3 & -1 \end{bmatrix} + \begin{bmatrix} -4 & -6 & -8 & 2 \\ 8 & 4 & -2 & 6 \end{bmatrix}$$

$$A + 2B' = \begin{bmatrix} -2 & -3 & -9 & 6 \\ 9 & 6 & 1 & 5 \end{bmatrix}$$

(iv) Since the order of the matrices A', B *and* C are same

$\therefore A' - 2B + C$ is possible.

$$A^i - 2B + C = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 3 \\ 4 & -1 \end{bmatrix} - \begin{bmatrix} -4 & 8 \\ -6 & 4 \\ -8 & -2 \\ 2 & 6 \end{bmatrix} + \begin{bmatrix} 0 & -4 \\ 1 & -3 \\ -2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} 6 & -11 \\ 10 & -5 \\ 5 & 6 \\ 5 & -5 \end{bmatrix}$$

Illustration 2 $A=\begin{bmatrix} 2 & -3 \\ 4 & 1 \\ 1 & 2 \end{bmatrix}$, $B = 4A$, $C = A + B - 3I$ then find matrix D such that $D = 6A + B - 2C$.

Solution

Here order of the matrices are equal so the addition/subtraction is possible.

$$\begin{aligned} \text{Now } C &= A + B - 3I \\ &= A + 4A - 3I \\ &= 5A - 3I \\ \text{and } D &= 6A + B - 2C \\ &= 6A + 4A - 2(5A - 3I) \\ &= 10A - 10A + 6I \\ D &= 6I \end{aligned}$$

Illustration 3 If $\begin{bmatrix} x & 2 \\ y & 1 \end{bmatrix} + \begin{bmatrix} y & x \\ 2x & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 4 & 2 \end{bmatrix}$ then find the value of x and y.

Solution

We have

$$\begin{bmatrix} x & 2 \\ y & 1 \end{bmatrix} + \begin{bmatrix} y & x \\ 2x & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} x + y & 3 \\ 2x + y & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 4 & 2 \end{bmatrix}$$

Since $x + y = 3$ and $2x + y = 4$

Solving equations $x + y = 3$

$$-2x + y = 4$$

$$-x = -1$$

$$-x = 1$$

Putting $x = 1$ in $x + y = 3$, we have $y = 2$

$x = 1$ and $y = 2$

Multiplication of Two Matrices

The necessary condition for the multiplication of two matrices is that the number of columns of first matrix must be equal to the number rows of second matrix. Consider, e.g.

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} \text{ and if}$$

We find *AB* as under

Here *A* has order 2×3 and *B* has order 2×2 since the number of rows of matrix *B* is not same so we cannot perform *AB*.

Now consider that we want to find *BA* since the number of columns of matrix *B* is equal to the number of rows of matrix *A* so *BA* is possible.

$$BA = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 1 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 2 + (-1)1 & 2 \times 3 + (-1) \times (-1) & 2 \times 4 + (-1) \times 3 \\ 4 \times 2 + 3 \times 1 & 4 \times 3 + 3 \times (-1) & 4 \times 4 + 3 \times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 - 1 & 6 + 1 & 8 - 3 \\ 8 + 3 & 12 - 3 & 8 + 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 7 & 5 \\ 11 & 9 & 17 \end{bmatrix}$$

Illustration 4 *If* $A = \begin{bmatrix} 2 & 1 & 3 \\ -4 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \end{bmatrix}$ then find *AB* and *BA*.

Solution

First we find *AB* since *A* has order 2×3 and *B* has order 3×2, so *AB* is possible and the result matrix has order 2×2.

$$AB = \begin{bmatrix} \boxed{\begin{matrix} 2 & 1 & 3 \end{matrix}} \\ \begin{matrix} -4 & 0 & -1 \end{matrix} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 + 2 + 9 & 0 + 1 + 0 \\ -4 + 0 + (-3) & 0 + 0 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 1 \\ -7 & 0 \end{bmatrix}$$

Now we find *BA* since matrix B has order 3×2 and matrix A has order 2×3 so the multiplication is possible and the resultant matrix will have order.

$$BA = \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \end{matrix}} \\ \begin{matrix} 2 & 1 \end{matrix} \\ \begin{matrix} 3 & 0 \end{matrix} \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ -4 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0 & 1+0 & 3+0 \\ -4+(-4) & 2+0 & 6+(-1) \\ 6+0 & 3+0 & 9+0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 6 \\ 6 & 3 & 9 \end{bmatrix}$$

Thus we can say that in matrix algebra, $AB \neq BA$. In multiplication, AB is said to have been post-multiplied by B and B is said to have been pre-multiplied by A. In short AB is called the post-multiplication of A by B or pre-multiplication of B by A.

Properties

(1) Matrix multiplication in general is not commutative. If A is order $m \times n$ and B is order $n \times p$ then we can see that the matrix AB is defined and has order $m \times p$ whereas matrix BA is not possible because number of columns of matrix B is not equal to the number of rows of matrix B.

Proof

Let us consider that matrix A is of the order of $m \times n$ and matrix B with order $n \times m$, then we can easily see that AB has order $m \times n$ where as BA has order $n \times n$. Thus in general we can say that matrix multiplication does not posses the commutative law.

(2) Multiplication is distributative with respect to addition.

Proof

Let $A = [aij]_{m \times n}$, $B = [bij]_{n \times p}$ and $C = [cij]_{n \times p}$

are three different matrices, then we have to prove that

$A(B + C) = AB + AC$

Since B and C have same order so

$$B + C = [bik + cik]_{n \times p} \text{ for } 1 \leq i \leq m \;\&\; 1 \leq k \leq p$$

$$\text{Now } A(B + C) = \sum_{j=i}^{n} aij\,(bik + cik)$$

$$= \sum_{j=i}^{j=i} (aij\ bik + aij\ cik)$$

$$= \sum_{j=i}^{n} aij\ bik + \sum_{j=1}^{n} aij\ cik$$

$$= (i,k)^{th} \text{ element of } AB + (i,k)^{th} \text{ element of } AC$$

$$= (i,k)^{th} \text{ element of} (AB + AC)$$

$$\therefore A(B + C) = AB + AC$$

(3) Matrix multiplication is associative, i.e. if A, B and C are any three matrices of order $m \times n$, $n \times p$ and $b \times q$ then $(AB)C = A(BC)$

Proof

$Let\ A = \left[aij\right]_{m \times n},\ B = \left[bik\right]_{n \times p}\ \ and\ C = \left[cki\right]_{p \times q}$

then the matrices $\left(AB\right)C$ and $A\left(BC\right)$ are defined and their order is $m \times q$ each.

$$AB = \left[uik\right]m\,x\,p \text{ and } BC = \left[vji\right]n\ xq$$

$$uik = \sum_{j=1}^{n} aij.bjk \qquad 1 \le i \le m,\ i \le k \le p$$

$$\text{and}\ vij = \sum_{k=1}^{p} bjk, \qquad 1 \le i \le n,\ i \le l \le q$$

Now $\left(i,\ l\right)$th element of $\left(AB\right)C$

$$= \sum_{k=1}^{p} uik\ ckl = \sum_{k=1}^{p}\left(\sum_{j=1}^{n} aij\ bjk\right)cki$$

$$= \sum_{j=1}^{n} aij\left(\sum_{k=1}^{p} bik\ ckl\right)$$

$$= \sum_{j=1}^{n} aij\ vjl$$

$$= \left(i,l\right) \text{the element of} A\left(BC\right)$$

$$\left(AB\right)C = A\left(BC\right)$$

(4) If a is a matrix of order $m \times n$ and O is also a null matrix of order $n \times m$ than we have $AO = OA = O$
(Proof is left as an exercise to the reader)

(5) If $AB = O$ then it does not mean that $A = O$ or $B = O$ or both are null matrices

$$e.g.\ A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} and\ B = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$then\ AB = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

(6) If A is a matrix of order n and IB a unit matrix of order n then
$AI = IA = A$.

(7) If A is a square matrix of order n, then
$A^2 = AA$
$A^3 = A\left(AA\right) = AA^2 = \left(AA\right)A = A^2A$
$A^n = AAA$................ n times A^n

Illustration 5 If A $\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$ then find A^2 and have identify the type of matrix A.

Solution

Here

$$A^2 = A\,A$$

$$= \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3-5 & -1-9+15 & -5-15+25 \\ -1-3+5 & 3+9-15 & 5+15-25 \\ 1+3-5 & -3-9+15 & -5-15+25 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$$

$$= A$$

Thus $A^2 = A$ given matrix A is idempotant matrix.

Illustration 6 Show that $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is nilpotant matrix.

Solution

We have

$$A^2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Since $A^2 = 0$ given matrix A is nilpotant matrix.

TRANSPOSE OF A MATRIX

If A is matrix of order $m \times n$ then the matrix obtained by converting rows into columns or columns into rows is called the *transpose of given matrix* A and it is denoted by A' or A^T.

Properties

(1) If A is a matrix of the order of $m \times n$ then its transpose A' has order $n \times m$.

(2) Transpose of the transpose matrix is a matrix itself i.e. $(A')' = A$.

(3) If k is a real number then $k\,(A)' = (kA)'$

(4) The transpose of the sum of two matrices is the sum of their transpose i.e. $(A+B)' = A' + B'$

(5) The transpose of the product of matrices is equal to the product transposes of the matrices taken in the reverse order, i.e. $(AB)' = B'A'$.

Illustration 7 $A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & -2 & 1 \\ -3 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -3 & 0 \\ 3 & 2 & 2 \end{bmatrix}$ then prove that

(i) $(A+B)' = A' + B'$ (ii) $(AB)' = B'A'$

Solution

(i) $$(A+B) = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -5 & 1 \\ 0 & 2 & 4 \end{bmatrix}$$

$$(A+B)' = \begin{bmatrix} 1 & 5 & 0 \\ -1 & -5 & 2 \\ 3 & 1 & 4 \end{bmatrix} \qquad (1)$$

$$\text{Now } A' + B' = \begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 0 \\ 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 3 \\ 0 & -3 & 2 \\ 3 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 & 0 \\ -1 & -5 & 2 \\ 3 & 1 & 4 \end{bmatrix} \qquad (2)$$

From (i) and (2) we can say that $(A+B)' = A' + B'$

(ii) Now

$$(AB) = \begin{bmatrix} 2 & -1 & 0 \\ 3 & -2 & 1 \\ -3 & 0 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 3 \\ 2 & -3 & 0 \\ 3 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 3 & 6 \\ -4 & 8 & 11 \\ 9 & 4 & -5 \end{bmatrix}$$

$$\therefore (AB)' = \begin{bmatrix} -4 & -4 & 9 \\ 3 & 8 & 4 \\ 6 & 11 & -5 \end{bmatrix}$$

Now

$$B'A' = \begin{bmatrix} -1 & 2 & 3 \\ 0 & -3 & 2 \\ 3 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -4 & 9 \\ 3 & 8 & 4 \\ 6 & 11 & -5 \end{bmatrix} \tag{4}$$

From eqs. (3) and (4) we can say that $(AB)' = B'A'$.

Illustration 8 For a skew symmetric matrix prove that the diagonal elements are always zero.

Solution

For a skew symmetric matrix we know that $aij = -aji$. Now for the diagonal elements we have $aij = aji = aii$. So consider

$$aij = -aij$$
$$\therefore aii = -aij$$
$$\therefore aii + aii = 0$$
$$\therefore 2aii = 0$$
$$\therefore aii = 0$$

Illustration 9 Prove that the matrix $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ is an orthogonal matrix.

Solution

We know if A is an orthogonal matrix then $AA' = A'A = I$.

Consider $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & 1 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & 1 \end{bmatrix}$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

Hence we can say that given matrix A is an orthogonal matrix.

CO-FACTOR AND ADJOINT OF A SQUARE MATRIX

A determinant obtained by omitting the row and the column in which an element lies is called minor of that element.

If we replace each element by its minor along with the proper sign then the resultant matrix is called co-factor matrix of the given matrix.

The transpose of co-factor matrix is called the adjoint matrix.

Illustration 10 Find the adjoint matrix for the following matrices:

$$\text{(i)} \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix} \qquad \text{(ii)} \begin{bmatrix} -2 & -2 & 1 \\ 1 & -4 & 2 \\ 0 & 4 & 3 \end{bmatrix}$$

Solution

(i) The co-factor matrix of $\begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix}$ is obtained as under

Co-factor of -2 is $= 4$

Co-factor of 1 is $= -3$

Co-factor of 3 is $= -1$

Co-factor of 4 is $= -2$

The co-factor matrix is $\begin{bmatrix} 4 & -3 \\ -1 & -2 \end{bmatrix}$

By taking transpose, we have $adj\, A = \begin{bmatrix} 4 & -1 \\ -3 & -2 \end{bmatrix}$

(ii) The co-factor matrix of given matrix is obtained as under

$$\begin{bmatrix} \begin{vmatrix} -1 & 2 \\ 4 & 3 \end{vmatrix} & {}^{(-1)}\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 0 & 4 \end{vmatrix} \\ {}^{(-1)}\begin{vmatrix} -3 & 1 \\ 4 & 3 \end{vmatrix} & \begin{vmatrix} -2 & 1 \\ 0 & 3 \end{vmatrix} & {}^{(-1)}\begin{vmatrix} -2 & -3 \\ 0 & 4 \end{vmatrix} \\ \begin{vmatrix} -3 & 1 \\ -1 & 2 \end{vmatrix} & {}^{(-1)}\begin{vmatrix} -3 & 1 \\ 4 & 3 \end{vmatrix} & \begin{vmatrix} -2 & -3 \\ 1 & -1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -11 & -3 & 4 \\ 13 & -6 & 8 \\ 7 & 5 & 5 \end{bmatrix}$$

By taking transpose

$$\text{adj} A = \begin{bmatrix} -11 & 13 & 7 \\ -3 & -6 & 5 \\ 4 & 8 & 5 \end{bmatrix}$$

Illustration 11 If $A \begin{bmatrix} -4 & 8 \\ 2 & 3 \end{bmatrix}$ then show that $A\left(adj\,A\right) = |A|\,I$

Solution

We have

$$|A| = \begin{vmatrix} -4 & 8 \\ 2 & 3 \end{vmatrix} = -28$$

$$\therefore |A| I_2 = -28 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

First we find co-factor matrix as under

$$\begin{bmatrix} 3 & -2 \\ -8 & -4 \end{bmatrix}$$

By taking transpose, we have

$$adj\,A = \begin{bmatrix} 3 & -8 \\ -2 & -4 \end{bmatrix}$$

$$\text{Now, } A(adj\,A) = \begin{bmatrix} -4 & 8 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -8 \\ -2 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -28 & 0 \\ 0 & -28 \end{bmatrix}$$

$$= -28 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |A| I_2$$

Hence $A\left(adj\,A\right) = |A| - I_2$

Illustration 12 If $A = \begin{bmatrix} -2 & 4 & -3 \\ 1 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix}$ then show that $\left(adj\,A\right)A = |A|\,I_3 = A\left(adj\,A\right)$

Solution

Here

$$|A| = \begin{bmatrix} -2 & 4 & -3 \\ 1 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix}$$

$$|A| = -2\left[0 - \left(-1\right)\right] - 4\left(2 - 3\right) + \left(-3\right)\left(0102\right)$$

$$= -2 + 4 + 3$$

$$= 5$$

$$|A|I_3 = 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Now the co-factor matrix is

$$= \begin{bmatrix} \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix} \\ \begin{vmatrix} 4 & -3 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} -2 & -3 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} -2 & 4 \\ 3 & -1 \end{vmatrix} \\ \begin{vmatrix} 4 & -3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} -2 & -3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} -2 & 4 \\ 1 & 0 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ -5 & 5 & 10 \\ 4 & -1 & -4 \end{bmatrix}$$

By taking transpose, $adj\,A = \begin{bmatrix} 1 & -5 & 4 \\ 1 & 5 & -1 \\ -4 & 10 & -4 \end{bmatrix}$

$$\text{Now}\,(adj\,A)\,A = \begin{bmatrix} 1 & -5 & 4 \\ 1 & 5 & -1 \\ -4 & 10 & -4 \end{bmatrix}\begin{bmatrix} -2 & 4 & -3 \\ 1 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 5I_3$$

$$= |A|\,I_3$$

Similarly $A\left(adj\,A\right)$

$$=\begin{bmatrix}-2 & 4 & -3\\ 1 & 0 & 1\\ 3 & -1 & 2\end{bmatrix}\begin{bmatrix}1 & -5 & 4\\ 1 & 0 & -1\\ -5 & -1 & -4\end{bmatrix}$$

$$=\begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 5\end{bmatrix}$$

$= 5\,I_3 = |A|\,I_3$

Hence $A\left(adj\,A\right) = \left(adj\,A\right)A = |\,A\,|I_3$

INVERSE OF A MATRIX

If A is any square matrix of order n and if there exists another matrix B of same order such that $AB = BA = I$ then B is called inverse of matrix A and is denoted by A^{-1}.

Some important properties are being stated (without proof) as follows:

(1) Inverse of matrix A is possible if A is a square matrix and it must be non-singular. A square matrix A of order n is said to be non-singular if $|\,A\,| \neq 0$, and it is said to be singular if $|\,A\,| = 0$.

(2) If B is inverse of A then A is inverse of B.

(3) Inverse of matrix A, if exists then it is unique and $AA^{-1} = A^{-1}A = I$.

(4) $\left(A^{-1}\right)^{-1} = A$ i.e. inverse of a matrix A is a matrix itself.

(5) $\left(A'\right)^{-1} = \left(A^{-1}\right)'$ i.e. inverse of a transpose of a matrix is same as the transpose of the inverse matrix of A.

(6) The inverse of a product of matrices is equal to the product of inverses of the matrices taken into inverse order, i.e.

$\left(AB\right)^{-1} = B^{-1}\,A^{1}$

or $\left(ABC\right)^{-1} = C^{-1}\,B^{-1}\,A^{-1}$

(7) The inverse of an identity matrix is an identity matrix itself.

(8) Since we know that $A\left(adj\,A\right) = \left(adj\,A\right)A = |\,A\,|I$

$$A.\left(\frac{1}{|\,A\,|}adj\,A\right) = \left(\frac{1}{|\,A\,|}adj\,A\right)\,A = I \quad \left(\because |\,A\,| \neq 0\right)$$

Thus we can say that $A^{-1} = \dfrac{1}{|\,A\,|}\,adj\,A$

Hence if A is any non–singular square matrix of order n then inverse of matrix A is obtained by using the formula

$$A^{-1} = \frac{1}{|\,A\,|}\,adj\,A$$

Illustration 13 If $A = \begin{bmatrix} 1 & -3 \\ 2 & 3 \end{bmatrix}$ then find A^{-1} and verify that $A^{-1}A = AA^{-1} = I_2$

Solution

$$|A| = \begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix} = 3 - (-6) = 9$$

$$= 9$$

$$\neq 0$$

$\therefore A^{-'}$ is possible

The co-factor matrix is

$$\begin{bmatrix} 3 & -2 \\ 3 & 1 \end{bmatrix}$$

By taking transpose, $adjA = \begin{bmatrix} 3 & 3 \\ -2 & 1 \end{bmatrix}$

$$\text{Hence } A^{-1} = \frac{1}{|A|} adj\,A$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$\text{Now } A^{-1}A = \frac{1}{9}\begin{bmatrix} 3 & 3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & 3 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$\text{and } AA^{-1} = \begin{bmatrix} 1 & -3 \\ 2 & 3 \end{bmatrix}\frac{1}{9}\begin{bmatrix} 3 & 3 \\ -2 & 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Thus $AA^{-1} = A^{-1}A = I_2$.

Illustration 14 If $A = \begin{bmatrix} 2 & 2 & 2 \\ -3 & 0 & 0 \\ 1 & -1 & 2 \end{bmatrix}$ then find A^{-1} and verify that $A^{-1}A = I$

Solution

$$|A| = \begin{vmatrix} 2 & 2 & 2 \\ -3 & 0 & 0 \\ 1 & 1 & 2 \end{vmatrix}$$

$= 2(0 - 0) - 2(-6 - 0) + 2(-3 - 0)$

$= 0 + 12 - 6$

$= 6$

$\neq 0 \therefore A^{-1}$ is possible

The co-factor matrix is

$$\begin{bmatrix} \begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} -3 & 0 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} -3 & 0 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 2 \\ 0 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ -3 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ -3 & 0 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 6 & -3 \\ -2 & 2 & 0 \\ 0 & -6 & 6 \end{bmatrix}$$

By taking transpose, $adjA = \begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$

Hence $A^{-1} = \frac{1}{|A|} adj\ A \frac{1}{6}\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix} A^{-1}$ and verify that $A^{-1}A = I$

$$\text{Now } A^{-1}A = \frac{1}{6}\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}\begin{bmatrix} 2 & 2 & 2 \\ -3 & 0 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

Illustration 15 If $A = \begin{bmatrix} -2 & -6 \\ 3 & 4 \end{bmatrix}$ then prove that $\left(A^{-\prime}\right) = \left(A'\right)^{-\prime}$

Solution

$$|A| = \begin{vmatrix} -2 & -6 \\ 3 & 4 \end{vmatrix} = -8 + 18 = 10 \neq 0$$

$\therefore A^{-1}$ is possible

The co-factor matrix is

$$\begin{bmatrix} -4 & -3 \\ 6 & 2 \end{bmatrix}$$

By taking tanspose, $adjA = \begin{bmatrix} -4 & 6 \\ -3 & 2 \end{bmatrix}$

$$\therefore A^{-\prime} = \frac{1}{|A|} adj\, A = \frac{1}{10}\begin{bmatrix} -4 & 6 \\ -3 & 2 \end{bmatrix}$$

$$\therefore \left(A^{-\prime}\right)' = \frac{1}{10}\begin{bmatrix} -4 & -3 \\ 6 & 2 \end{bmatrix} \qquad (1)$$

$$Now\left(A^{1}\right) = \begin{bmatrix} -2 & 3 \\ -6 & 4 \end{bmatrix}$$

$$\therefore \left|A^{1}\right| = \begin{vmatrix} -2 & 3 \\ -6 & 4 \end{vmatrix} = -8 + 18 = 10$$

The co-factor matrix of A' is $\begin{bmatrix} 4 & 6 \\ -3 & -2 \end{bmatrix}$

Now by taking transpose, $adj\left(A'\right) = \begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix}$

$$\text{Hence } \left(A'\right)^{-\prime} = \frac{1}{|A|} adj(A') \frac{1}{10}\begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix} \qquad (2)$$

From (1) and (2) we have $\left(A^{-\prime}\right)' = \left(A'\right)^{-\prime}$

Illustration 16 If $A = \begin{bmatrix} 6 & -3 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 1 \\ 8 & 0 \end{bmatrix}$ then show that $\left(AB\right)^{-1} = B^{-1}A^{-1}$

Solution

We have

$$\left(AB\right) = \begin{bmatrix} 6 & -3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -5 & 1 \\ 8 & 0 \end{bmatrix} = \begin{bmatrix} -54 & 6 \\ -5 & 1 \end{bmatrix}$$

$$\therefore |AB| = \begin{vmatrix} -54 & 6 \\ -5 & 1 \end{vmatrix}$$

$$= -54 - \left(-30\right)$$

$$= -24$$

Now the co-factor matrix is $\begin{bmatrix} 1 & 6 \\ -6 & -54 \end{bmatrix}$

By taking transpose, $adj(AB) = \begin{bmatrix} 1 & 6 \\ -5 & -54 \end{bmatrix}$

$$\therefore (AB)^{-1} = \frac{1}{|AB|}\ adj(AB)$$

$$= \frac{1}{24}\begin{bmatrix} -1 & -6 \\ 6 & 54 \end{bmatrix}$$

Now

$$|A| = \begin{vmatrix} 6 & -3 \\ 1 & 0 \end{vmatrix} = 0 - (-3) = 3$$

The co-factor matrix is

$$\begin{bmatrix} 0 & -1 \\ 3 & 6 \end{bmatrix}$$

$\therefore$ By taking transpose, $adj\, A = \begin{bmatrix} 0 & -1 \\ 3 & 6 \end{bmatrix}$

Hence $A^1 = \frac{1}{|A|}\ adj\ A$

$$= \frac{1}{3}\begin{bmatrix} 0 & 3 \\ -1 & 6 \end{bmatrix}$$

Similarly

$$|B| = \begin{bmatrix} -5 & 1 \\ 8 & 0 \end{bmatrix} = -8$$

and $adj\ B = \begin{vmatrix} -5 & 1 \\ 8 & 0 \end{vmatrix}$

$$\therefore B^{-1} = \frac{1}{8}\begin{bmatrix} 0 & 3 \\ 8 & 6 \end{bmatrix}$$

$$= \frac{1}{24}\begin{bmatrix} -1 & 6 \\ -5 & 54 \end{bmatrix}$$

From eqs. (1) and (2)

$(AB)^{-1} = B^{-1}\, A^{-1}$

Illustration 17 If $A^{-1} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$ then find A

Solution

We know that

$(A^{-1})^{-1} = A$

$$\text{Now } |A^{-1}| = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$$

$$= 2(-5) - 2(1) + 0$$
$$= -12$$

The co-factor matrix is

$$\begin{bmatrix} -5 & -1 & 11 \\ 6 & -6 & -18 \\ 2 & -2 & -2 \end{bmatrix}$$

$$\therefore adj(A^{-1}) = \begin{bmatrix} -5 & 6 & 2 \\ -1 & -6 & -2 \\ 11 & -2 & -2 \end{bmatrix}$$

$$(A^{-1})^{-1} = \frac{1}{|A^{-1}|} adj(A^{-1}) = A$$

$$\therefore A = \frac{1}{-12}\begin{bmatrix} -5 & 6 & 2 \\ -1 & -6 & -2 \\ 11 & -2 & -2 \end{bmatrix}$$

TO FIND INVERSE OF A MATRIX THROUGH AUGMENTED MATRIX

Pivotal Method or Reduction Method

Let A be any non–singular matrix of order n. For finding its inverse, we consider the augmented matrix $[A \mid I]$, where I is the identity matrix of matrix A's order. Now by using elementary row or column transformation (strictly) we reduce this augmented matrix into the form $[I \mid B]$ then the matrix B is called inverse of matrix A i.e. $B = A^{-1}$. For quick result the following steps may be followed.

(1) Transform one column completely to the desired form before moving on to the next column.

(2) While transforming any column, the diagonal elements in that column into 1 and rest of the elements in that column into 0 (zero).

(3) The form of the previously transformed column should not be allowed to change by the subsequent operations.

Illustration 18 Find the inverse of following matrix by using pivotal reduction method.

$$\text{(i)} \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 2 & 3 \\ -1 & -4 \end{bmatrix}$$

Solution

(i) Consider

$$\left[\begin{array}{cc|cc} 4 & -1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \frac{1}{4}R_1$$

$$= \left[\begin{array}{cc|cc} 1 & -\frac{1}{4} & \frac{1}{4} & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] R_2 - 2R_1$$

$$= \left[\begin{array}{cc|cc} 1 & -\frac{1}{4} & \frac{1}{4} & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] R_2 - 2R_1$$

$$= \left[\begin{array}{cc|cc} 1 & -\frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{1}{7} & 1 \end{array}\right] R_1 + \frac{1}{4}R_2$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{14} & \frac{1}{14} \\ 0 & 1 & -\frac{1}{7} & \frac{2}{7} \end{array}\right]$$

$$\text{Thus, } A^{-1} = \begin{bmatrix} \frac{3}{14} & \frac{1}{14} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$$

(ii) Consider

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ -1 & -4 & 0 & 1 \end{array}\right] \frac{1}{2}R_1$$

$$= \left[\begin{array}{cc|cc} 1 & \frac{3}{2} & \frac{1}{2} & 0 \\ -1 & -4 & 0 & 1 \end{array}\right] R_2 + R_1$$

$$= \left[\begin{array}{cc|cc} 1 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{5}{2} & \frac{1}{2} & 1 \end{array}\right] -\frac{2}{5}R_2$$

$$= \left[\begin{array}{cc|cc} 1 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{5} & -\frac{2}{5} \end{array}\right] R_1 - \frac{3}{2}R_2$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & \frac{4}{5} & \frac{3}{5} \\ 0 & 1 & -\frac{1}{5} & -\frac{2}{5} \end{array}\right]$$

$$\text{Thus } A^{-1} = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ -\frac{1}{5} & -\frac{2}{5} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 & 3 \\ -1 & -2 \end{bmatrix}$$

Illustration 19 Obtain inverse for the following matrices by using Gauss elimination method.

$$\text{(i)} \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 0 \\ -2 & 2 & -3 \end{bmatrix} \qquad \text{(ii)} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -3 \\ 2 & 1 & -1 \end{bmatrix}$$

Solution

(i) Consider

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 1 & 2 & -3 \\ -2 & 2 & -3 & 2 & 1 & -1 \end{array}\right] \frac{1}{2}R_1$$

$$= \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -2 & 2 & -3 & 0 & 0 & 1 \end{array}\right] R_2 - 2R_1$$

$$= \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ -2 & 2 & -3 & 0 & 0 & 1 \end{array}\right] R_3 + 2R_1$$

$$= \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 3 & -2 & 1 & 0 & 1 \end{array}\right] R_1 - \frac{1}{2}R_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 3 & -2 & 1 & 0 & 1 \end{array}\right] R_3 - 3R_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 4 & -3 & 1 \end{array}\right] R_2 + R_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 3 & -2 & 1 \\ 0 & 0 & 1 & 4 & -3 & 1 \end{array}\right] R_1 - R_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & \frac{5}{2} & -1 \\ 0 & 1 & 0 & 3 & -2 & 1 \\ 0 & 0 & 1 & 4 & -3 & 1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -3 & \frac{5}{2} & -1 \\ 3 & -2 & 1 \\ 4 & -3 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -6 & 5 & -2 \\ 6 & -4 & 2 \\ 8 & -6 & 2 \end{bmatrix}$$

(ii) Consider

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -2 & -3 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right] R_2 - R_1$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & -4 & -1 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right] R_3 - 2R_1$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & -4 & -1 & 1 & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \end{array}\right] \frac{-1}{3} R_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{4}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \end{array}\right] R_1 - R_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{4}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \end{array}\right] R_3 + R_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{4}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{5}{3} & -\frac{5}{3} & -\frac{1}{3} & 1 \end{array}\right] -\frac{3}{5} R_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{4}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 1 & \frac{1}{5} & -\frac{3}{5} \end{array}\right] R_2 - \frac{4}{3} R_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 0 & -1 & -\frac{3}{5} & \frac{4}{5} \\ 0 & 0 & 1 & 1 & \frac{1}{5} & -\frac{3}{5} \end{array}\right] R_1 + \frac{1}{3} R_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \frac{2}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & -1 & \frac{3}{5} & \frac{4}{5} \\ 0 & 0 & 1 & 1 & \frac{1}{5} & -\frac{3}{5} \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & \frac{2}{5} & -\frac{1}{5} \\ -1 & -\frac{3}{5} & \frac{4}{5} \\ 1 & \frac{1}{5} & -\frac{3}{5} \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 5 & 2 & -1 \\ -5 & -3 & 4 \\ 5 & 1 & 3 \end{bmatrix}$$

USE OF MATRIX ALGEBRA TO SOLVE SIMULTANENOUS LINEAR EQUATIONS

Consider the system of linear equations as

$$a_{11}x + a_{12}y = b_1$$
$$a_{21}x + a_{22}y = b_2$$

These equations can be written as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{13} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\therefore AX = b \text{ where } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{13} & a_{22} \end{bmatrix} X = \begin{bmatrix} x \\ y \end{bmatrix} \text{and } b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Now by pre-multiplying both sides by A^{-1}, we have

$$A^{-1} AX = A^{-1} b$$
$$\therefore IX = A^{-1} b$$
$$\therefore X = A^{-1} b$$

which gives solution to the above given equations.

Illustration 20 Solve the following by using matrix. $2x - 3y = 13$, $4x + y = 5$.

Solution

Consider the coefficient matrix

$$A = \begin{bmatrix} 2 & -3 \\ 2 & 1 \end{bmatrix}$$

$$\text{Now} |A| = \begin{vmatrix} 2 & -3 \\ 2 & 1 \end{vmatrix} = 2 + 12 = 14 \neq 0$$

$\therefore A^{-1}$ is possible and given equations are consistent

The co-factor matrix is $\begin{bmatrix} 1 & -4 \\ 3 & 2 \end{bmatrix}$

By taking transpose, $adj\ A = \begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}$

$$\therefore A^{-1} = \frac{1}{|A|} adj\ A = \frac{1}{4}\begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}$$

Hence the solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 13 \\ 5 \end{bmatrix}$$

$$= \frac{1}{14}\begin{bmatrix} 13 + 15 \\ -52 + 10 \end{bmatrix}$$

$$= \frac{1}{14}\begin{bmatrix} 28 \\ -42 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$\therefore x = 2$ and $y = 3$.

Illustration 21 Solve the following equations by using matrix algebra.

(i) $\frac{3}{x} - 3y - 3 = 0$ (ii) $3x + 2y = 2xy$

$\frac{4}{x} + 2y - 10 = 0$ $2x + 3y = \frac{13}{6}xy$

Solution

(i) Let $x_1 = \frac{1}{x}$ then the given equations are

$$3x_1 - 3y = 3$$
$$4x_1 + 2y = 10$$

The coefficient matrix is $A = \begin{bmatrix} 3 & -3 \\ 4 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & -3 \\ 4 & 2 \end{vmatrix}$$
$$= 6 + 12 = 18$$

$\therefore A^{-1}$ is possible and given equations are consistent.

The co-factor matrix is $\begin{bmatrix} 2 & -4 \\ 3 & +3 \end{bmatrix}$

By taking transpose, $adj\, A = \begin{bmatrix} 2 & 3 \\ -4 & +3 \end{bmatrix}$

$$\therefore A^{-1} = \frac{1}{|A|} adj\, A = \frac{1}{18}\begin{bmatrix} 2 & 3 \\ -4 & +3 \end{bmatrix}$$

Hence the solution to the given system is

$$\begin{bmatrix} x_1 \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} 3 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 2 & 3 \\ -4 & +3 \end{bmatrix}\begin{bmatrix} 3 \\ 10 \end{bmatrix}$$

$$= \frac{1}{18}\begin{bmatrix} 6+30 \\ -12+30 \end{bmatrix}$$

$$= \frac{1}{18}\begin{bmatrix} 36 \\ 18 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\therefore x_1 = 2$ and $y = 1$

$\therefore \frac{1}{x} = 2$ and $y = 1$

$\therefore x = \frac{1}{2}$ and $y = 1$

is the solution to the given equations.

(ii) Given equations can be written as $\frac{3x}{xy} + \frac{2x}{xy} = \frac{2xy}{xy}$

$$\therefore \frac{3}{y}+\frac{2}{x}=2 \text{ and}$$

$$\frac{2x}{xy}+\frac{3x}{xy}=\frac{13xy}{6xy}$$

$$\therefore \frac{2}{y}+\frac{3}{x}=\frac{13}{6}$$

Consider $x_1=\frac{1}{x}$ and $y_1=\frac{1}{y}$ then we have

$$2x_1+3y_1=2 \times 3x_1+2y_2=\frac{13}{6}$$

The coefficient matrix is

$$A=\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

$$\text{Now}\left|A\right|=\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}=4-9=-5 \neq 0$$

The co-factor matrix is $\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$

By taking tranpose, $adj\,A\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$

$$\therefore A^{-1}=\frac{1}{|A|}adj\,A=\frac{1}{-5}\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}=\frac{1}{5}\begin{bmatrix} -2 & 3 \\ 3 & -2 \end{bmatrix}$$

Hence the solution is

$$\therefore A^{-1}=\frac{1}{|A|}adj\,A=\frac{1}{-5}\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}=\frac{1}{5}\begin{bmatrix} -2 & 3 \\ 3 & -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}=\frac{1}{5}\begin{bmatrix} -2 & 3 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ \frac{13}{6} \end{bmatrix}$$

$$=\frac{1}{5}\begin{bmatrix} -4+\frac{13}{2} \\ 6-\frac{13}{3} \end{bmatrix}=\frac{1}{5}\begin{bmatrix} \frac{5}{2} \\ \frac{5}{3} \end{bmatrix}$$

$$x_1=\frac{1}{2} \quad y_1=\frac{1}{3} \qquad \therefore x=2 \text{ and } y=3$$

Illustration 22 By using Gauss reduction method find the solution of the following equations.

(i) $2x + 3y = 14 \qquad 7x - 10y = 8$

(ii) $x - y = 2 \qquad 7x - 5y + 1 = 0$

Solution

(i) Consider

$$\left[\begin{array}{cc|c} 2 & 3 & 14 \\ 7 & -10 & 8 \end{array}\right] \frac{1}{2}R_1$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 7 \\ 7 & -10 & 8 \end{array}\right] R_2 - 7R_1$$

$$= \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 7 \\ 0 & -\frac{41}{2} & -41 \end{array}\right] \frac{-2}{41} \times R_2$$

$$= \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 7 \\ 0 & 1 & 2 \end{array}\right] R_1 - \frac{3}{2}R_2$$

$$= \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 2 \end{array}\right]$$

$x = 4$ and $y = 2$

(ii) Consider the given equations as

$x - y = 2$

$4x + 5y = -1$

Now consider

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 4 & 5 & -1 \end{array}\right] R_2 - 4R_1$$

$$= \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 9 & -9 \end{array}\right] \frac{1}{9}R_2$$

$$= \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 1 & -1 \end{array}\right] R_1 + R_2$$

$$= \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right]$$

$x = 1$ and $y = -1$.

Illustration 23 Solve the following equations by using Pivotal reduction method.

(i) $2x - 3y\ 4z = 11 \qquad 3x - 4z = 2 \qquad 3y + 4z = 1$

(ii) $x - 2y\ 3z = 10 \qquad 2x - y - 4 = 0 \qquad 3y + 2z = 8$

Solution

(i) Consider the given equations as

$2x - 3y + 4z = 11$
$3x + 0 - 4z = 2$
$0x + 3y + 4z = 1$

$$\text{Now} \left[\begin{array}{ccc|c} 2 & -3 & 4 & 11 \\ 3 & 0 & -4 & 2 \\ 0 & 3 & 4 & 1 \end{array}\right] \frac{1}{2}R_1$$

$$= \left[\begin{array}{ccc|c} 1 & -\frac{3}{2} & 2 & \frac{11}{2} \\ 3 & 0 & -4 & 2 \\ 0 & 3 & 4 & 1 \end{array}\right] R_2 - 3R_1$$

$$= \left[\begin{array}{ccc|c} 1 & -\frac{3}{2} & 2 & \frac{11}{2} \\ 0 & \frac{9}{2} & -10 & -\frac{29}{2} \\ 0 & 3 & 4 & 1 \end{array}\right] R_2 \times \frac{2}{9}$$

$$= \left[\begin{array}{ccc|c} 1 & -\frac{3}{2} & 2 & \frac{11}{2} \\ 0 & 1 & -\frac{20}{9} & -\frac{29}{9} \\ 0 & 3 & 4 & 1 \end{array}\right] R_1 + \frac{3}{2}R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & \frac{2}{3} \\ 0 & 1 & -\frac{20}{9} & -\frac{29}{9} \\ 0 & 3 & 4 & 1 \end{array}\right] R_3 - 3R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & \frac{2}{3} \\ 0 & 1 & -\frac{20}{9} & -\frac{29}{9} \\ 0 & 0 & \frac{32}{3} & \frac{32}{3} \end{array}\right] \frac{3}{32}R_3$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & \frac{2}{3} \\ 0 & 1 & -\frac{20}{9} & -\frac{29}{9} \\ 0 & 0 & 1 & 1 \end{array}\right] R_2 + \frac{20}{9}R_3$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & \frac{2}{3} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] R_1 + \frac{4}{3}R_3$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\therefore x = 2, y = -1, z = 1$

(ii) The given equations are

$x + 2y + 3z = 10$

$2x - y + 0z = 4$

$0x + 3y + 2z = 8$

Now, $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 2 & -1 & 0 & 4 \\ 0 & 3 & 2 & 8 \end{array}\right] R_2 - 2R_1$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & -5 & -6 & -16 \\ 0 & 3 & 2 & 8 \end{array}\right] \frac{1}{5}R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & \frac{6}{5} & \frac{16}{5} \\ 0 & 3 & 2 & 8 \end{array}\right] R_1 - 2R_2$$

$$=\left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{5} & \frac{18}{5} \\ 0 & 1 & \frac{6}{5} & \frac{16}{5} \\ 0 & 3 & 2 & 8 \end{array}\right] R_3 - 3R_2$$

$$=\left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{5} & \frac{18}{5} \\ 0 & 1 & \frac{6}{5} & \frac{16}{5} \\ 0 & 0 & -\frac{8}{5} & -\frac{8}{5} \end{array}\right] \left(-\frac{5}{8}\right) R_3$$

$$=\left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{5} & \frac{18}{5} \\ 0 & 1 & \frac{6}{5} & \frac{16}{5} \\ 0 & 0 & -\frac{8}{5} & -\frac{8}{5} \end{array}\right] \left(-\frac{5}{8}\right) R_3$$

$$=\left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{5} & \frac{18}{5} \\ 0 & 1 & \frac{6}{5} & \frac{16}{5} \\ 0 & 0 & 1 & 1 \end{array}\right] R_2 - \frac{6}{5} R_3$$

$$=\left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{5} & \frac{18}{5} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] R_1 - \frac{3}{5} R_3$$

$$=\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$x = 3$, $y = 2$ and $z = 1$

Illustration 24 If x, y and z are non-negative constants then solve the following equations by using pivotal reduction method (or triangular form reduction method)

$x + 3y + 4z = 8,\ 2x + y + 2z = 5,\ 5x + 2x = 7$

Solution

Consider the following

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 8 \\ 2 & 1 & 2 & 5 \\ 5 & 0 & 2 & 7 \end{array}\right] R_2 - 2R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 3 & 4 & 8 \\ 0 & -5 & -6 & -11 \\ 5 & 0 & 2 & 7 \end{array}\right] R_3 - 5R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 3 & 4 & 8 \\ 0 & -5 & -6 & -11 \\ 0 & -15 & -18 & -33 \end{array}\right] -\frac{1}{5}R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 3 & 4 & 8 \\ 0 & 1 & \frac{6}{5} & \frac{11}{5} \\ 0 & -15 & -18 & -33 \end{array}\right] R_1 - 3R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{5} & \frac{7}{5} \\ 0 & 1 & \frac{6}{5} & \frac{11}{5} \\ 0 & -6 & -6 & -9 \end{array}\right] R_3 + 15R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{5} & \frac{7}{5} \\ 0 & 1 & \frac{6}{5} & \frac{11}{5} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Multiplying the triangular matrix by $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and equating the product to the revised matrix $\begin{bmatrix} \frac{7}{5} \\ \frac{11}{6} \\ 0 \end{bmatrix}$ we get the following.

$$x + \frac{2z}{5} = \frac{7}{5}$$

$$\text{and } y + \frac{6z}{5} = \frac{11}{5}$$

Now suppose $z = k$ where $k \in R$ then

$$\text{We have } x = \frac{7}{5} - \frac{2k}{5} = \frac{7-2k}{5}$$

$$\text{and } y = \frac{11}{5} + \frac{6k}{5} = \frac{11-6k}{5}$$

$$:\!- \text{ Then } x = \frac{7-2k}{5}, y = \frac{11-6k}{5} \text{ and } z = k \; k \in R$$

Since x, y and z are non-negative constants we have

$$z = k > 0, x = \frac{7-2k}{5} \geq 0, \; y = \frac{11-6k}{5} \geq 0$$

$$k \leq \frac{7}{2} \quad k \leq \frac{11}{6}$$

Hence we can say that $k \leq \frac{11}{6}$ because it covers the constant $k \leq \frac{7}{2}$.

Illustration 25 A manufacturing unit produces there different types of food articles. Food type A contains 6 kg of flour 1.5 kg of fat and 1.5 kg of sugar. Food type B contain 6 kg of flour, 2 kg of fat and 1 kg of sugar. Food type C contains 4.5 kg of flour, 3 kg of fat and 2.5 kg of sugar. Now due to changes in consumer's taste it has been decided to change the mix with the following amendment (in kg)

	Food Type		
Contents	A	B	C
Flour	0	+0.5	−0.5
Fat	−1	0	+1
Sugar	+1.5	−1	+0.5

Using matrix algebra

(i) Find the matrix of product mix and for the new product mix.

(ii) If the manufacturing unit receives an order of 50 units of type *A*, 40 units of food type *B* and 30 units of food type *C* then find the total requirement of the ingredients.

(iii) If the unit has 700 kg of flour, 730 kg of fat and 800 kg of sugar in stock then find how many units of each type of food can be produced?

Solution

(i) The given data can be represented in the matrix form as under

$$\begin{array}{cc} \text{Food Type} & \begin{array}{ccc} \textit{Flour} & \textit{Fat} & \textit{Sugar} \end{array} \\ X = \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 6 & 1.5 & 1.5 \\ 6 & 2 & 1 \\ 4.5 & 3 & 2.5 \end{bmatrix} \end{array}$$

Now the amendment matrix can be expressed as under

$$\begin{array}{cc} \text{Food Type} & \begin{array}{ccc} \textit{Flour} & \textit{Fat} & \textit{Sugar} \end{array} \\ Y = \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & +0.5 & -0.5 \\ -1 & 0 & +1 \\ +1.5 & -1 & 0.5 \end{bmatrix} \end{array}$$

Hence the matrix of new product mix can be obtained as

$$X + Y = \begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$

(ii) Now the requirement matrix is

$$Z = \begin{array}{ccc} A & B & C \\ \hline \end{array}$$

$$Z = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 50 & 40 & 30 \end{bmatrix} \end{array}$$

Hence the total requirement of the ingredients can be obtained as

$$Z_{1\times 3}(x+y)_{3\times 2} = \begin{bmatrix} 50 & 40 & 30 \end{bmatrix} \begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 300+200+180 & 100+80+60 & 50+40+30 \end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{ccc} \textit{Flour} & \textit{Fat} & \textit{Sugar} \end{array} \\ \begin{bmatrix} 680 & 240 & 120 \end{bmatrix} \end{array}$$

∴ To meet the demand of 680 kg of Flour, 240 kg of fat and 120 kg of sugar is required.

(iii) Suppose the manufacturing unit produces x units of food A, y units of food B and z units of food C then we have

$$\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3700 \\ 1700 \\ 800 \end{bmatrix}$$

First we find A^{-1}.

$$|A| = \begin{vmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{vmatrix} = 6\begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 5 & 2 \\ 6 & 3 \end{vmatrix} + 1\begin{vmatrix} 5 & 2 \\ 6 & 2 \end{vmatrix}$$

$$= 6(6-4) - 2(15-12) + 1(10-12)$$

$$= 12 - 6 - 2$$

$$= 4$$

The co-factor matrix is

$$\begin{bmatrix} 2 & -3 & -2 \\ -4 & 12 & 0 \\ 2 & -7 & 2 \end{bmatrix}$$

By taking transpose, $adj\, A = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$

Hence the solutions is

$$\therefore A^{-1} = \frac{1}{|A|} = \frac{1}{4}\begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 80 \\ 1{,}060 \\ 200 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$x = 20$, $y = 265$, and $z = 50$

Illustration 26 The demand equation for the commodity A is $D_A = 85 - 3P_A + 9P_B$ and the respective supply production is $S_A = -5 + 15P_A$. The demand and supply transactions for another commodity B are respectively $D_B = 56 + 2P_A - 4P_B$ and $S_B = -6 + 33P_B$. Find the market equilibrium price and quantity.

Solution

For market equilibrium we must have demand equal to the supply. So by equating both the functions we have

$85 - 3P_A + 9P_B = -5 + 15P_A$

$18P_A - 9P_A = +90$

and $56 + 2P_A - 4P_B = -6 + 33 + P_B$

$\therefore 37P_B - 2P_A = 62$

$\therefore -2P_A + 37P_B = 62$

The given two equations are

$\therefore 18P_A - 9P_B = 95$

$\therefore -2P_A + 37P_B = 62$

$$\therefore \begin{bmatrix} 18 & -9 \\ -2 & 37 \end{bmatrix} \begin{bmatrix} P_A \\ P_B \end{bmatrix} = \begin{bmatrix} 90 \\ 62 \end{bmatrix}$$

Consider $A = \begin{bmatrix} 18 & -9 \\ -2 & 37 \end{bmatrix}$

$$|A| = \begin{vmatrix} 18 & -9 \\ -2 & 37 \end{vmatrix} \begin{matrix} = 666 - 18 \\ = 648 \end{matrix}$$

Now the co-factor matrix is $\begin{bmatrix} 37 & 2 \\ 9 & 18 \end{bmatrix}$

By taking transpose, *adj* $A = \begin{bmatrix} 37 & 9 \\ 2 & 18 \end{bmatrix}$

$$\therefore A^{-1} = \frac{1}{|A|} = \frac{1}{648} \begin{bmatrix} 37 & 9 \\ 2 & 18 \end{bmatrix}$$

Hence the solution to the given equations is

$$\begin{bmatrix} P_A \\ P_B \end{bmatrix} = \frac{1}{648} \begin{bmatrix} 37 & 9 \\ 2 & 18 \end{bmatrix} \begin{bmatrix} 90 \\ 62 \end{bmatrix}$$

$$= \frac{1}{648} \begin{bmatrix} 3,330 + 558 \\ 180 + 1,116 \end{bmatrix} = \frac{1}{648} \begin{bmatrix} 3,888 \\ 1,296 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

$\therefore P_A = 6$ *and* $P_B = 2$ and the quantity $D_A = 85 - 3(6) + 9(2)$
$= 85$

Illustration 27 A manufacturer produces three products X, Y, Z and sells in two markets. Annual sales are as under

Market	Products		
	X	Y	Z
I	10,000	20,000	18,000
II	16,000	18,000	10,000

(i) If unit sale prices of X, Y, Z are Rs. 2.50, Rs. 1.50 and Rs. 1.25 respectively then find the total revenue in each market and hence find the total revenue.

(ii) If the unit costs of the above three commodities are Rs. 1.25, Rs. 0.50 and Rs. 0.25 respectively then find total profit in each market and hence find the total profit.

Solution

From the given information we can form

the sales volume matirx $A = \begin{bmatrix} 10{,}000 & 20{,}000 & 18{,}000 \\ 16{,}000 & 18{,}000 & 10{,}000 \end{bmatrix}$

the sales price matrix $B = \begin{bmatrix} 2.50 & 1.50 & 1.25 \end{bmatrix}$

the cost price matrix $C = \begin{bmatrix} 1.25 & 0.50 & 0.25 \end{bmatrix}$

(i) Total revenue in each market can be obtained as

$$BA^1 = \begin{bmatrix} 2.50 & 1.50 & 1.25 \end{bmatrix} \begin{bmatrix} 10{,}000 & 16{,}000 \\ 20{,}000 & 18{,}000 \\ 18{,}000 & 10{,}000 \end{bmatrix}$$

$$= \begin{bmatrix} 77{,}500 & 79{,}500 \end{bmatrix}$$

$$\text{Total revenue} = \begin{bmatrix} 77{,}500 & 79{,}500 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1{,}57{,}000 \end{bmatrix}$$

(ii) Now the profit per unit of the three commodities is

$B - C = \begin{bmatrix} 1.25 & 1.00 & 1.00 \end{bmatrix}$

$$\text{Profit} = (B - C)\, A^1 = \begin{bmatrix} 1.25 & 1.00 & 1.00 \end{bmatrix} \begin{bmatrix} 10{,}000 & 16{,}000 \\ 20{,}000 & 18{,}000 \\ 18{,}000 & 10{,}000 \end{bmatrix}$$

$$= \begin{bmatrix} 50{,}500 & 48{,}000 \end{bmatrix}$$

$$\text{Total profit} = \begin{bmatrix} 50{,}500 & 48{,}000 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 98{,}500 \end{bmatrix}$$

Illustration 28 The total number of units produced (P) is a linear function of amount of over times in labour (in hours) (L), amount of additional machine time (M) and finishing time fixed time (a) $P = a + b + l + cm$.

From the data given below find the values of constants a, b and c by using the pivotal reduction method

Day	Production (In Units) (*P*)	Labour (In Hrs)(*l*)	Additional Machine Time (In Hrs)(*m*)
Monday	6,950	40	10
Tuesday	6,725	35	9
Wednesday	7,100	40	12

Estimate the production when overtime in labour is 50 hrs and additional machine time is 15 hrs.

Solution

We have

$P = a + bl + cm$

Putting above values we have

$6{,}950 = a + 40b + 10c$

$6{,}725 = a + 35b + 9c$

$7{,}100 = a + 40b + 12c$

By using pivotal reduction method, we have

$$\left[\begin{array}{ccc|c} 1 & 40 & 10 & 6{,}950 \\ 1 & 35 & 9 & 6{,}725 \\ 1 & 40 & 12 & 7{,}100 \end{array}\right] R_2 - R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 40 & 10 & 6{,}950 \\ 0 & -5 & -1 & -225 \\ 1 & 40 & 12 & 7{,}100 \end{array}\right] R_3 - R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 40 & 10 & 6{,}950 \\ 0 & -5 & -1 & -225 \\ 0 & 0 & 2 & 150 \end{array}\right] -\frac{1}{5}R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 40 & 10 & 6{,}950 \\ 0 & 1 & \frac{1}{5} & 45 \\ 0 & 0 & 2 & 150 \end{array}\right] R_1 - 40R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5{,}150 \\ 0 & 1 & \frac{1}{5} & 45 \\ 0 & 0 & 2 & 150 \end{array}\right] \frac{1}{2}R_3$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5{,}150 \\ 1 & 1 & \frac{1}{5} & 45 \\ 0 & 0 & 1 & 75 \end{array}\right] R_2 - \frac{1}{5}R_3$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5{,}150 \\ 0 & 1 & 0 & 30 \\ 0 & 0 & 1 & 75 \end{array}\right] R_1 - 2R_3$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5{,}000 \\ 0 & 1 & 0 & 30 \\ 0 & 0 & 1 & 75 \end{array}\right]$$

$\therefore a = 5{,}000,\ b = 30,\ c = 75$

$\therefore$ The production equation is $P = 5{,}000 + 30L + 75M$. When $l = 50$ and $m = 15$, the estimated production is

$$P = 5{,}000 + 30(50) + 75(15)$$
$$= 7{,}625 \text{ units}$$

$a = 5{,}000,\ b = 30,\ c = 75$

Illustration 29 The price of three commodities X, Y and Z are x, y and z, respectively. Mr Anand purchases 6 units of Z and sells 2 units of x and 3 units of Y. Mr Amar purchases a unit of Y and sells 3 units of x and 2 units of Z. Mr Amit purchases a unit of x and sells 3 units of Y and 9 units of Z. In the process they earn Rs. 5,000, Rs. 2,000 and Rs. 5,500, respectively. Find the prices per unit of three commodities.

Solution

Here we consider selling of units as positive earning whereas purchasing of units as negative earning. Then from the given information

$$2x + 3y - 6z = 5{,}000$$
$$3x - y + 2z = 2{,}000$$
$$-x + 3y + z = 5{,}500$$

Consider the coefficient matrix

$$A = \begin{bmatrix} 2 & 3 & -6 \\ 3 & -1 & 2 \\ -1 & 3 & 1 \end{bmatrix}$$

$$Now|A| = \begin{vmatrix} 2 & 3 & -6 \\ 3 & -1 & 2 \\ -1 & 3 & 1 \end{vmatrix}$$

$$= 2(-1-6) - 3[3-(-2)] + (-6)(9-1)$$
$$= -14 - 15 - 48$$
$$= -77$$

The co-factor matrix is

$$\begin{bmatrix} -7 & -5 & 8 \\ -21 & -4 & -9 \\ 0 & -22 & -11 \end{bmatrix}$$

By taking transpose, $adj\,A = \begin{bmatrix} -7 & -21 & 0 \\ -5 & -4 & -22 \\ 8 & -9 & -11 \end{bmatrix}$

By using the definition

$$A^{-1} = \frac{1}{|A|}\ adj\,A = \frac{1}{77}\begin{bmatrix} 7 & 21 & 0 \\ 5 & 4 & 22 \\ -8 & 9 & 11 \end{bmatrix}$$

Hence the solution is

$$\begin{bmatrix} x \\ y \\ x \end{bmatrix} = A^{-1}\begin{bmatrix} 5{,}000 \\ 2{,}000 \\ 5{,}500 \end{bmatrix}$$

$$= \frac{1}{77}\begin{bmatrix} 7 & 21 & 0 \\ 5 & 4 & 22 \\ -8 & 9 & 11 \end{bmatrix}\begin{bmatrix} 5{,}000 \\ 2{,}000 \\ 5{,}500 \end{bmatrix}$$

$$= \frac{1}{77}\begin{bmatrix} 35{,}000 + 42{,}000 + 0 \\ 25{,}000 + 8{,}000 + 1{,}21{,}000 \\ -40{,}000 + 18{,}000 + 60{,}500 \end{bmatrix}$$

$$= \frac{1}{77}\begin{bmatrix} 77{,}000 \\ 1{,}54{,}000 \\ 38{,}500 \end{bmatrix}$$

$$= \begin{bmatrix} 1{,}000 \\ 2{,}000 \\ 500 \end{bmatrix}$$

$x = 1{,}000$, $y = 2{,}000$ and $z = 500$.

Illustration 30 An amount of Rs. 5,000 is to be deposited in three different bonds bearing 6%, 7% and 8% per year respectively. Total annual income is Rs. 358. If the income from first two investments is Rs. 70 more than the income from the third, then find the amount of investment by using Gauss elimination method.

Solution

Suppose the amount is to be invested in three different bonds x, y and z, then the equations are

$x + y + z = 5{,}000$

$0.06x + 0.07y + 0.08z = 358$

$0.06x + 0.07y + 0.08z = 70$

Now consider

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5{,}000 \\ 0.06 & 0.07 & 0.08 & 358 \\ 0.06 & 0.07 & -0.08 & 70 \end{array}\right] R_2 - 0.06R_1 \quad R_3 - 0.06R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5{,}000 \\ 0 & 0.01 & 0.02 & 58 \\ 0 & 0.01 & -0.14 & -230 \end{array}\right] \frac{1}{0.01}R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5{,}000 \\ 0 & 1 & 2 & 5{,}800 \\ 0 & 0.01 & -0.14 & -230 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 0.01R_2 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -1 & -800 \\ 0 & 1 & 2 & 5{,}800 \\ 0 & 0.01 & -0.16 & -288 \end{array}\right] \frac{1}{0.16}R_3$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -1 & -800 \\ 0 & 1 & 2 & 5,800 \\ 0 & 0 & 1 & 1,800 \end{array}\right] R_1 + R_2 \quad R_2 - 2R_3$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1,000 \\ 0 & 1 & 0 & 2,200 \\ 0 & 0 & 1 & 1,800 \end{array}\right]$$

$x = 1,000$, $y = 2,200$ and $z = 1,800$

Illustration 31 If $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $AB = \begin{bmatrix} -13 & 8 \\ -8 & 5 \end{bmatrix}$ then find matrix B.

Solution

Here we have $AB = \begin{bmatrix} -13 & 8 \\ -8 & 5 \end{bmatrix}$

Pre-multiplying both sides by A^{-1} we have

$$A^{-1}\,AB = A^{-1}\begin{bmatrix} -13 & 8 \\ -8 & 5 \end{bmatrix}$$

$$\therefore B = A^{-1}\begin{bmatrix} -13 & 8 \\ -8 & 5 \end{bmatrix}$$

Now we find A^{-1}

$$|A| = \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} = 6 - 5 = 1$$

and the co-factor matrix is $= \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

By taking transpose, $adj\,A = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

and hence $A^{-1} = \dfrac{1}{|A|}\,adj\,A = \dfrac{1}{1}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

Hence $B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -13 & 8 \\ -8 & 5 \end{bmatrix}$

$$= \begin{bmatrix} -39+40 & 24-25 \\ 13-16 & -8+10 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix}$$

Illustration 32 If $\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$ then $A = \begin{bmatrix} -4 & 8 & 4 \\ -1 & 2 & 1 \\ -3 & 6 & 3 \end{bmatrix}$

Solution

Here resultant matrix is of order 3×3 hence the order of matrix A must be of 1×3, say $A = \begin{bmatrix} a\ b\ c \end{bmatrix}$

Now $\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} \text{a b c} \end{bmatrix} = \begin{bmatrix} -4 & 8 & 4 \\ -1 & 2 & 1 \\ -3 & 6 & 3 \end{bmatrix}$

$$\begin{bmatrix} 4a & 4b & 4c \\ a & b & c \\ 3a & 3b & 3c \end{bmatrix} = \begin{bmatrix} -4 & 8 & 4 \\ -1 & 2 & 1 \\ -3 & 6 & 3 \end{bmatrix}$$

$\therefore$ By comparing we have $a = -1$, $b = 2$ and $c = 1$

$\therefore$ Matrix $A = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}$

Illustration 33 If $A^2 = \begin{bmatrix} 65 & 56 \\ 56 & 65 \end{bmatrix}$ then find matrix A.

Solution

Suppose $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$

then $A^2 = AA = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} 65 & 56 \\ 56 & 65 \end{bmatrix}$

$\therefore a^2 + b^2 = 65$ and $2ab = 56$

$$\text{Now } (a+b)^2 = a^2 + b^2 + 2ab$$
$$= 65 + 56$$
$$= 121$$

$$a + b = 11 \qquad (1)$$

$$\text{and } (a-b)^2 = a^2 + b^2 - 2ab$$
$$= 65 - 56$$
$$= 9$$

$a - b = 3$ (2)

Solving (1) and (2) we have $a = 7$ and $b = 4$

The matrix A = $\begin{bmatrix} 7 & 4 \\ 4 & 7 \end{bmatrix}$

Illustration 34 If $A = \begin{bmatrix} 5 & 3 & 1 \\ 2 & -1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$ then prove that $A^3 - 7A^3 - 5A + 13I = 0$ and hence find A^{-1}.

Solution

$$\text{Now } A^2 = AA = \begin{bmatrix} 5 & 3 & 1 \\ 2 & -1 & 2 \\ 4 & 1 & 3 \end{bmatrix}\begin{bmatrix} 5 & 3 & 1 \\ 2 & -1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 35 & 13 & 14 \\ 16 & 9 & 6 \\ 34 & 14 & 15 \end{bmatrix}$$

$$\text{and } A^3 = A^2A = \begin{bmatrix} 35 & 13 & 14 \\ 16 & 9 & 6 \\ 34 & 14 & 15 \end{bmatrix}\begin{bmatrix} 5 & 3 & 1 \\ 2 & -1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 257 & 106 & 103 \\ 122 & 45 & 52 \\ 258 & 103 & 107 \end{bmatrix}$$

$$\therefore A^3 - 7A^2 - 5A + 13I = \begin{bmatrix} 257 & 106 & 103 \\ 122 & 45 & 52 \\ 258 & 103 & 107 \end{bmatrix} - \begin{bmatrix} 245 & 91 & 98 \\ 112 & 45 & 52 \\ 238 & 98 & 105 \end{bmatrix}$$

$$- \begin{bmatrix} 25 & 15 & 5 \\ 10 & -5 & 10 \\ 20 & 5 & 15 \end{bmatrix} + \begin{bmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore A^3 - 7A^2 - 5A + 13I = 0$

$\therefore A^3 - 7A^2 - 54 = -13I$

$\therefore A\left(A^2 - 7A - 5I\right) = -13I$

$$\therefore A^{-1}\left(A^2 - 7A - 5I\right) = -13A^{-1}\,I$$

$$\therefore -13A^{-1} = A^2 - 7A - 5I$$

$$= \begin{bmatrix} 35 & 13 & 14 \\ 16 & 9 & 6 \\ 34 & 14 & 15 \end{bmatrix} - \begin{bmatrix} 35 & 21 & 7 \\ 14 & -7 & 14 \\ 28 & 7 & 21 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$-13A^{-1} = \begin{bmatrix} -5 & -8 & 7 \\ 2 & 11 & -8 \\ 6 & 7 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{13}\begin{bmatrix} 5 & 8 & -7 \\ 2 & -11 & 8 \\ 6 & 7 & 1 \end{bmatrix}$$

Illustration 35 If A is symmetric matrix and B is skew symmetric matrix and if $A + B = \begin{bmatrix} 1 & 0 & 6 \\ -4 & 4 & -2 \\ 0 & 8 & 6 \end{bmatrix}$ then find matrix A and matrix B.

Solution

Suppose $A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -p & -q \\ p & 0 & -r \\ q & r & 0 \end{bmatrix}$

$$Now\ A + B = \begin{bmatrix} a & b-p & c-q \\ b+p & d & e-r \\ c+q & e+r & f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ -4 & 4 & -2 \\ 0 & 8 & 6 \end{bmatrix}$$

By comparison we have $a = 1$, $d = 4$, $f = 6$

and $b - p = 0$ $\quad c - 9 = 6$ $\quad$ and $\quad e - r = 2$

$\underline{b + p = -4}$ $\quad \underline{c - 9 = 0}$ $\quad \underline{e + r = 8}$

$2b = -4$ $\quad 2c = 6$ $\quad 2e = 6$

$\therefore b = -2$ $\quad \therefore c = 3$ $\quad \therefore e = 3$

and $p = 2$ $\quad q = -3$ $\quad r = 5$

$$\text{Hence } A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & 3 \\ 3 & 3 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -5 \\ -3 & 5 & 0 \end{bmatrix}$$

Illustration 36 Solve the equations. $x - y = a, y - z = b$ and $x + z = c$.

Solution

The given equations are

$x - y + 0z = a$

$0x + y - z = b$

$x + 0y + z = c$

$$\therefore \text{ The co-efficient matrix A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\text{Now}|A| = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= 1(1-0) - (-1)(0-1-1) + 0(0-1)$$

$$= 1 + 1 + 0$$

$$= 2$$

The co-factor matrix is

$$\begin{bmatrix} 1 & -1 & -1 \\ +1 & 1 & -1 \\ 1 & +1 & 1 \end{bmatrix}$$

$$\therefore \text{ By taking transpose, } adj\,A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adj\,A = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

Hence the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} a & +b & +c \\ -a & +b & +c \\ -a & -b & +c \end{bmatrix}$$

$$\therefore\ x = \frac{a+b+c}{2}, y \frac{-a+b+c}{2}, z = \frac{-a-b+c}{2}$$

ANALYTICAL EXERCISES

1. Explain the following matrices with suitable illustrations.

(1) Equal matrices (2) Square matrix

(3) Transpose of matrix (4) Null matrix

2. Define matrix and give general form of a matrix of the order of $m \times n$.

3. Give differences between the following matrices.

(1) Row matrix and Column matrix

(2) Identity matrix and Diagonal matrix

(3) Scalar matrix and Diagonal matrix

4. Define the following matrices with appropriate illustrations.

(1) Triangular matrices (2) Symmetric matrix

(3) Skew symmetric matrix (4) Sub–matrices

5. Give definitions of the following matrices.

(1) Orthogonal matrix (2) Nilpotent matrix

(3) Idempotant matrix (4) Inverse of a matrix

6. What is adjoint matrix? Expalin it by taking a matrix of order 2×2.

7. Write the necessary condition for adding two or more matrices.

8. Write the necessary condition for multiplying two matrices.

9. Write the necessary conditions for finding inverse of a matrix.

10. How is matrix algebra used to solve the simultaneous linear equations of three unknown?

11. Classify the following matrices.

$$(i)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (ii)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (iii)\begin{bmatrix} 1 & 4 & 1 \\ 2 & 0 & 0 \\ 3 & 3 & 1 \end{bmatrix}$$

$$(iv)\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (v)\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 1 & 5 \end{bmatrix} \quad (vi)\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix}$$

$$(vii)\begin{bmatrix} 4 & 2 & 5 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad (viii)\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (ix)\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

12. If $\begin{bmatrix} x-y & 3 \\ 4 & 2x+3y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 14 \end{bmatrix}$ then find x and y

13. During the calendar year 2008, the centuries made by four cricket players of Indian team at the first, second, third and fourth batting positions are given below. The row represents batting position and column represents the player.

$$\begin{array}{c|cccc} & A & B & C & D \\ I & 4 & 0 & 2 & 1 \\ II & 2 & 1 & 3 & 0 \\ III & 0 & 3 & 1 & 0 \\ IV & 0 & 1 & 2 & 4 \end{array}$$

On the basis of this information answer the following

1) How many centuries player A has made?
2) At first position how many centuries are made?
3) How many centuries player A had made at second position?
4) At third position which batsman has made maximum centuries?
5) Write the column vector which represents the centuries made by player A and player D at different positions?
6) Write a submatrix representing the number of centuries made by different batsmen at first and third position.
7) In a row vector $\begin{bmatrix} 4 & 0 & 2 & 1 \end{bmatrix}$ what 0 represents?
8) On the basis of above matrix, how you assign the different batting position to different batsmen?

14. If $A = \begin{bmatrix} 2 & 4 \\ -3 & 2 \end{bmatrix}$ and $B - 2A$, $C = 4A - B + 5I$ then find matrix D such that

$D = 6A - (2B + C) + 6I$

15. If $A = \begin{bmatrix} x & 4 \\ y & 2 \end{bmatrix}, B = \begin{bmatrix} -y & 2 \\ x & 3 \end{bmatrix}$ $and\ A + 2B = \begin{bmatrix} 2 & 8 \\ 9 & 8 \end{bmatrix}$ then find x and y.

16. If $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & 4 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 4 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix}$ find the following operations

(which are possible)

(i) $2A + B$ (ii) $2A + 3B^1 - 2C$ (iii) $A^1 + B^1 - C^1$

(iv) $A^1 - 2B + 2C^1$ (v) $(A + 2C)^1 - 2B$.

17. If $x = \begin{bmatrix} 2 & 1 \\ 4 & -2 \end{bmatrix}$ then find matrix y such that $x - 2y = \begin{bmatrix} -2 & -3 \\ 0 & -6 \end{bmatrix}$

18. If $A = \begin{bmatrix} 3 & -1 & 1 \\ -2 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 & -3 \\ 2 & -2 & 6 \\ -6 & 0 & 6 \end{bmatrix}$ find AB and BA.

19. If $A = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ 6 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & -3 \\ 0 & 1 \end{bmatrix}$

Then prove the followings

(i) $(AB)\ C = (AB)\ C$

(ii) $(ABC)' = C^1B'A'$.

20. Evaluate $\begin{bmatrix} x\, y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

21. If $A = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$ then find A^2 and have identity the type of matrix A.

22. If $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ then prove that A is nilpotent matrix.

23. If $A^2 = \begin{bmatrix} 29 & 20 \\ 20 & 29 \end{bmatrix}$ then find matrix A.

24. If $A \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 8 \end{bmatrix} = \begin{bmatrix} 25 & 66 & 100 \\ 4 & 12 & 16 \\ 7 & 19 & 28 \end{bmatrix}$ then find matrix A and hence find $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 8 \end{bmatrix} A$.

25. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & x \\ 4 & y \end{bmatrix}$ and $(A + B)^2 = A^2 + B^2$ then find x and y.

26. If $A=\begin{bmatrix}4 & 1 & 0\\ 1 & -2 & 2\end{bmatrix}, B=\begin{bmatrix}2 & 0 & -1\\ 3 & 1 & 2\end{bmatrix}$ and $C=\begin{bmatrix}1\\ 2\\ -0\end{bmatrix}$ then find matrix x such that $(3A-2B)C+3x=0.$

27. Find the value of x, if $\begin{bmatrix}x & 4 & 13\end{bmatrix}\begin{bmatrix}2 & 1 & 0\\ 1 & 0 & 5\\ 0 & 2 & 4\end{bmatrix}\begin{bmatrix}x\\ 4\\ -1\end{bmatrix}=0$

28. Let $f(x)=x^3-2x^2+4x-18$ then find $f(A)$ if $A=\begin{bmatrix}1 & 2 & 1\\ -1 & 0 & 3\\ 2 & -1 & -1\end{bmatrix}$

29. $A=\begin{bmatrix}1 & 1 & -1\\ 2 & -3 & 4\\ 3 & -2 & 3\end{bmatrix}, B=\begin{bmatrix}-1 & -2 & -1\\ 6 & 12 & 6\\ 5 & 10 & 5\end{bmatrix}$ and $C=\begin{bmatrix}-1 & -1 & 1\\ 2 & 2 & -2\\ 3 & 3 & 0\end{bmatrix}$ then find AB and CA.

30. If $AA=\begin{bmatrix}2 & -3 & -5\\ -1 & 4 & 5\\ 1 & -3 & -4\end{bmatrix}, B=\begin{bmatrix}-1 & 3 & 5\\ 1 & -3 & -5\\ -1 & 3 & 5\end{bmatrix}$ and $C=\begin{bmatrix}2 & -2 & -4\\ -1 & 3 & 4\\ 1 & -2 & -3\end{bmatrix}$ then prove that

(i) $AB=BA=0$

(ii) $AC=A$ and $CA=C$

31. If $2x-y\begin{bmatrix}3 & 1 & 0\\ 0 & -8 & 6\\ 3 & 3 & 1\end{bmatrix}$ and $x+3y=\begin{bmatrix}5 & -21 & 0\\ 7 & 10 & 10\\ -9 & 12 & 4\end{bmatrix}$ then find matrix x and y.

32. If $A=\frac{1}{3}\begin{bmatrix}a & 1 & 2\\ 2 & 1 & b\\ c & 2 & -1\end{bmatrix}$ is an orthogonal matrix, then find the value of constants a, b and c.

33. Prove that $A=\frac{1}{9}\begin{bmatrix}-8 & 1 & 4\\ 4 & 4 & 7\\ 1 & -8 & 4\end{bmatrix}$ is an orthogonal matrix.

34. If $A=\frac{1}{9}\begin{bmatrix}x & 1 & 4\\ 4 & 4 & z\\ y & x & 4\end{bmatrix}$ v is an orthogonal matrix. Then find the value of of x, y and z.

35. If $\begin{bmatrix} a & 2 & -3 \\ b & b & -4 \\ c & e & f \end{bmatrix}$ is a skew–symmetric matrix then find the value of *a,b,c,d,e* and *f*.

36. Sum of a symmetric and skew symmetric matrix is given as $\begin{bmatrix} 2 & 1 & -3 \\ 5 & -2 & -2 \\ -5 & 4 & 0 \end{bmatrix}$ Find both the matrices.

37. If $f(x) = 5x + 7x^2 - x^3$ and if $A = \begin{bmatrix} 5 & 3 & 1 \\ 2 & -1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$ then find $f(A)$

38. If $x^2 + y^2 + z^2 = 1$ then prove that $A^3 + A = 0$ where $A = \begin{bmatrix} o & -z & -x \\ z & o & -y \\ y & x & o \end{bmatrix}$

39. Evaulate $\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x & p & q \\ p & y & r \\ q & r & z \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

40. If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then prove that $(a + b)A - A^2 = (ad - bc)I$

41. If A is a matrix of order $2x \cdot x$ and B is also a matrix of order such that AB and BA are defined then find the value of x and y.

42. If $A = \begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix} = \begin{bmatrix} 16 & 57 \\ 25 & 56 \end{bmatrix}$ then find matrix Δ

43. For the following find the adjoint matrix.

(i) $\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & 6 & -5 \\ -2 & 12 & 10 \\ -1 & -6 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$

44. If $A = \begin{bmatrix} 2 & -1 \\ 6 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -1 \\ 0 & 3 \end{bmatrix}$ then prove the following

(i) $adj(AB) = adjB\, adjA.$

(ii) $(adjA)A = A(adjA) = |A|\, I_2$

45. If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ then prove the following

(i) $(adj\, A)\, A = A(adj\, A) = |A|\, I_3$

(ii) *If* $B = \begin{bmatrix} 0 & 2 & -1 \\ 0 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ then show that $adj\ (AB) = ad\ B\ \ ad\, A.$

46. For the following matrices find the co-factor matrix.

(i) $\begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & -6 & -1 \\ 7 & 4 & 2 \\ 2 & 1 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 4 & -1 \\ 3 & 1 & 2 \\ 1 & 3 & -3 \end{bmatrix}$

47. Find the inverse of the following matrices and verify the result $A^{-1}\, A = I.$

(i) $\begin{bmatrix} 4 & 3 \\ -2 & -4 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 4 \\ 0 & 1 & -2 \end{bmatrix}$ (iii) $\begin{bmatrix} -2 & 1 & -3 \\ 0 & -1 & 2 \\ -1 & 2 & 0 \end{bmatrix}$

48. A trust has Rs. 1,00,000 that must be invested in two different types of bonds. The first bond pays 14% interest per year and second bond pays 16% interest per year. Using matrix multiplication determine how to distribute the money among two types of bonds to obtain Rs. 15,200 interest.

49. A company has to choose the best way of producing three goods *A*, *B* and *C*. The amount of each good produced by each methods I, II and III is given by the following matrix.

$$\begin{array}{c|ccc} & A & B & C \\ \hline \text{I} & 4 & 8 & 2 \\ \text{II} & 5 & 7 & 1 \\ \text{III} & 5 & 3 & 9 \end{array}$$

The profit per unit for these goods is represented as $\begin{bmatrix} 10 & 4 & 6 \end{bmatrix}$. Find which method is profitable?

50. A man buys 5 dozens of oranges, 6 dozens of apples and 3 dozens of bananas. Oranges cost Rs. 15 per dozen, apples Rs. 12 per dozen and banana Rs. 8 per dozen. Represent the information in matrix form and hence obtain total cost.

51. The amount of Rs. 5,000 is put into three investments at the rate of interest 6%, 7% and 8% per annum respectively. The total annual income is Rs. 358. If the combined income from first two investment is Rs. 70 more than the income from the third, find the amount of investment by using matrix.

52. Mr X has invested a part of his investment in 10% bond *A* and part in 15% bond *B*. His interest income during the first year is Rs. 4,000. If he invest 20% more in bond *A* and 10% more in bond *B* his income during second year increases by Rs. 500. Find his initial investment and the new investment in bond *A* and *B* using matrix.

ANSWERS

(11) (i) Null matrix (ii) Identity matrix (iii) Square matrix (iv) Nilpotent matrix (v) Lower triangular matrix (vi) Skew–symmetric matrix (vii) Upper triangular matrix (viii) Diagonal matrix (ix) Scalar matrix

(12) $x = 4, y = 2$

(13) (1) 6 (2) 7 (3) 2 (4) Player B

(5) $\begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \end{bmatrix}$ (6) $\begin{bmatrix} 4 & 0 & 2 & 1 \\ 0 & 3 & 1 & 0 \end{bmatrix}$

(7) Player had made no century at first position

(8) First position player A, Second position player C, third position player B and at fourth position player D.

(14) I

(15) $x = 4, y = 1$

(16) (i) Not possible (ii) $\begin{bmatrix} 2 & 8 & 3 \\ 14 & -1 & 0 \end{bmatrix}$

(iii) Not possible (iv) $\begin{bmatrix} 6 & -7 \\ 1 & 5 \\ 1 & 0 \end{bmatrix}$ (v) $\begin{bmatrix} 8 & -3 \\ 0 & 4 \\ 2 & 0 \end{bmatrix}$

(17) $y = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

(18) $AB = \begin{bmatrix} -5 & -2 \\ 2 & 8 \end{bmatrix}$ and $BA = \begin{bmatrix} -1 & 1 & -3 \\ 2 & -2 & 6 \\ -6 & 0 & 6 \end{bmatrix}$

(20) $ax^2 + 2bxy + cy^2$

(21) $A^2 = I$:. A is an idempotant matrix.

(23) $A = \begin{bmatrix} 2 & 5 \\ 5 & 2 \end{bmatrix}$

(24) $A = \begin{bmatrix} 7 & 9 \\ 4 & 0 \\ 3 & 2 \end{bmatrix}$ and $\begin{bmatrix} 31 & 17 \\ 58 & 34 \end{bmatrix}$

(25) $x = 1, y = -1$

(26) $x = \begin{bmatrix} 2 \\ -7 \end{bmatrix}$

(27) $x = -4$

(28) 0

(29) $AB = 0$ and $CA = 0$

(31) $x = \begin{bmatrix} 1 & -3 & 0 \\ 2 & -2 & 4 \\ 0 & 3 & 1 \end{bmatrix}$ and $y = \begin{bmatrix} 1 & -6 & 0 \\ 2 & 4 & 2 \\ -3 & 3 & 1 \end{bmatrix}$

(32) $a = 1, b = -2, c = -2$

(34) $x = -8, y = 1, z = 7$

(35) $a = d = f = 0, b = 2, c = 3, e = 4$

(36) $\begin{bmatrix} 2 & 3 & -4 \\ 3 & -2 & 1 \\ -4 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$

(37) $3I_3$

(39) $a^2x + b^2y + c^2z + 2(pab + rbc + qac)$

(41) $x = 3$ and $y = 3$

(42) A = $A = \begin{bmatrix} 9 & 1 \\ 4 & 3 \end{bmatrix}$

(43) (i) $\begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 120 & -60 & 120 \\ 20 & 10 & -20 \\ 24 & -12 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 12 & 12 & 0 \\ 0 & -6 & -6 \\ -4 & 0 & 4 \end{bmatrix}$

(46) (i) $\begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix}$ (ii) $\begin{bmatrix} 27 & -48 & -1 \\ 40 & 22 & -17 \\ 2 & 43 & 6 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 4 & -1 \\ 3 & 1 & 2 \\ 1 & 3 & -3 \end{bmatrix}$

(48) 40,000, 60,000

(49) 1st method, 84 2nd method 84, 3rd method 116. 3rd method is profit table.

(50) Rs. 171

(51) 1,000, 2,200, 1,800

(52) 10,000, Rs. 20,000

23

Permutation and Combination

LEARNING OBJECTIVES

After studying this chapter, the student will be able to understand:

- Difference between permutation and combination
- Different cases of permutation such as word formation
- Different cases of combination such as selection of a team
- Application of permutation and combination to derive more formulae

INTRODUCTION

Permutation and combination has lately emerged as an important topic for many entrance examinations. This is primary because questions from the topic require analytical skill and a logical bend of mind. Even students who do not have mathematics as a subject can handle them if they have a fairly good understanding of the concepts and their application. Hence anyone who is well-versed in different methods of counting and basic calculations will be able to solve these problems easily.

IMPORTANT NOTATION

$n!$ (Read as n factorial)

Product of first n positive integers is called n factorial

$$n! = 1 \times 2 \times 3 \times 4 \times 5 \times . . . n$$

$$n! = (n-1)! \;\; n \in \mathrm{N}$$

In special case $0! = 1$

MEANING OF PERMUTATION AND COMBINATION

Permutation

The arrangement made by taking some or all elements out of a number of things is called a permutation.

The number of permutations of n things taking r at a time is denoted by ${}^nP_r$ and it is defined as under:

$${}^nP_r = \frac{n!}{(n-r)!}\,; r \le n \quad \left(\text{Read } {}^np_r\right)$$

Combination

The group or selection made by taking some or all elements out of a number of things is called a combination.

The number of combinations of n things taking r at a time is denoted by ^nC_r or $n\underline{\lfloor r}$ and it is defined as under:

$$^nC_r = \frac{n!}{r!(n-r)!}\,; r \le n$$

Here $n!$ = Multiple of n natural number

Some Important Results of Permutations

(1) $^nP_{n-1} = {}^nP_n$

(2) $^nP_n = n!$

(3) $^nP_r = n\left({}^{n-1}P_{r-1}\right)$

(4) $^nP_r = (n-r+1) \times {}^nP_{r-1}$

(5) $^nP_r = {}^{n-1}P_r + r\left({}^{n-1}P_{r-1}\right)$

Types of Permutations

Case 1

When in a permutation of n things taken r at a time, a particular thing always occurs, then the required number of permutations = $r\left({}^{n-1}P_{r-1}\right)$.

Case 2

The number of permutations of n different things taken r at a time, when a particular thing is never taken in each arrangements = $^{n-1}P_r$.

Case 3 (Permutation of Like Things)

The number of n things taken all at a time, given that p_1 things are first alike, p_2 things are second alike and p_r things are rth alike is

$$\frac{n!}{p_1! \cdot p_2! \ldots p_n!}$$

Case 4 (Permutation with Repetitions)

The number of n different things taken r at a time when each may be repeated any number of times in each arrangements = n.

Case 5 (Circular Permutations)

Circular permutations are the permutations of things along the circumference of a circle. It is important to note that in a circular arrangement we have to consider the relative position of the different things.

If we have to arrange the five letters P, Q, R, S and T, the two of the arrangements would be PQRST and TPQRS. These two arrangements are obviously different if the things are to be placed in a straight line. But if the arrangements are written along the circumference of a circle, then the two arrangements PQRST and TPQRS are one and the same.

As the number of circular permutations depends on the relative position of the objects, we fix the position of one object and then arrange the remaining $(n-1)$ objects in all possible ways. This can be done in $(n-1)!$ ways. Thus the circular arrangements of five letters P, Q, R, S, T will be

$(5-1)! = 4! = 4 \times 3 \times 2 \times 1 = 24$ ways

The number of circular permutations of n different objects $= (n-1)!$

Some Important Results of Combinations

(1) ${}^nC_0 = 1 = {}^nC_n$

(2) ${}^nP_r = r!\left({}^nC_r\right)$

(3) ${}^nC_r = {}^nC_{n-r}$

(4) ${}^nC_p = {}^nc_q \Leftrightarrow p = q$ or $p + q = n$

(5) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ (Pascal's law)

(6) $\dfrac{{}^nC_r}{(n-r+1)} = \dfrac{{}^nC_{n-r}}{r}$

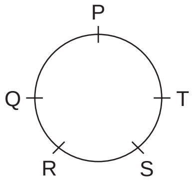
(7) $n \times {}^{n-1}C_{r-1} = (n-r+1) \times {}^nC_{r-1} \; (\because 1 \le r \le n)$

Types of Combinations

To find the total number of combinations of dissimilar things taking any number of them at a time, when all the things are different

(1) The number of combinations of n items taken r at a time in which given p perticular items will always occurs is ${}^{n-p}C_{r-p}$

(2) The number of combinations of n items taken r at a time in which p particular items never occur is ${}^{(n-p)}C_r$.



ILLUSTRATIONS

Illustration 1 If ${}^nC_{10} = {}^nC_{14}$ then find the value of n

Solution

$${}^nC_{10} = {}^nC_{14} \Rightarrow n = (10 + 14) = 24 \qquad (\because n = p + q)$$

Illustration 2 If ${}^nC_3 = 220$ then find the value of n.

Solution

$${}^nC_3 = 220$$

$$\therefore \frac{n!}{(n-3)!\,3!} = 220$$

$$\therefore \frac{n(n-1)(n-2)(n-3)!}{(n-3)!\,6} = 220$$
$$\therefore n(n-1)(n-2) = 1320$$
$$\therefore n(n-1)(n-2) = 12 \times 11 \times 10$$
$$\therefore n = 12$$

Illustration 3 If ${}^{n}P_{5} = 20\,{}^{n}P_{3}$ then find the value of n.

Solution

$${}^{n}P_{5} = 20\,{}^{n}P_{3}$$
$$\therefore \frac{n!}{(n-5)!} = 20\frac{n!}{(n-3)!}$$
$$\therefore \frac{1}{(n-5)!} = 20\frac{1}{(n-3)!}$$
$$\therefore (n-3)! = 20(n-5)!$$
$$\therefore (n-3)(n-4)(n-5)! = 20(n-5)!$$
$$\therefore (n-3)(n-4) = 20$$
$$\therefore n^2 - 7n + 12 = 20$$
$$\therefore n^2 - 7n - 8 = 0$$
$$\therefore (n-8)(n+1) = 0$$
$$\therefore n = 8 \text{ or } n = -1 \notin N$$
$$\therefore n = 8$$

Illustration 4 If ${}^{n}C_{r} + {}^{n}C_{r+1} = {}^{n+1}C_{x}$ then find the value of x.

Solution

$${}^{n}C_{r} + {}^{n}C_{r+1} = {}^{n+1}C_{r+1} \text{ (formula)}$$

and ${}^{n}C_{r} + {}^{n}C_{r+1} = {}^{n+1}C_{x}$ (given)

By comparing above two statements we can say that

$x = r + 1$

Illustration 5 If ${}^{15}P_{r-1} : {}^{16}P_{r-2} = 3:4$ then find the value of r

Solution

$$\frac{{}^{15}P_{r-1}}{{}^{16}P_{r-2}} = \frac{3}{4}$$
$$\therefore \frac{15!/(16-r)!}{16!/(18-r)!} = \frac{3}{4}$$
$$\therefore \frac{15!}{(16-r)!} \times \frac{(18-r)(17-r)(16-r)!}{16 \times 15!} = \frac{3}{4}$$

$$\therefore (18-r)(17-r) = 3 \times 4 = 12$$
$$\therefore r^2 - 35r + 294 = 0$$
$$\therefore (r-21)(r-14) = 0$$
$$\therefore r = 14 \qquad (\therefore r \le 16)$$

Illustration 6 In how many ways can the letters of the word 'APPLE' be arranged?

Solution

There are in all 5 letters; out of these 2 are P, A, L and E

Required number of ways $= \dfrac{5!}{2! \times 1! \times 1! \times 1!} = 60$

Illustration 7 In how many ways can 8 students be arranged in a row?

Solution

8 students may be arranged in a row in ${}^8p_8 = 8!$ ways.

Illustration 8 In how many ways 6 rings of different types can be had in a finger?

Solution

The first ring can be worn in any of the 4 fingers.
So there are 4 ways of wearing this ring similarly each one of the other rings may be worn in 4 ways.
$\therefore$ Required number of ways = 4^6

Illustration 9 In how many ways a committee of 5 members can be selected from 6 men and 5 women consisting of 3 men and 2 women?

Solution

(3 men out of 6) and (2 women out of 5) can be selected in
${}^6C_3 \times {}^5C_2 = 200$ ways

Illustration 10 How many diagonals are there in a decagon?

Solution

$$\text{No. of diagonals } = \frac{1}{2}n(n-3)$$
$$= \frac{1}{2}10(10-3)$$
$$= 5 \times 7 = 35$$

Illustration 11 How many triangles can be drawn through n given points on a circle?

Solution

Any three points on a circle are non-collinear.
$\therefore$ Required number of triangles = nc_3

Illustration 12 How many 3-digit even numbers can be formed with no digit repeated by using the digits 0,1,2,3,4 and 5?

Solution

Numbers with 0 at unit place $= 5 \times 4 \times 1 = 20$
Numbers with 2 at unit place $= 4 \times 4 \times 1 = 16$
Numbers with 4 at unit place $= 4 \times 4 \times 1 = 16$
$\therefore$ Total numbers = 20 + 16 + 16 = 52

Illustration 13 In how many ways can 10 books be arranged on a shelf so that a particular pair of book shall be together?

Solution

Number of ways in which 10 books may be arranged $=$ 10!
Number of ways in which 10 books may be arranged with two particular books together $= 2 \times 9!$

Illustration 14 Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?

Solution

Number of ways of selecting 3 consonants out of 7 and 2 vowels out of 4
$= {}^7C_3 \times {}^4C_2 \;\; = {}^7C_3 \times {}^4C_2 = 210$ (Q By calculating)

Illustration 15 How many words can be formed from the letters of the word 'DAUGHTER' so that the vowels always come together?

Solution

Take all the vowels A,U,E together and take them as one letter
Then the letters to be arranged are P, G, H, T, R (AUE)
These 6 letters can be arranged at 6 places in 6! ways
Now 3 letters A,U,E among themselves can be arranged in 3! ways
$\therefore$ Required no. of words $= 6! \times 3!$ = 4,320 ways

ANALYTICAL EXERCISES

1. If $\dfrac{{}^{n+3}P_6}{{}^{n+2}P_4} = 14$ then find the value of n.

2. If $\dfrac{{}^{2n}P_{n+1}}{{}^{2n-2}P_n} = \dfrac{56}{3}$ then find the value of n.

3. If $\dfrac{{}^{12}P_x}{{}^{10}P_x} = \dfrac{3}{4}$ then find the value of x.

4. If ${}^nP_3 = 60$ then find the value of n.

5. If $\frac{{}^{56}P_{r+6}}{{}^{54}P_{r+3}} = 30{,}800$ then find the value of r.

6. Find the value of $(n+1)! - n!$

7. If $\frac{{}^{n}P_{3}}{{}^{n}P_{2}} = 3$ then find n.

8. If ${}^{5}P_{r} = 60$ then find r.

9. If ${}^{18}C_{r} = {}^{18}C_{r+2}$ then find r.

10. Find the value of ${}^{n-2}C_{r} + 2\ {}^{n-2}C_{r-1} + {}^{n-2}C_{r-2}$.

11. Find the value of ${}^{12}C_{3} + 2\ {}^{12}C_{4} + {}^{12}C_{5}$.

12. Find the value of ${}^{n}C_{r} + {}^{n}C_{r-1}$.

13. Find the value of ${}^{n-1}P_{r} + r \cdot {}^{n-1}P_{r-1}$.

14. Find the value of $\sum_{r=1}^{n} {}^{n}C_{r}$.

15. Find the value of ${}^{12}C_{4} + {}^{12}C_{3}$.

16. If ${}^{499}C_{92} + {}^{n}C_{91} = {}^{500}C_{92}$ then find the value of n

17. If $\frac{{}^{n-1}P_{3}}{{}^{n+1}P_{3}} = \frac{5}{12}$ then find the value of n.

18. If ${}^{7}P_{n} + {}^{7}P_{n-3} = 60$ then find the value of n.

19. If $\frac{{}^{n}P_{4}}{{}^{n}P_{2}} = 12$ then find the value of n.

20. If ${}^{10}P_{r} = 720$ then find the value of r.

21. A man has 6 friends to invite. In how many ways can he send invitation cards to them if he has three servants to carry the cards?

22. A tool item has 5 arms and each arm is capable of 4 distinct positions including the position of rest. Then how many total number of signals that can be?

23. How many 3-digit numbers divisible by 5 can be formed using any number from 0 to 9 without repetition?

24. A corporate has 11 electronic engineers and 7 mechanical engineers. In how many ways can they be seated in a row so that no two of the mechanical engineers may sit together?

25. In how many ways can 6 lecturers and 5 professors be seated in a row so that they are positioned alternately?

26. A high school has 11 assistant teachers and 7 teachers. In how many ways can they be seated in a row so that all the teachers are together?

27. In how many ways 5 men and 5 women can be seated in a round table so that they are positioned alternately?

28. Three dices are rolled. Find the number of possible outcomes in which at least one dice shows 5?

29. How many numbers not exceeding 1,000 can be made using the digit of first nine natural numbers if no number is repeated?

30. How many words can be formed with the letters of the word 'MATHEMATICS' if all the vowels are together?

31. Find the number of ways in which 5 identical marbles can be distributed among 10 identical boxes if not more than one marble can go into a box.

32. There are 10 points in a plane out of which 4 are collinear. Find the number of triangles.

33. A film-star has 12 friends and he wants to invite 9 of them at the wedding party. How many parties to each of 9 fellows can be given?

34. A box has 10 bananas. Then in how many ways 6 bananas can be selected if each banana may be repeated any numbers of times?

35. There are 10 red, 8 white and 6 green balls in a box. In how many ways one or more balls can be put in the box?

36. There are 5 different blue colour; 4 different yellow colour and three different golden colour then how many ways the same colour are different be selected?

37. In how many ways can 20 identical TV-sets be divided among 5 persons?

38. In how many ways can 20 books be arranged on a shelf so that particular 3 books shall be always together?

39. If 20 friends were invited for a party, in how many ways can they and the host be seated at a round table if there is no restriction?

40. How many different numbers of four digits can be formed with the digits 2, 3, 4, 5, 6, 7, none of the digits being repeated in any of the numbers so formed?

41. In how many ways the letters of the word "PETROL" be arranged?

42. How many numbers of four different digits each greater than 5,000 can be formed from the digits 2, 4, 5, 7, 8, 0?

43. In how many ways can the letters of the word 'DAUGHTER' be arranged so that the vowels may appear in the odd places?

44. How many numbers of six digits can be formed from the digits 1, 2, 3, 4, 5, 6 (no digit being repeated)? How many of these are not divisible by 5?

45. How many 5-digit numbers can be formed by taking the digits 5, 6, 0, 7, 8?

46. In how many ways can the letters of the word 'STRANGE' be arranged so that the vowels may appear in the odd places?

47. There are 6 books of mathematics, 6 of physics and 2 of chemistry. In how many ways can these be placed on a shelf if the books on the same subject are to be together?

48. In how many of the permutations of 10 different things, taken 4 at a time. Will one particular thing always occur?

49. Four men and three women are to be seated for a dinner such that no two women sit together and no two men sit together. In how many number of ways this can be arranged?

50. Find the value of ${}^{n-1}P_r + r \cdot \left({}^{n-1}P_{r-1}\right)$.

51. In how many different ways can 10 examination papers be arranged in a row so that the best and the worst papers may never come together?

52. How many numbers each lying between 100 and 1,000 can be formed with the digits 2,0,3,4,5 (no digit being repeated)? How many of these are odd?

53. In how many ways can the letters of the word 'ASSASSINATION' be arranged?

54. There are three copies of four different books each. In how many ways can they be arranged on a shelf?

55. There are 5 blue, 4 white and 3 red balls in a box. They are drawn one by one and arranged in a row. Assuming that all the 12 balls are drawn, determine the number of different arrangements.

56. How many 7-digit numbers can be formed using the digits 1, 2, 0, 2, 4, 2 and 4?

57. How many arrangements can be made out of the letters of the word COMMITTEE taken all at a time, such that the four vowels do not come together?

58. How many numbers each lying between 9 and 1,000 can be formed with the digit 0, 1, 2, 3, 7, 8 (numbers can be repeated)?

59. How many natural numbers not exceeding 4 then 4,321 can be formed with the digit 1, 2, 3, 4? (repetition are allowed)

60. In how many ways can 7 boys form a ring?

61. In how many ways can 6 different beads be strung into a necklace?

62. Simplify ${}^{n}C_r + {}^{n-1}C_{r-1} + {}^{n-1}C_{r-2}$

63. How many different committees of 6 members may be formed from 7 men and 5 women?

64. A man has 6 friends. In how many ways can he invite one or more of them to the dinner?

65. If ${}^{n}C_x = 56$ and ${}^{n}P_x = 336$ then find the value of n and x

66. In an election a voter may vote for any number of candidates not greater than the number to be chosen. There are 7 candidates and 4 members are to be chosen. In how many ways can a person vote?

67. Find the number of diagonals that can be formed by joining the vertices of a hexagon?

68. A box contains 7 red, 6 white and 4 blue balls. How many selections of three balls can be made so that there is one ball of each colour?

69. In how many ways 5 members forming a committee out of 10 be selected so that 2 particular members must be included?

70. In how many ways can a committee of 5 members be selected from 6 men and 5 women consisting of 3 men and 2 women?

71. A committee of 7 is to be chosen from 13 students of whom 6 are science students and 7 are arts students. In how many ways can the selection be made so as to retain a majority on the committee for arts students?

72. A cricket team of 11 players is to be formed from 16 players including 4 bowlers and 2 wicket keepers. In how many ways can a team be formed so that the team consists of at least 3 bowlers and at least one wicket keeper?

73. Out of 3 books of mathematics, 4 books of physics and 5 books of political science, how many collections can be made if each collection consists of at least one book on each subject?

74. A committee of 3 experts is to be selected out of a panel of 7 persons. Three of them are lawyers, three of them are C.As. and one is both C.A. and lawyer. In how many ways can the committee be selected if it must have at least a C.A. and a lawyer?

75. How many numbers of different words can be formed from 12 consonants and 5 vowels by taking 4 consonants and 3 vowels in each word?

76. How many ways 4 letters from the word EXAMINATION

77. In how many ways can 3 ladies and 3 gentlemen be seated at a round table so that any two of gentlemen and only two of ladies sit together?

78. Four letters are written and four envelopes are addressed. In how many ways can all the letters be placed in the wrong envelopes?

79. In how many ways 12 students may be equally divided into three groups?

80. Find the number of ways in which 12 mangoes may be equally divided among 3 boys?

81. Five balls of different colours are to be placed in three boxes of different sizes. Each box can hold all the five balls. In how many ways can we place the balls so that no box remains empty?

82. Find the simplification of $\frac{(n-r+1)}{r}\ {}^{n}C_{r-1}$

83. If all the permutation of the word AGAIN are arranged in a dictionary, what is the fifteenth word?

84. In how many ways can 5 things be divided between two persons?

85. Find out the number of ways in which 52 playing cards be placed in 4 heaps of 13 cards each is given?

ANSWERS

(1) 4
(2) 4
(3) 4
(4) 5
(5) 41
(6) $n \cdot n!$
(7) 5
(8) 3
(9) 8
(10) ${}^{n}C_{r}$
(11) ${}^{14}C_{5}$
(12) ${}^{n+1}C_{r}$
(13) ${}^{n}P_{r}$
(14) $2^{n} - 1$
(15) 715
(16) 499
(17) 8
(18) 5
(19) 7

(20) 3
(21) 729
(22) 1,023
(23) 72
(24) $\dfrac{11!\cdot 12!}{5!}$
(25) 8,640
(26) $12!\times 7!$
(27) $5!\times 4!$
(28) 91
(29) 819
(30) $\dfrac{8!\cdot 4!}{(2!)^3}$
(31) 210
(32) 116
(33) 220
(34) 5,005
(35) 692
(36) $2^{12}-1$
(37) 10,626
(38) $19!\times 3!$
(39) $20!$
(40) 360
(41) 720
(42) 180
(43) 4,320
(44) 720, 600
(45) 96
(46) 1,440
(47) 51,840
(48) 2,016
(49) 144
(50) ${}^nP_r$
(51) 29,03,040
(52) 48, 18
(53) 1,08,10,800
(54) $\dfrac{12!}{(3!)^4}$
(55) 27,720
(56) 360
(57) 43,200
(58) 210
(59) 229
(60) 720
(61) 60
(62) ${}^nC_r$
(63) 924
(64) 63
(65) 8, 3
(66) 98
(67) 9
(68) 168
(69) 56
(70) 200
(71) 1,057
(72) 2,472
(73) 31
(74) 33
(75) 2,49,48,000
(76) 136
(77) 72
(78) 9
(79) 5,775
(80) 34,650
(81) 150
(82) ${}^nC_r$
(83) NAAGI
(84) 32
(85) $\dfrac{52!}{4!\times(13!)^4}$

24

Binomial Expansion

LEARNING OBJECTIVES

After studying this chapter, the student will be able to understand:

- Binomial expansion
- Binomial theorem
- Characteristics of binomial expansion
- General term of binomial expansion
- Illustrations
- Exercises

INTRODUCTION

The expression $x + y$, $2x + y$, $3x - 4y$, etc. are known as binomial expressions. Such expressions are represented as the sum or difference of two terms. When such binomial expressions are given with some index, the method of their expansions is known to us to some extent. For example,

$$(x + b)^1 = x + b$$
$$(x + b)^2 = x^2 + 2bx + b^2$$
$$(x + b)^3 = x^3 + 3x^2b + 3xb^2 + b^3$$
$$(x + b)^4 = x^4 + 4x^3b + 6x^2b^2 + 4xb^3 + b^4$$

These types of expansions are known as binomial expressions. If the index of binomial expression is 2, 3 or 4 the expansion is simple.

In mathematics we have binomial theorem to obtain the expansion of binomial expression $(x + b)$ with positive integer power n.

We saw that

$$\begin{aligned}(x + b)^3 &= x^3 + 3x^2b + 3xb^2 + b^3\\ &= 3c_0x^3 + 3c_1x^2b + 3c_2xb^2 + 3c_3b^3\end{aligned}$$

Similarly

$$(x + b)^5 = 5c_0x^5 + 5c_1x^4b + 5c_2x^3b^2 + 5c_3x^2b^3 + 5c_4x^1b^4 + 5c_5b^5$$

Following the same pattern we can easily expand the expansion of $(x + a)^n$ as follows:

$$(x + a)^n = nc_0x^n + nc_1x^{x-1}a + nc_2x^{n-2}a^2 + nc_3x^{n-3}a^3 + \ldots + nc_nx^{n-n}a^n$$

This expression is known as binomial expansion of binomial theorem.

BINOMIAL THEOREM

For any positive integral value

$$(x+a)^n = nc_0x^n + nc_1x^{n-1}a + nc_2x^{n-2}a^2 + \ldots + nc_na^n$$

Proof

We can prove this theorem by using principle of mathematical Induction.
First we shall verify the theorem for $n = 1$

$$\text{L.H.S.} = (x+a)^1 \qquad \text{R.H.S.} = 1c_0x^1 + 1c_1a^1$$

$$= x + a \qquad = x + a$$

$\therefore$ L.H.S. = R.H.S.

$\therefore$ The result is true for some positive integral value of k or n.

$$\therefore (x+a)^k = kc_0x^k + kc_1x^{k-1}a + kc_2x^{k-2}a^2 + \ldots + kc_ka^k$$

Now multiplying both sides by x and a and adding the two results

$$x(x+a)^k = kc_0x^{k+1} + kc_1xa + kc_2x^{k-1}a^2 + \ldots + kc_kxa^k$$

and

$$a(x+a)^k = kc_0x^ka + kc_1x^{k-1}a^2 + kc_2x^{k-2}a^3 + \ldots + kc_kxa^{k+1}$$

$$\therefore (x+a)^{k+1} = kc_0x^{k+1} + (kc_1 + kc_0)x^ka + (kc_2 + kc_1)x^{k-1}a^2 +$$

$$(kc_3 + kc_2)x^{k-2}a^3 + \ldots + kc_ka^{k+1}$$

By using ${}^nc_r + {}^nc_{r-1} = {}^{n+1}c_r$ we can say that

$$(x+a)^{k+1} = {}^kc_0x^{k+1} + {}^{(k+1)}c_1x^ka + {}^{(k+2)}c_2x^{k-1}a^2 + \ldots + {}^kc_ka^{k+1}$$

$\therefore$ We can say that $(x+a)^{k+1}$

$$= {}^{k+1}c_0x^{k+1} + {}^{k+1}c_1x^{(k+1)-1}a + {}^{k+1}c_2x^{(k+1)-2}a^2 + \ldots + {}^{k+k}c_{k+k}a^{k+1}$$

Thus it is seen that the given result is also true for $n = k + 1$

Thus on the basis of the principle of mathematical induction it can be said that the given result is true for all positive integral values of n.

$$\therefore (x+a)^n = {}^nc_0x^n + {}^nc_1x^{n-1}a + {}^nc_2x^{n-2}a^2 + {}^nc_3x^{n-3}a^3 + {}^nc_na^n$$

Characteristics of Binomial Expansion

We observe the following features in the binomial expansion.

(1) In all there are $(n+1)$ terms in the expansion.
(2) The coefficient of the terms are ${}^nc_0, {}^nc_1, {}^nc_2, \ldots {}^nc_n$ respectively.
(3) In each term of the expansion the sum of the powers of x and a is n.
(4) The first term is ${}^nc_0x^n = x^n$ and the last term is ${}^nc_0a^n = a^n$.

(5) The power of x in various terms decreases by 1 and the power of a increases by 1.

(6) The coefficients of the first and the last terms are 1 each and the coefficients of terms at equal distance from the middle term are equal. The coefficient of binomial expansion can also be obtained from the following Pascal's triangle.

Pascal's Triangle for Binominal Coefficient

Power	Coefficients
1	1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
7	1 7 21 35 35 21 7 1

GENERAL TERM OF BINOMIAL EXPANSION

We have derived binomial expansion as

$$(x+a)^n = {}^nc_0x^n + {}^nc_1x^{n-1}a + {}^nc_2x^{n-2}a^2 + \ldots {}^nc_na^n$$

Here $T_1 = {}^nc_0x^n$

$T_2 = {}^nc_2x^{n-1}a$

$T_3 = {}^nc_2x^{n-2}a^2$

$\therefore$ The $(r+1)$ th term of the binomial expansion can be written as follows:

$$T_{r+1} = {}^nc_rx^{n-r}\,a^r$$

The $(r+1)$ th term of binomial expansion is called the general term of expansion.

ILLUSTRATIONS

Illustration 1 Expand $(x+2y)^5$

Solution

$$(x+2y)^5$$

$$= {}^5c_0x^5 + {}^5c_1x^4(2y) + {}^5c_2x^3(2y)^2 + {}^5C_3x^2(2y)^3 + {}^5C_4(x)(2y)^4 + {}^5C_5(2y)^5$$

$$= x^5 + 5x^4(2y) + 10x^3(4y^2) + 10x^2(8y^3) + 5(16y^4) + 32y^5$$

$$= x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5$$

Illustration 2 Expand $\left(\frac{a}{b} - \frac{b}{a}\right)^4$

Solution

$$= {}^4C_0\left(\frac{a}{b}\right)^4 - {}^4C_1\left(\frac{a}{b}\right)^3\left(\frac{b}{a}\right) + {}^4C_2\left(\frac{a}{b}\right)^2\left(\frac{b}{a}\right)^2 - {}^4C_3\left(\frac{a}{b}\right)^3\left(\frac{b}{a}\right)^3 + {}^4C_4\left(\frac{b}{a}\right)^4$$

$$= \frac{a^4}{b^4} - 4\frac{a^3}{b^3}\cdot\frac{b}{a} + 6\left(\frac{a^2}{b^2}\right)\left(\frac{b^2}{a^2}\right) - 4\left(\frac{a}{b}\right)\left(\frac{b^3}{a^3}\right) + \frac{b^4}{a^4}$$

$$= \frac{a^4}{b^4} - \frac{4a^2}{b^2} + 6 - \frac{4b^2}{a^2} + \frac{b^4}{a^4}$$

Illustration 3 Expand $\left(2a - \frac{1}{a}\right)^5$

Solution

$$\left(2a - \frac{1}{a}\right)^5$$

$$= {}^5c_0(2a)^5 - {}^5c_1(2a)^4\left(\frac{1}{a}\right) + {}^5c_2(2a)^3\left(\frac{1}{a}\right)^2 - {}^5C_3(2a)^2\left(\frac{1}{a}\right)^3 + {}^5C_4(2a)\left(\frac{1}{a}\right)^4 - {}^5C_5\left(\frac{1}{a}\right)^5$$

$$= 32a^5 - 5\left(16a^4\right)\left(\frac{1}{a}\right) + 10\left(8a^3\right)\left(\frac{1}{a^2}\right) - 10\left(4a^2\right)\left(\frac{1}{a^3}\right) + 5(2a)\left(\frac{1}{a^4}\right) - \frac{1}{a^5}$$

$$= 32a^5 - 80a^3 + 80a - \frac{40}{a} + \frac{10}{a^3} - \frac{1}{a^5}$$

Illustration 4 Find the value of $(101)^5$

Solution

$(101)^5$

$= (101)^5$

$= {}^5c_0(100)^5 + {}^5c_1(10)^4\cdot(1) + {}^5c_2(100)^3(1)^2 + {}^5C_3(100)^2(1)^3 + {}^5C_4(100)(1)^4 + {}^5C_5(1)^5$

$= 1(10{,}00{,}00{,}00{,}000) + 5(10{,}00{,}00{,}000) + 10(10{,}00{,}000) + 10(10{,}000) + 5(100) + 1$

= 10,00,00,00,000 + 50,00,00,000 + 1,00,00,000 + 1,00,000 + 500 + 1

= 1,05,10,10,501

Illustration 5 Find the value of $(9.9)^5$ by using binomial expansion.

Solution

$(9.9)^5$

$= (10 - 0.1)^5$

$$= {}^5c_0(10)^5 - {}^5c_1(10)^4(0.1) + {}^5c_2(10)^3(0.1)^2 - {}^5C_3(10)^2(0.1)^3 + {}^5C_4(10)(0.1)^4 -$$
$$\quad {}^5C_5(0.1)^5$$
$$= (1{,}00{,}000) - 5(10{,}000)(0.1) + 10(1{,}000)(0.01) - 10(100)(0.001) -$$
$$\quad 5(10)(0.0001) - 1(0.00001)$$
$$= (1{,}00{,}000) - (50{,}000)(0.1) + (10{,}000)(0.01) - (1{,}000)(0.001) +$$
$$\quad 50(0.0001) - (0.00001)$$
$$= 1{,}00{,}000 - 50{,}00 + 100 - 1 + 0.005 - 0.00001$$
$$= 95{,}099.00499$$

Illustration 6 Evaluate $(19)^4$

Solution

$$(19)^4 = (20-1)^4$$
$$= {}^4c_0(20)^4 - {}^4c_1(20)^3(1) + {}^4c_2(20)^2(1)^2 - {}^4C_3(20)^1(1)^3 + {}^4C_4(1)^4$$
$$= (1{,}60{,}000) - 4(8{,}000) + 6(400) - 4(20) - 1$$
$$= 16{,}00{,}000 - 32{,}000 + 2{,}400 - 80 - 1$$
$$= 1{,}30{,}321$$

Illustration 7 Evaluate $\left(\sqrt{3}+1\right)^6 + \left(\sqrt{3}-1\right)^6$

Solution

$$\left(\sqrt{3}+1\right)^6$$
$$= {}^6c_0\left(\sqrt{3}\right)^6 + {}^6c_1\left(\sqrt{3}\right)^5(1) + {}^6c_2\left(\sqrt{3}\right)^4(1)^2 + {}^6c_3\left(\sqrt{3}\right)^3(1)^3 + {}^6c_4\left(\sqrt{3}\right)^2(1)^4 +$$
$$\quad {}^6c_5\left(\sqrt{3}\right)(1)^5 + {}^6c_6(1)^6$$
$$= 1(27) + 6\left(9\sqrt{3}\right) + 15(9) + 20\left(3\sqrt{3}\right) + 15(3) + 6\sqrt{3} + 1$$
$$= 27 + 54\sqrt{3} + 135 + 60\sqrt{3} + 45 + 6\sqrt{3} + 1 \qquad (1)$$

Similarly we can say that

$$\left(\sqrt{3}-1\right)^6 = 27 - 54\sqrt{3} + 135 - 60\sqrt{3} + 45 - 6\sqrt{3} + 1 \qquad (2)$$

Now adding results of Eqs. (1) and (2) we get

$$\left(\sqrt{3}+1\right)^6 + \left(\sqrt{3}-1\right)^6$$
$$= 2(27 + 135 + 45 + 1)$$
$$= 2(208)$$
$$= 416$$

Illustration 8 Evaluate $\left(\sqrt{5}+1\right)^5 + \left(\sqrt{5}-1\right)^5$

Solution

$$\left(\sqrt{5}+1\right)^5$$

$$= {}^5c_0\left(\sqrt{5}\right)^5 + {}^5c_1\left(\sqrt{5}\right)^4(1) + {}^5c_2\left(\sqrt{5}\right)^3(1)^2 + {}^5c_3\left(\sqrt{5}\right)^2(1)^3 + {}^5c_4\left(\sqrt{5}\right)(1)^4 + {}^5c_5(1)^5$$

$$= 25\sqrt{5} + 5(25) + 10\left(5\sqrt{5}\right) + 10(5) + 5\left(\sqrt{5}\right) + 1$$

$$= 25\sqrt{5} + 125 + 50\sqrt{5} + 50 + 5\sqrt{5} + 1 \qquad (1)$$

Similarly we can say that

$$\left(\sqrt{5}-1\right)^5 = 25\sqrt{5} - 125 + 50\sqrt{5} - 50 + 5\sqrt{5} - 1 \qquad (2)$$

Now subtracting result of Eq. (2) from Eq. (1) we get

$$\left(\sqrt{5}+1\right)^5 - \left(\sqrt{5}-1\right)^5$$

$$= 2(125 + 50 + 1)$$

$$= 2(176)$$

$$= 352$$

Illustration 9 Obtain 6th term of the expansion of $(2x+y)^9$

Solution

In the expansion of $(2x+y)^9$

$$T_{r+1} = {}^nc_r x^{n-r} a^r$$

$$\therefore T_{r+1} = {}^9c_r (2x)^{9-r}(y)^r$$

$$\therefore T_6 = {}^9c_5 (2x)^{9-5}(y)^5 \qquad \text{For the sixth term put } r = 5$$

$$= 126(16x)^4(y)^5$$

$$= 2{,}016x^4y^5$$

Illustration 10 Obtain the fifth term in the expansion of $\left(\frac{5a}{4}+\frac{4}{5a}\right)^{12}$

Solution

In the expansion of $\left(\frac{5a}{4}+\frac{4}{5a}\right)^{12}$

$$T_{r+1} = {}^{12}c_r\left(\frac{5a}{4}\right)^{12-r}\left(\frac{4}{5a}\right)^r$$

$$\therefore T_5 = {}^{12}c_4\left(\frac{5a}{4}\right)^{12-4}\left(\frac{4}{5a}\right)^4 \qquad \text{For the fifth term put } r = 4$$

$$= 495\,\frac{5^8a^8}{4^8}\,\frac{4^4}{5^4a^4}$$

$$= \frac{495\times 5^{8-4}}{4^{8-4}}a^{8-4}$$

$$= \frac{495\times 5^4}{4^4}a^4$$

$$= \frac{(495)(625)}{256}a^4$$

$$= \frac{3{,}09{,}375}{256}a^4$$

Illustration 11 Obtain the fourth term in the expansion of $\left(\frac{2x}{3} - \frac{3}{2x}\right)^9$

Solution

In the expansion of $\left(\frac{2x}{3} - \frac{3}{2x}\right)^9$ we have

$$T_{r+1} = {}^nc_r x^{n-r} a^r$$

$$\therefore T_{r+1} = {}^9c_r \left(\frac{2x}{3}\right)^{9-r} \left(\frac{-3}{2x}\right)^r$$

$$\therefore T_4 = {}^9c_3 \left(\frac{2x}{3}\right)^{9-3} \left(\frac{-3}{2x}\right)^3 \qquad \text{For the fourth term put } r = 3$$

$$= 84 \frac{2^6 \times x^6}{3^6} \frac{(-1)^3 \times 3^3}{2^3 \times x^3}$$

$$= -\frac{84 \times 2^{6-3}}{3^{6-3}} x^3$$

$$= -\frac{84(8)}{27} x^3$$

$$= -\frac{672}{27} x^3$$

Illustration 12 Obtain the middle term in the expansion of $\left(\frac{a}{x} - \frac{x}{a}\right)^{10}$

Solution

In the expansion of $\left(\frac{a}{x} - \frac{x}{a}\right)^{10}$ there will be 10 + 1 = 11 terms in the expansion

$$\therefore \frac{10}{2} + 1 = \text{Sixth term will be the middle term}$$

$$\text{Now } T_{r+1} = {}^{10}c_r \left(\frac{a}{x}\right)^{10-r} \left(\frac{-x}{a}\right)^r$$

$$\therefore T_6 = {}^{10}c_5 \left(\frac{a}{x}\right)^{10-5} \cdot \left(\frac{-x}{a}\right)^5 \qquad \text{For the sixth term put } r = 5$$

$$= 252 \frac{a^5}{x^5} (-1)^5 \frac{x^5}{a^5}$$

$$= -252$$

Illustration 13 Obtain the middle term in the expansion of $\left(\frac{2x}{3}-\frac{3}{2y}\right)^9$

Solution

In the expansion of $\left(\frac{2x}{3}-\frac{3}{2y}\right)^9$ there will be 9 + 1 = 10 terms in the expansion

$\therefore$ 5th and 6th term will be the two middle terms

Now $\quad T_{r+1} = {}^nc_r\, x^{n-r} a^r$

Put $r = 4$

$$T_5 = {}^9c_4\left(\frac{2x}{3}\right)^{9-4}\left(\frac{-3}{2y}\right)^4$$

$$= 126\frac{2^5 x^5}{3^5}\frac{(-1)^4 3^4}{2^4 y^4}$$

$$= \frac{126\times 2^{5-4}}{3^{5-4}}\frac{x^5}{y^4}$$

$$= -\frac{126(2)}{3}\frac{x^5}{y^4}$$

$$= \frac{84x^5}{y^4}$$

Put $r = 5$

$$T_6 = {}^9c_5\left(\frac{2x}{3}\right)^{9-5}\left(\frac{-3}{2y}\right)^5$$

$$= 126\frac{2^4 x^4}{3^4}\frac{(-1)^5 3^5}{2^5 y^5}$$

$$= -\frac{126\times 3^{5-4}}{2^{5-4}}\frac{x^4}{y^5}$$

$$= -\frac{126(3)}{2}\frac{x^4}{y^5}$$

$$= \frac{-189x^4}{y^5}$$

Illustration 14 Obtain the coefficient of x^2 in the expansion of $\left(\frac{x}{2}+\frac{2}{x}\right)^8$

Solution

In the expansion of $\left(\frac{x}{2}+\frac{2}{x}\right)^8$

$$T_{r+1} = {}^nc_r x^{n-r} a^r$$

$$\therefore T_{r+1} = {}^8c_r\left(\frac{x}{2}\right)^{8-r}\left(\frac{2}{x}\right)^r$$

$$= {}^8c_r\frac{x^{8-r}}{2^{8-r}}\frac{2^r}{x^r}$$

$$= \frac{{}^8c_r}{2^{8-2r}}x^{8-2r}$$

For the coefficient of x^2

$$x^{8-2r} = x^2$$

$$\therefore\ 8 - 2r = 2$$

$$\therefore\ 6 = 2r$$

$$\therefore\ r = 3$$

$$\therefore T_4 = {}^8c_3\frac{x^2}{2^2}$$

$$= \frac{56}{4}x^2$$

$$= 14x^2$$

$\therefore$ Coefficient of x^2 = 14

Illustration 15 Obtain the coefficient of x in the expansion of $\left(2x - \frac{1}{x}\right)^5$

Solution

In the expansion of $\left(2x - \frac{1}{x}\right)^5$ we have

$$T_{r+1} = {}^nc_r x^{n-r} a^r$$

$$= {}^5c_r \left(2x\right)^{5-r} \left(\frac{-1}{x}\right)^r$$

$$= {}^5c_r \left(2\right)^{5-r} x^{5-r} \left(-1\right)^r \frac{1}{x^r}$$

$$T_{r+1} = \left[{}^5c_r 2^{5-r} \left(-1\right)^r\right] x^{5-2r}$$

$$\therefore T_3 = {}^5c_3 2^{5-2} \left(-1\right)^2 x$$

$$= 10\left(8\right)\left(1\right)x$$

$$= 80x$$

For the coefficient of x

$$x^{5-2r} = x$$

$$\therefore\ 5 - 2r = 1$$

$$\therefore\ 4 = 2r$$

$$\therefore\ r = 2$$

$\therefore$ Coefficient of $x = 80$

Illustration 16 Obtain the coefficient of x^{-8} in the expansion of $\left(2x^2 - \frac{1}{3x}\right)^{14}$

Solution

In the expansion of $\left(2x^2 - \frac{1}{3x}\right)^{14}$ we have

$$T_{r+1} = {}^nc_r x^{n-r} a^r$$

$$\therefore T_{r+1} = {}^{14}c_r \left(2x^2\right)^{14-r} \left(\frac{-1}{3x}\right)^r$$

$$= {}^{14}c_r 2^{14-r} x^{28-2r} \frac{\left(-1\right)^r}{3^r x^r}$$

$$\therefore T_{r+1} = \frac{\left[{}^{14}c_r 2^{14-r} \left(-1\right)^r\right]}{3^r} x^{28-2r-r}$$

$$\therefore T_{r+1} = \left[\frac{{}^{14}c_r 2^{14-r} \left(-1\right)^r}{3^r}\right] x^{28-3r}$$

$$\therefore T_{13} = \left[\frac{{}^{14}c_{12} 2^{14-12} \left(-1\right)^{12}}{3^{12}}\right] x^{-8}$$

$$= \left(\frac{{}^{14}c_{12} 2^2}{3^{12}}\right) x^{-8}$$

For the coefficient of x^{-8}

$$x^{28-3r} = x^{-8}$$

$$28 - 3r = -8$$

$$36 = 3r$$

$$r = 12$$

$$= \frac{91 \times 4}{3^{12}} \cdot x^{-8}$$

$$= \frac{364}{3^{12}} x^{-8}$$

Coefficient of x^{-8} is $\frac{364}{3^{12}}$

Illustration 17 Find the constant term in the expansion of $\left(x + \frac{2}{x}\right)^4$

Solution

In the expansion of $\left(x + \frac{2}{x}\right)^4$ we have

$$T_{r+1} = {}^{n}c_r x^{n-r} a^r$$

$$\therefore T_{r+1} = {}^{4}c_r (x)^{4-r} \left(\frac{2}{x}\right)^r$$

$$= {}^{4}c_r x^{4-r} \frac{2^r}{x^r}$$

$$\therefore T_{r+1} = \left({}^{4}c_r 2^r\right) x^{4-2r}$$

$$\therefore T_3 = \left({}^{4}c_r 2^2\right) x^0$$

$$= (6 \times 4) 1$$

$$= 24$$

For the constant term

$$x^{4-2r} = x^0$$

$$\therefore 4 - 2r = 0$$

$$\therefore 4 = 2r$$

$$\therefore r = 2$$

$\therefore$ The constant term = 24

Illustration 18 Find the constant term in the expansion of $\left(\frac{4x^2}{3} - \frac{3}{x}\right)^9$

Solution

In the expansion of $\left(\frac{4x^2}{3} - \frac{3}{x}\right)^9$ we have

$$\therefore T_{r+1} = {}^{n}c_r x^{n-r} a^r$$

$$\therefore T_{r+1} = {}^{9}c_r \left(\frac{4x^2}{3}\right)^{9-r} \left(\frac{-3}{x}\right)^r$$

$$= {}^{9}c_r \frac{4^{9-r} x^{18-2r}}{3^{9-r}} \frac{(-1)^r 3^r}{x^r}$$

$$\therefore T_{r+1} = \left[\frac{{}^{9}c_r 4^{9-r} (-1)^r 3^r}{3^{9-r}}\right] x^{18-3r}$$

$$\therefore T_7 = \left[\frac{{}^{9}c_6 4^{9-6} (-1)^6 3^6}{3^{9-6}}\right] x^0$$

$$\therefore T_7 = {}^{9}c_6 \times 4^3 \times 3^{6-3}$$

For the constant term $x^{18-3r} = 0$

$$\therefore 18 - 3r = 0$$

$$\therefore 18 = 3r$$

$$\therefore r = 6$$

$$= {}^9c_6 \times 4^3 \times 3^3$$
$$= {}^9c_6 (64)(27)$$
$$= (84)\,(64)\,(27)$$
$$= 1{,}45{,}152$$

Illustration 19 Prove that there is no constant term in the expansion of $\left(3x^2 - \frac{2}{x}\right)^7$

Solution

In the expansion of $\left(3x^2 - \frac{2}{x}\right)^7$ we have

$$T_{r+1} = {}^nc_r x^{n-r} a^r$$
$$= {}^7c_r (3x)^{7-r} \left(\frac{-2}{x}\right)^r$$
$$= {}^7c_r 3^{7-r} x^{14-2r} \frac{(-1)^r 2^r}{x^r}$$
$$= \left[{}^7c_r 3^{7-r} (-1)^r 2^r\right] x^{14-2r-r}$$
$$= \left[{}^7c_r \cdot 3^{7-r} \cdot (-1)^r \cdot 2^r\right] x^{14-3r}$$

$\therefore$ For the constant terms

$$x^{14-3r} = x^0$$
$$\therefore 14 - 3r = 0$$
$$\therefore\ 3r = 14$$
$$\therefore r = \frac{14}{3} \notin N$$

There is no constant term in the expansion of $\left(3x^2 - \frac{2}{x}\right)^7$

Illustration 20 If the middle term in the expansion of $\left(\frac{k}{2} + 2\right)^8$ is 1,120 then find the value of k.

Solution

In the expansion of $\left(\frac{k}{2} + 2\right)^8$ we have 8 + 1 = 9 terms. Then the 5th term is the middle term

$$T_{r+1} = {}^nc_r x^{n-r} a^r$$
$$T_5 = {}^8c_4 \left(\frac{k}{2}\right)^{8-4} \times (2)^4 \qquad \text{Put } r = 4$$
$$= {}^8c_4 \frac{k^4}{2^4} 2^4$$

$$= {}^{8}c_{4}k^{4}$$
$$= 70k^{4}$$
$$\therefore\ T5 = 1{,}120$$
$$70k^{4} = 1{,}120$$
$$\therefore k^{4} = \frac{1{,}120}{70}$$
$$\therefore\ k^{4} = 16$$
$$\therefore\ k^{4} = (2)^{4}$$
$$\therefore\ k = 2$$

Illustration 21 If the ratio of coefficient of rth term to $(r + 1)$th term in the expansion of $(1 + x)^{20}$ is 1 : 2, find the value of r.

Solution

In the expansion of $(1 + x)^{20}$

$$(r + 1)\text{ th term } T_{r+1} = {}^{20}c_{r}(1)^{20-r}\ x^{r}$$
$$= {}^{20}c_{r-1} \times x^{r}$$

$$\text{and } r\text{th term } T_{r} = {}^{20}c_{r-1}(1)^{20-(r-1)}\ x^{r-1}$$
$$= {}^{20}c_{r-1} \times x^{r}$$

$$\text{Now } \frac{\text{coefficient of } r\text{th term}}{\text{coefficient of } (r+1)\text{th term}} = \frac{1}{2}$$

$$\therefore \frac{{}^{20}c_{r-1}}{{}^{20}c_{r}} = \frac{1}{2}$$

$$\therefore \frac{20!/(20-r+1)!(r-1)!}{20!/(20-r)!r!} = \frac{1}{2}$$

$$\therefore \frac{20!}{(20-r+1)!(20-r)!(r-1)!}\ \frac{(20-r)!\,r(r-1)!}{20!} = \frac{1}{2}$$

$$\therefore \frac{r}{(20-r+1)} = \frac{1}{2}$$

$$\therefore \frac{r}{21-r} = \frac{1}{2}$$

$$\therefore\ 2r = 21 - r$$
$$\therefore\ 3r = 21$$
$$\therefore\ r = 7$$

Illustration 22 Prove that the sum of coefficient of x^{32} and x^{-17} in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$ is zero.

Solution

In the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$ we have

$$T_{r+1} = {}^{n}c_r \cdot x^{n-r} \times a^r$$

$$\therefore T_{r+1} = {}^{15}c_r \left(x^4\right)^{15-r} \left(\frac{-1}{x^3}\right)^r$$

$$= {}^{15}c_r x^{60-4r} \left(-1\right)^r \frac{1}{x^{3r}}$$

$$= \left(-1\right)^r \times {}^{15}c_r x^{60-7r}$$

$$\therefore T_5 = \left(-1\right)^4 \times {}^{15}c_4 x^{32}$$

$\therefore$ Coefficient of x^{32} is

${}^{15}c_4$ (1)

For the coefficient of x^{32}

$$x^{60-7r} = x^{32}$$

$$\therefore 60 - 7r = 32$$

$$\therefore 7r = 28$$

$$\therefore r = 4$$

Similarly for the coefficient of x^{-17}

$$\therefore T_{12} = \left(-1\right)^{11} \times {}^{15}c_{11} \left(x\right)^{-17}$$

$\therefore$ coefficient of $\left(x\right)^{-17}$ is

$-{}^{15}c_{11}$ (2)

$$x^{60-7r} = x^{-17}$$

$$60 - 7r = -17$$

$$\therefore 77 = 7r$$

$$\therefore r = 11$$

$\therefore$ The sum of both coefficients

$= {}^{15}c_4 - {}^{15}c_{11}$

$= {}^{15}c_4 - {}^{15}c_4 \qquad \left(\because 15c_{11} = 15c_4\right)$

$= 0$

Illustration 23 Find the value of $\left(0.49\right)^5$

Solution

$\left(0.49\right)^5$

$= \left(0.50 - 0.01\right)^5$

$= {}^{5}c_0 \left(0.5\right)^5 - {}^{5}c_1 \left(0.5\right)^4 \left(0.01\right) + {}^{5}c_2 \left(0.5\right)^3 \left(0.01\right)^2 - {}^{5}c_3 \left(0.5\right)^2 \left(0.01\right)^3 +$
${}^{5}c_4 \left(0.5\right)^1 \left(0.01\right)^4 - {}^{5}c_5 \left(0.5\right)^0 \left(0.01\right)^5$

$= 1\left(0.03125\right) - 5\left(0.0625\right)\left(0.01\right) + 10\left(0.125\right)\left(0.0001\right) -$
$10\left(0.25\right)\left(0.000001\right) + 5\left(0.5\right)\left(0.00000001\right) - 1\left(0.0000000001\right)$

$= (0.03125) - (0.3125)(0.01) + (0.125)(0.001) - (0.25)(0.00001) +$
$(0.5)(0.00000005) - (0.0000000001)$
$= 0.028247524$

Illustration 24 Evaluate $(1.05)^4$

Solution

$(1.05)^4$
$= (1 + 0.05)^4$
$= {}^4c_0(1)^4 + {}^4c_1(1)^3(0.05)^1 + {}^4c_2(1)^2(0.05)^2 + {}^4c_3(1)(0.05)^3 + {}^4c_4(0.05)^4$
$= 1 + 4(0.05) + 6(0.0025) + 4(0.000125) + 1(0.00000625)$
$= 1 + 0.20 + (0.0150) + (0.000500) + (0.00000625)$
$= 1.21550625$

Illustration 25 Find the coefficient of x^4 in the expansion of $(1 - x)^6(1 + 2x)^5$

Solution

$(1 - x)^6$
$= {}^6c_0 - {}^6c_1x + {}^6c_2x^2 - {}^6c_3x^3 + {}^6c_4x^4 - {}^6c_5x^5 + {}^6c_6x^6$
$= 1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$ (1)
and $(1 + 2x)^5$
$= {}^5c_0 + {}^5c_1(2x) + {}^5c_2(2x)^2 + {}^5c_3(2x)^3 + {}^5c_4(2x)^4 - {}^5c_5(2x)^5$
$= 1 + 5(2x) + 10(4x^2) + 10(8x^3) + 5(16x^4) + 1(32x^5)$
$= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$ (2)
From (1) and (2) we can say that
$(1 - x)^6(1 + 2x)^5$
Now sum of the coefficients of x^4
$= (1)(80x^4) + (-6x)(80x^3) + (15x^2)(40x^2) - (20x^3)(10x) + 15x^4$
$\therefore$ Coefficient of x^4
$= (80 - 480 + 600 - 200 + 15)$
$= 15$

ANALYTICAL EXERCISES

1. Expand $\left(2x + \dfrac{1}{x}\right)^4$
2. Expand $\left(1 - \dfrac{x^2}{2}\right)^5$
3. Expand $\left(2x - \dfrac{2}{x}\right)^5$
4. Expand $\left(3x + \dfrac{2}{y}\right)^5$

5. Evaluate $(11)^5$

6. Evaluate $(51)^5$

7. Evaluate $(10.1)^4$

8. Evaluate $(101)^4$

9. Evaluate $(1.05)^5$

10. Simplify $\left(2+\sqrt{5}\right)^5+\left(2-\sqrt{5}\right)^5$

11. Simplify $\left(\sqrt{3}+1\right)^5-\left(\sqrt{3}-1\right)^5$

12. Simplify $\left(\sqrt{2}+1\right)^6+\left(\sqrt{2}-1\right)^6$

13. Simplify $\left(2+\sqrt{3}\right)^7+\left(2-\sqrt{3}\right)^7$

14. Simplify $\left(\sqrt{5}+\sqrt{3}\right)^6+\left(\sqrt{5}-\sqrt{3}\right)^6$

15. Simplify $\left(\sqrt{5}+\sqrt{2}\right)^5-\left(\sqrt{5}-\sqrt{2}\right)^5$

16. Find the fourth term in the expansion of $\left(3x-\frac{1}{2}\right)^7$

17. Find the seventh term in the expansions of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^9$

18. Find the fourth term in the expansion of $\left(\frac{2x}{3}-\frac{3}{2x}\right)^6$

19. Find the third term in the expansion of $\left(x+\frac{1}{y}\right)^7$

20. Find the fourth term in the expansion of $(x-2y)^8$

21. Find the third term in the expansion of $\left(2x-\frac{5}{x}\right)^5$

22. Obtain the middle term of $\left(x^2-\frac{1}{x}\right)^6$

23. Obtain the middle term of $\left(\frac{x}{3}-\frac{3}{x}\right)^5$

24. Find the middle term of $\left(\frac{2x}{3}-\frac{3y}{2}\right)^9$

25. Obtain the middle term in the expansion of $\left(x-\frac{1}{x}\right)^{11}$

26. Obtain the middle term in the expansion of $\left(2x+\frac{3}{x}\right)^{20}$

27. Find the coefficient of x^7 in the expansion of $\left(x^2+\frac{1}{x}\right)^{11}$

28. Find the coefficient of x^{10} in the expansion of $\left(x^4 - \frac{1}{2x}\right)^{10}$

29. Find the coefficient of x in the expansion of $\left(2x - \frac{1}{3x}\right)^9$

30. Find the coefficient of x^{16} in the expansion of $\left(2x^2 - x\right)^{10}$

31. Obtain the coefficient of x^{-12} in the expansion of $\left(x - \frac{1}{x^3}\right)^{12}$

32. Obtain the coefficient of x^{18} in the expansion of $\left(ax^4 - bx\right)^9$

33. In the expansion of $\left(\frac{x}{3} - \frac{3}{x}\right)^{10}$ find the term containing x^4

34. Find the constant term in the expansion of $\left(3x^2 - \frac{1}{x}\right)^9$

35. Find the constant term in the expansion of $\left(2x + \frac{1}{3x^2}\right)^9$

36. Find the term independent of x in the expansion of $\left(x^3 + \frac{1}{x^8}\right)^{11}$

37. Obtain the constant term in the expansion of $\left(\frac{3x^2}{5} - \frac{1}{2x}\right)^9$

38. Find the coefficient of x^{-19} in the expansion of $\left(\sqrt{x} - \frac{1}{x^2}\right)^{12}$

39. Find the constant term in the expansion of $\left(\sqrt{x} - \frac{3}{x}\right)^{12}$

40. Prove that there is no constant term in the expansion of $\left(2x^2 - \frac{1}{4x}\right)^{11}$

41. If the coefficient of x in the expansion of $\left(x^2 + \frac{k}{x}\right)^5$ is 270, find the value of k.

42. If the coefficients of x^2 and x^3 in the expansion of $\left(3 + kx\right)^9$ are equal then find the value of k.

43. If the coefficients of $\left(2r + 1\right)$th and $\left(4r + 5\right)$th term in the expansion of $\left(1 + x\right)^{10}$ are equal then find the value of r.

44. The fifth term in the expansion of $\left(1 + x\right)^{11}$ is 24 times its third. Find the value of x.

45. If the sixth term in the expansion of $\left(1 + x\right)^{10}$ is $\frac{63}{8}$ find the value of x.

46. Prove that the middle term in the expansion of $\left(x + \frac{1}{2x}\right)^{2n}$ is

$$\frac{1 \times 3 \times 5 \times 7 \cdots \left(2n - 1\right)}{n!}$$

47. Find the middle term in the expansion of $\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^{10}$

48. Find the value of $\left(\sqrt{6}+\sqrt{2}\right)^4+\left(\sqrt{6}-\sqrt{2}\right)^4$

49. Find the coefficient of x^{-9} in the expansion of $\left(\frac{x}{4}-\frac{2}{x^2}\right)^9$

ANSWERS

(1) $16x^4+32x^2+24+\frac{8}{x^2}+\frac{1}{x^4}$

(2) $1-\frac{5x^2}{2}+\frac{5x^4}{2}-\frac{5x^6}{4}+\frac{5x^8}{16}-\frac{x^{10}}{32}$

(3) $32x^5-160x^3+320x-\frac{320}{x}+\frac{160}{x^3}-\frac{32}{x^5}$

(4) $243x^5+\frac{810x^4}{y}+\frac{1080x^3}{y^2}+\frac{720x^3}{y^3}+\frac{240x}{y^4}+\frac{32}{y^5}$

(5) 1,61,051

(6) 34,50,25,251

(7) 10,406.0401

(8) 10,40,60,401

(9) 1.0510100501

(10) 1,364

(11) 152

(12) 198

(13) 10,084

(14) 3,904

(15) $458\sqrt{2}$

(16) $-\frac{2,835}{8}x^4$

(17) $\frac{10,500}{x^3}$

(18) −20

(19) $\frac{21x^5}{y^2}$

(20) $-448x^5y^3$

(21) $2,000x$

(22) $-20x^3$

(23) $\frac{10x}{3};\frac{30}{x}$

(24) $89x^5y^4;\ -189x^{4y^5}$

(25) $-462x;\ \frac{462}{x}$

(26) ${}^{20}C_{10}\times 2^{10}\times 3^{10}$

(27) 462

(28) $\frac{105}{32}$

(29) $\frac{448}{9}$

(30) 13,440

(31) 924

(32) $84a^3b^6$

(33) $\frac{-120}{81}x^4$

(34) 2,268

(35) $\frac{1,792}{9}$

(36) 165

(37) $\frac{567}{2,000}$

(38) 66

(39) 40,095

(41) $k=3$

(42) $k=\frac{9}{7}$

(43) $r=1$

(44) $x=\pm 2$

(45) $x=\frac{1}{2}$

(47) −252

(48) 224

(49) 84

25

Principle of Mathematical Induction

LEARNING OBJECTIVES

After studying this chapter, the student will be able to understand:

- Sequence
- Series
- Principle of mathematical induction
- Power series

INTRODUCTION

Consider the set N of natural numbers. N has two characteristic properties.

(1) N contains the natural number 1.

(2) N is closed with respect to addition of 1 to each of its numbers.

Therefore to determine whether a set k consisting of natural numbers is the set of all natural numbers, we have to verify the following two conditions on K:

(1) Does $1 \in K$?

(2) For each natural number $K \in K$; is it true that $K \in K$?

When answer to both the questions is 'yes' then K is N. It gives several important principles for establishing the truth of certain classes of statements.

HISTORICAL INTRODUCTION

The discovery of the principle of mathematical induction is generally attributed to the French mathematician Blaise Pascal (1623–1662). However, the principal has been used by the Italian mathematician Francesco Maurolycus (1445–1575) in his writings. The writings of Bhaska racharya (1150 A.D.) also lead us to believe that he knew of this principle.

The first one to use the term 'induction' was the English mathematician John Walls (1616–1703). Later on the Swiss mathematician James Barnouli (1655–1705) used the principle to provide a proof of the binomial theorem about which we shall learn later in this book.

The term *mathematical induction* in the modern sense was used by English mathematician Augustus De' Morgan (1806–1871) in his article 'induction'

(Mathematics) in Penny Cyclopaedia, London 1938. The term gained immediate acceptance by the mathematical community and during the next fifty years or so it was universally accepted.

PRINCIPLE OF FINITE INDUCTION (PFI)

If we denote the given statement or formula by $P(n)$, for all positive integral values of n, then the proof of this statement with the help of the principle of mathematical induction consists of three steps.

Step 1: Verify that the result is true for the first available value of n, generally for $n = 1$, i.e. verify that $p(1)$ is true.

Step 2: Assume that the result is true for a positive integral value K of n,. i.e. assume that $p(k)$ is true.

Step 3: Now using the result that $p(k)$ is true, prove that the result is also true for $n = k + 1$, i.e. prove that $p(k+1)$ is also true.

Having followed the above three steps it can be said that theresult is proved with the help of principle of mathematical induction.

Let us take some illustrations to understand the principle of mathematical induction.

ILLUSTRATIONS

Illustration 1 For positive integer value of n, prove that
$1 + 3 + 5 + 7 + \dots (2n - 1) = n^2$

Solution

Here $p(n) = 1 + 3 + 5 + 7 + \dots (2n - 1) = n^2$

Let us verify the given result for $n = 1$

L.H.S. = 1 R.H.S. = $(1)^2 = 1$

$\therefore$ L.H.S. = R.H.S.

Now we shall assume that the result is true for $n = k$

i.e. we shall assume that $p(k)$ is true.

$\therefore\ p(k) = 1 + 3 + 5 + 7 + \dots (2k - 1) = k^2$

Now we shall prove that the result is true for $n = k + 1$

i.e. we shall prove that

$1 + 3 + 5 + 7 + \dots (2k - 1) + (2k + 1) = (k + 1)^2$

$$\begin{aligned} \text{L.H.S.} &= 1 + 3 + 5 + 7 + \dots (2k - 1) + (2k + 1) \\ &= [1 + 3 + 5 + 7 + \dots (2k - 1)] + (2k + 1) \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2 \end{aligned}$$

Thus it is proved that the result is also true for $n = k + 1$ i.e. if $p(k)$ is true, $p(k + 1)$ is also true.

Thus we can say that $p(n)$ is true for $n \in N$

Illustration 2 With the help of mathematical induction principle prove that

$$1.2 + 2.3 + 3.4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

Solution

Here $p(n) = 1.2 + 2.3 + 3.4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$

Verify that $p(1)$ is true.

L.H.S. $= 1.2$ $= 2$

R.H.S. $= \frac{1(1+1)(1+2)}{3}$ $= 2$

$\therefore$ L.H.S. = R.H.S.

Suppose that $p(k)$ is true

$$\therefore\ 1.2 + 2.3 + 3.4 + \ldots + k(k+1) = \frac{k(k+1)(k+2)}{3}$$

Now we shall prove that

$$p(k+1) = 1.2 + 2.3 + \ldots + k(k+1) + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

$$\therefore\ \text{R.H.S.} = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

$$= (k+1)(k+2)\left(\frac{k}{3}+1\right)$$

$$= (k+1)(k+2)\left(\frac{k+3}{3}\right)$$

$$= \frac{(k+1)(k+2)(k+3)}{3}$$

$$= \frac{(k+1)\left[k+(k+1)\right]\left[(k+1)+2\right]}{3}$$

$\therefore$ $p(k+1)$ is true.

$\therefore$ Statement $p(n)$ is true for $n \in N$

Illustration 3 With the help of mathematical induction principle prove that

$$2 + 5 + 8 + \ldots + (3n-1) = \frac{n(3n+1)}{2}$$

Solution

Here $p(n) = 2 + 5 + 8 + \ldots + (3n-1) = \frac{n(3n+1)}{2}$

Let us verify that $p(1)$ is true.

L.H.S. $= 2$ R.H.S. $= \frac{1(3+1)}{2} = 2$

$\therefore$ L.H.S. = R.H.S.

$\therefore$ $p(1)$ is true

Suppose that $p(k)$ is true

$$2+5+8+\ldots+(3k-1)=\frac{k(3k+1)}{2}$$

Now we shall prove that $p(k+1)$ is true i.e. we shall prove that

$$2+5+8+\ldots+(3k-1)+[3(k+1)-1]=\frac{(k+1)[3(k+1)+1]}{2}$$

$\therefore$ L.H.S.

$$=[2+5+8+\ldots+(3k-1)]+[3(k+1)-1]$$

$$=\frac{k(3k+1)}{2}+[3(k+1)-1]$$

$$=\frac{k(3k+1)+2[3(k+1)-1]}{2}$$

$$=\frac{k(3k+1)+2(3k+3-1)}{2}$$

$$=\frac{k(3k+1)+2(3k+2)}{2}$$

$$=\frac{3k^2+k+6k+4}{2}$$

$$=\frac{3k^2+7k+4}{2}$$

$$=\frac{(k+1)(3k+4)}{2}$$

$$=\frac{(k+1)[3(k+1)+1]}{2}$$

= R.H.S.

$p(k+1)$ is true

Statement is true for $n \in N$

Illustration 4 By using the principle of mathematical induction prove that

$$1.3+2.4+3.5+\ldots+n(n+2)=\frac{n(n+1)(2n+7)}{6}$$

Solution

Here $p(n)=1.3+2.4+3.5+\ldots+n(n+2)=\dfrac{n(n+1)(2n+7)}{6}$

Let us verify that $p(1)$ is true

L.H.S. = 1.3 R.H.S. = $\dfrac{1(1+1)(2+7)}{6}$

= 3 = 3

L.H.S. = R.H.S.

$p(1)$ is true verified.

Now suppose $p(k)$ is true

$$p(k) = 1.3 + 2.4 + 3.5 + \ldots + k(k+2) + (k+1)[(k+1)+2]$$

$$= \frac{(k+1)[(k+1)+1][2(k+1)+7]}{6}$$

$$\text{L.H.S.} = [1.3 + 2.4 + 3.5 + \ldots + k(k+2)] + (k+1)[(k+1)+2]$$

$$= \frac{k(k+1)(2k+7)}{6} + (k+1)[(k+1)+2]$$

$$= (k+1)\left[\frac{k(2k+7)}{6} + (k+3)\right]$$

$$= (k+1)\left[\frac{2k^2 + 7k + 6k + 18}{6}\right]$$

$$= \frac{(k+1)(2k^2 + 13k + 18)}{6}$$

$$= \frac{(k+1)(2k^2 + 4k + 9k + 18)}{6}$$

$$= \frac{(k+1)[2k(k+2) + 9(k+2)]}{6}$$

$$= \frac{(k+1)(k+2)(2k+9)}{6}$$

$$= \frac{(k+1)[(k+1)+1][2(k+1)+7]}{6}$$

= R.H.S.

$p(k+1)$ is true.

we can say that $p(n)$ is true for $n \in N$

Illustration 5 Using mathematical induction principle prove that

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}; n \in Na$$

Solution

Here $p(n) = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

First we shall verify that $p(1)$ is true.

$\text{L.H.S.} = \frac{1}{1.2}$ $\qquad$ $\text{R.H.S.} = \frac{1}{1+1}$

$= \frac{1}{1.2}$ $\qquad$ $= \frac{1}{1.2}$

$p(1)$ is true is verified.

We assume that $p(k)$ is true.

Now we shall prove that $p(k+1)$ is also true.

$$p(k+1) = \frac{1}{1.2} + \frac{1}{2.3} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)[(k+1)+1]} = \frac{k+1}{(k+1)+1}$$

$$\text{L.H.S.} = \left[\frac{1}{1.2} + \frac{1}{2.3} + \ldots + \frac{1}{k(k+1)}\right] + \frac{1}{(k+1)[(k+1)+1]}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)[(k+1)+1]}$$

$$= \frac{1}{k+1}\left[k + \frac{1}{k+2}\right]$$

$$= \frac{1}{k+1}\left(\frac{k^2+2k+1}{k+2}\right)$$

$$= \frac{1}{k+1}\,\frac{(k+1)^2}{(k+2)}$$

$$= \frac{k+1}{k+2}$$

$$= \frac{k+1}{(k+1)+1}$$

Thus we have proved that $p(k+1)$ is true.

Statement is true for $n \in N$.

Illustration 6 By mathematical induction principle prove that

$$\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \ldots + \frac{1}{(2n+1)(2n+3)} = \frac{n}{3(2n+3)}$$

Solution

$$p(n) = \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \ldots + \frac{1}{(2n+1)(2n+3)} = \frac{n}{3(2n+3)}$$

Let us verify that $p(1)$ is true.

$$\text{L.H.S.} = \frac{1}{3.5} \qquad \text{R.H.S.} = \frac{1}{3(2+3)}$$

$$= \frac{1}{15} \qquad = \frac{1}{15}$$

$p(1)$ is true is verified.

We assume that $p(k)$ is true.

$$\therefore \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2k+1)(2k+3)} = \frac{k}{3(2k+3)}$$

Now we shall prove that $p(k+1)$ is also true.

$$p(k+1) = \frac{1}{3.5}+\frac{1}{5.7}+......+\frac{1}{(2k+1)(2k+3)}+\frac{1}{[2(k+1)+1][2(k+1)+3]}$$

$$= \frac{k+1}{3[2(k+1)+3]}$$

L.H.S.

$$= \left[\frac{1}{3.5}+\frac{1}{5.7}+\ldots+\frac{1}{(2k+1)(2k+3)}\right]+\left[\frac{1}{[2(k+1)+1][2(k+1)+3]}\right]$$

$$= \frac{k}{3(2k+3)} + \frac{1}{[2(k+1)+1][2(k+1)+3]}$$

$$= \frac{k}{3(2k+3)} + \frac{1}{(2k+3)(2k+5)}$$

$$= \frac{1}{2k+3}\left(\frac{k}{3}+\frac{1}{2k+5}\right)$$

$$= \frac{1}{2k+3}\left[\frac{2k^2+5k+3}{3(2k+5)}\right]$$

$$= \frac{1}{2k+3}\left[\frac{(k+1)(2k+3)}{3(2k+5)}\right]$$

$$= \frac{k+1}{3(2k+5)}$$

$$= \frac{k+1}{3[2(k+1)+3]}$$

Thus we have proved that $p(k+1)$ is true.
We can say that $p(n)$ is true for $n \in N$.

Illustration 7 By using the principle of mathematical induction prove that

$$a + ar + ar^2 + \ . \ . \ . \ + ar^{n-1} = \frac{a(r^n-1)}{r-1}; \ n \in N$$

Solution

Here $p(n) = a + ar + ar^2 + \ . \ . \ . \ + ar^{n-1} = \dfrac{a(r^n-1)}{r-1}; \ n \in N$

Let us verify that $p(1)$ is true.

L.H.S. $= a$ R.H.S. $= \dfrac{a(r^1-1)}{r-1} = a$

We assume that $p(k)$ is true.

$$a + ar + ar^2 + \ldots + ar^{k-1} = \frac{a(r^k-1)}{r-1}$$

Now we shall prove that $p(k+1)$ is also true.

$$p(k+1) = a + ar + ar^2 + \ldots + ar^{k-1} + ar^k = \frac{a(r^{k+1}-1)}{r-1}$$

$$\text{L.H.S.} = (a + ar + ar^2 + \ldots + ar^{k-1}) + ar^k$$

$$= \frac{a(r^k-1)}{r-1} + ar^k$$

$$= \frac{a(r^k-1) + ar^k(r-1)}{r-1}$$

$$= \frac{ar^k - a + ar^{k+1} - ar^k}{r-1}$$

$$= \frac{a(r^{k+1}-1)}{r-1}$$

= R.H.S.

Thus we have proved that $p(k+1)$ is true.

We can say that statement is true for $n \in N$.

Illustration 8 With the help of principle of mathematical induction prove that $3^n - 1$ is divisible by 2; $n \in N$.

Solution

Here $p(n) = 3^n - 1$

Putting $n = 1$

$$p(1) = 3 - 1$$
$$= 2$$
$$= 2 \times 1$$

which is divisible by 2

Thus $p(1)$ is true is verified

Now we assume that $p(k)$ is true

i.e. $3^k - 1$ is divisible by 2

$3^k - 1 = 2m;\ n \in N$

Now we shall prove that

$p(k+1)$ is true

$$p(k+1) = 3^{k+1} - 1$$
$$= 3^k 3^1 - 1$$
$$= (2m+1)3 - 1$$
$$= 6m + 3 - 1$$
$$= 6m + 2$$
$$= 2(3m+1)$$
$$= 2(\text{a positive integer})$$

$3^{k+1} - 1$ is divisible by 2

We can say that $p(k+1)$ is true.

Statement $p(n)$ is true for $n \in N$.

Illustration 9 With the help of mathematical induction principle prove that $7^{2n} + 16n - 1$ is divisible by 64.

Solution

Here $p(n) = 7^{2n} + 16n - 1$
First let us verify that $p(1)$ is true.
Substitute $n = 1$ in $p(n)$

$$
\begin{aligned}
p(1) &= 7^2 + 16 - 1 \\
&= 49 + 16 - 1 \\
&= 64 \times 1
\end{aligned}
$$

$p(1)$ is true.
We assume that $p(k)$ is true.
$p(k) = 7^{2k} + 16k - 1 = 64m;\ m \in n$
Now we shall prove that $p(k+1)$ is true.

$$
\begin{aligned}
p(k+1) &= 7^{2(k+1)} + 16(k+1) - 1 \\
&= 7^{2k}\,49 + 16k + 16 - 1 \\
&= (64k - 16m + 1)49 + 16k + 15 \\
&= 64(49k) - 768m + 64 \\
&= 64(49k - 12m + 1) \\
&= 64 \text{ (a positive integer)}
\end{aligned}
$$

Thus it is proved that $p(k+1)$ is true.
The statement is true for $n \in N$.

Illustration 10 By using principle of mathematical induction check that $2 + 6 + 10 + 14 + \ldots + 2(2n-1) = 2n^2 + 1;\ n \in N$

Solution

$p(n) = 2 + 6 + 10 + 14 + \ldots + 2(2n-1) = 2n^2 + 1$

Let L.H.S.	R.H.S.
$= 2(2-1)$	$= 2(1)^2 + 1$
$= 2$	$= 3$

L.H.S. R.H.S.
It is clear that $p(1)$ is not true.
Statement $p(1)$ is not true for $n \in N$.

Illustration 11 By using principle of mathematical induction prove that $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$.

Solution

$p(n) = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
Let us verify that $p(1)^2$ is true.

L.H.S. $= 1$ R.H.S. $= \frac{1(1+1)}{2} = 1$

L.H.S. = R.H.S.

$p(1)$ is true.

Suppose $p(k)$ is true

i.e. $1 + 2 + 3 + \ldots + k = \dfrac{k(k+1)}{2}$

Now we shall prove that $p(k+1)$ is true.

$$p(k+1) = 1 + 2 + 3 + \ldots + k + (k+1) = \frac{(k+1)[(k+1)+1]}{2}$$

$$\begin{aligned}
\text{L.H.S.} &= 1 + 2 + 3 + \ldots + k + (k+1) \\
&= (1 + 2 + 3 + \ldots + k) + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \\
&= (k+1)\left(\frac{k}{2} + 1\right) \\
&= \frac{(k+1)(k+2)}{2} \\
&= \frac{(k+1)[(k+1)+1]}{2} \\
&= \text{R.H.S.}
\end{aligned}$$

$p(k+1)$ is true.

Statement $p(n)$ is true for $n \in N$.

POWER SERIES

$$1 + 2 + 3 + \ldots + n = \sum n = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \sum n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\sum n\right)^2 = \frac{n^2(n+1)^2}{4} = \left(\sum n^3\right)$$

Illustration 12 By using mathematical induction law prove that

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \sum n^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution

Here $p(n) = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \sum n^2 = \dfrac{n(n+1)(2n+1)}{6}$

We want to verify that $p(1)$ is true.

L.H.S. $p(n) = n^2$ R.H.S. $= p(n) = \dfrac{n(n+1)(2n+1)}{6}$

$p(1) = 1^2 = 1$ $p(1) = \dfrac{1(1+1)(2+1)}{6} = 1$

$p(1)$ is true

Suppose that $p(k)$ is true.

i.e. $1^2 + 2^2 + 3^2 + \ldots + k^2 = \dfrac{k(k+1)(2k+1)}{6}$

Now we shall prove that $p(k+1)$ is true i.e. we shall prove

$$1^2 + 2^2 + \ldots + k^2 + (k+1)^2 = \frac{(k+1)(k+1+1)[2(k+1)+1]}{6}$$

$$\begin{aligned}
\text{L.H.S.} &= 1^2 + 2^2 + \ldots + k^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right] \\
&= (k+1)\left(\frac{2k^2 + k + 6k + 6}{6}\right) \\
&= (k+1)\left(\frac{2k^2 + 7k + 6}{6}\right) \\
&= (k+1)\left(\frac{2k^2 + 4k + 3k + 6}{6}\right) \\
&= (k+1)\left[\frac{2k(k+2) + 3(k+2)}{6}\right] \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)(k+1+1)[2(k+1)+1]}{6}
\end{aligned}$$

= R.H.S. Statement is true for $n \in N$.

Illustration 13 By the principle of mathematical induction prove that

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\sum n\right)^2 = \frac{n^2(n+1)^2}{4}$$

Solution

$$p(n) = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\sum n\right)^2 = \frac{n^2(n+1)^2}{4}$$

First we shall verify that $p(1)$ is true.

L.H.S. $= p(n) = n^3$

$p(1) = 1^3 = 1$

R.H.S. $\left(\sum n\right)^2 = p(n) = \dfrac{n^2(n+1)^2}{4}$

$p(1) = \dfrac{1(1+1)^2}{4} = 1$

L.H.S. = R.H.S.

Suppose $p(k)$ is true.

i.e. $1^3 + 2^3 + 3^3 + \ldots + k^3 = \dfrac{k^2(k+1)^2}{4}$

Now we shall prove that $p(k+1)$ is true i.e. we shall prove that

$$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \frac{(k+1)^2(k+1+1)^2}{4}$$

$$\begin{aligned}
\text{L.H.S.} &= (1^3 + 2^3 + 3^3 + \ldots + k^3) + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= (k+1)^2\left[\frac{k^2}{4} + (k+1)\right] \\
&= (k+1)^2\left[\frac{k^2 + 4(k+1)}{4}\right] \\
&= (k+1)^2\left(\frac{k^2 + 4k + 4}{4}\right) \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \frac{(k+1)^2(k+1+1)^2}{4}
\end{aligned}$$

= R.H.S. Statement is true for $n \in N$.

Illustration 14 Find the sum of n terms $n \in N$ of the following series 38 + 511 + 714 + . . .

Solution

$$\begin{aligned}
Tn &= 38 + 511 + 714 + \ldots \\
&= (3,5,7\ldots)(11,14\ldots) \\
&= [a + (n-1)d]\,[a + (n-1)d] \\
&= [3 + (n-1)2]\,[8 + (n-1)3] \\
&= (3 + 2n - 2)\,(8 + 3n - 3) \\
Tn &= (2n+1)\,(3n+5) \\
&= 6n^2 + 3n + 10n + 5 \\
Tn &= 6n^2 + 13n + 5
\end{aligned}$$

$$\begin{aligned}
\text{Now } Sn &= \sum Tn = \sum(6n^2 + 13n + 5) \\
&= 6\sum n^2 + 13\sum n + 5n \\
&= \frac{6n(n+1)(2n+1)}{6} + \frac{13n(n+1)}{2} + 5n \\
&= \frac{2n(n+1)(2n+1) + 13n(n+1) + 10n}{2}
\end{aligned}$$

$$= \frac{n}{2}\left[2(n+1)(2n+1)+13n+13+10\right]$$
$$= \frac{n}{2}\left(4n^2+2n+4n+2+13n+13+10\right)$$
$$= \frac{n}{2}\left(4n^2+19n+25\right)$$

Illustration 15 Find the sum of n terms $(n \in N)$ of the following series
257 + 599 + 81311 + 1,11,713 + . . .

Solution

257 + 599 + 81,311 + 1,11,713 + . . .

$$\text{Here } Tn = (2,5,8\ \ldots)(5,9,13\ \ldots)(7,9,11\ \ldots)$$
$$= \left[a+(n-1)d\right]\left[a+(n-1)d\right]\left[a+(n-1)d\right]$$
$$= \left[2+(n-1)3\right]\left[5+(n-1)4\right]\left[7+(n-1)2\right]$$
$$= (2+3n-3)(5+4n-4)(7+2n-2)$$
$$= (3n-1)(4n+1)(2n+5)$$
$$= (3n-1)(8n^2+20n+2n+5)$$
$$= (3n-1)(8n^2+22n+5)$$
$$= 24n^3-8n^2+66n^2-22n+15n-5$$
$$Tn = 24n^3+58n^2-7n-5$$
$$\text{Now } Sn = \sum Tn$$
$$= \sum\left(24n^3+58n^2-7n-5\right)$$
$$= 24\sum n^3+58\sum n^2-7\sum n-5$$
$$= \frac{24n^2(n+1)^2}{4}+\frac{58n(n+1)(2n+1)}{6}-\frac{7n(n+1)}{2}-5n$$
$$= 6n^2(n+1)^2+\frac{29n(n+1)(2n+1)}{3}-\frac{7n(n+1)}{2}-5n$$
$$= \frac{n\left[36n(n+1)^2+58(n+1)(2n+1)-21(n+1)-30\right]}{6}$$
$$= \frac{n}{6}\left[36n\left(n^2+2n+1\right)+58\left(2n^2+3n+1\right)-21n-21-30\right]$$
$$= \frac{n}{6}\left(36n^3+188n^2+189n+7\right)$$

Illustration 16 Find the sum of n terms $(n \in N)$
$4\times1^2+7\times3^2+10\times5^2+13\times7^2+\ \ldots$

Solution

$$4\times1^2+7\times3^2+10\times5^2+13\times7^2+\ \ldots$$
$$Tn = (4,71,013\ \ldots)\ (1^23^25^27^2\ \ldots)$$
$$= \left[a+(n-1)d\right]\left[a+(n-1)d\right]\left[a+(n-1)d\right]$$

$$= \left[4 + (n-1)3\right]\left[1 + (n-1)2\right]^2$$
$$= (3n+1)\,(2n+1)^2$$
$$= (3n+1)\,(4n^2+4n+1)$$
$$= 12n^3 - 12n^2 + 3n + 4n^2 - 4n + 1$$
$$Tn = 12n^3 - 8n^2 - n + 1$$
$$\text{Now } Sn = \sum Tn$$
$$= \sum\left(12n^3 - 8n^2 - n + 1\right)$$
$$= 12\sum n^3 - 8\sum n^2 - \sum n + \sum 1$$
$$= \frac{12n^2(n+1)^2}{4} - \frac{8n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + n$$
$$= 3n^2(n+1)^2 - \frac{4n(n+1)(2n+1)}{3} - \frac{n(n+1)}{2} + n$$
$$= \frac{18n^2(n+1)^2 - 8n(n+1)(2n+1) - 3n(n+1) + 6n}{6}$$
$$= \frac{n}{6}\left[18n(n^2+2n+1) - 8(2n^2+3n+1) - 3n - 3 + 6\right]$$
$$= \frac{n}{6}\left(18n^3 + 36n^2 + 18n - 16n^2 - 24n - 8 - 3n - 3 + 6\right)$$
$$= \frac{n}{6}\left(18n^3 + 20n^2 - 9n - 5\right)$$

Illustration 17 Find the sum of n terms $(n \in N)$

$$\frac{1^2}{1} + \frac{1^2+2^2}{2} + \frac{1^2+2^2+3^2}{3} + \,.\,.\,.$$

Solution

$$Tn = \frac{1^2+2^2+3^2+\,.\,.\,.\,+n^2}{3}$$
$$= \frac{\sum n^2}{n}$$
$$= \frac{n(n+1)(2n+1)/6}{n}$$
$$= \frac{n(n+1)(2n+1)}{6}$$
$$= \sum\frac{1}{6}\left(2n^2+3n+1\right)$$
$$= \frac{1}{6}\left(2\sum n^2 + 3\sum n + \sum 1\right)$$
$$= \frac{1}{6}\left[\frac{2n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} + n\right]$$

$$= \frac{1}{6}\left[\frac{2n(n+1)(2n+1)+9n(n+1)+6n}{6}\right]$$

$$= \frac{1}{6}\frac{n}{6}\left[2(n+1)(2n+1)+9(n+1)+6\right]$$

$$= \frac{n}{36}\left(4n^2+15n+7\right)$$

Illustration 18 Find the sum of the following series $51^3 + 52^3 + \ldots + 70^3$.

Solution

The required sum = $\sum_{n=1}^{70} n^3 = \sum_{n=1}^{70} n^3 - \sum_{n=1}^{50} n^3$

Now substitute = 70 and = 50 in the formula

$$\sum n^3 = \frac{n^2(n+1)^2}{4}$$

$$\text{Required sum} = \frac{(70)^2(70+1)^2}{4} - \frac{(50)^2(50+1)^2}{4}$$

$$= \frac{(70)^2(71)^2}{4} - \frac{(50)^2(51)^2}{4}$$

$$= \frac{(4{,}900)(5{,}041) - (2{,}500)(2{,}601)}{4}$$

$$= \frac{2{,}47{,}00{,}900 - 65{,}02{,}500}{4}$$

$$= 45{,}49{,}600$$

Illustration 19 Prove by mathematical induction principle that $\frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$ is an integer for every integer *n*.

Solution

Let us denote that given expression of $f(n)$ and $T(n)$ be the statement.

$Tn = f(n) = \frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$ is an integer for every positive integer.

We shall prove the above statement in two steps:

Step 1

$T(1)$ is true because

$f(1) = T(1) = \frac{1}{7} + \frac{1}{5} + \frac{2}{3} - \frac{1}{105} = 1$ which is an integer

Step 2

Suppose $T(n)$ is true when $n = k$. We shall show that $T(n)$ is true when $n = k + 1$ i.e. given that $f(k)$ is an integer. We shall show that $f(k+1)$ is an integer.

$f(k+1) - f(k)$

$$= \frac{1}{7}\left(k+10^7-k^7\right) + \frac{1}{5}\left[\left(k+1\right)^5-k^5\right]+\frac{2}{3}\left[\left(k+1\right)^3-k^3\right] - \frac{1}{105}\left[\left(k+1\right)-k\right] \quad (1)$$

We shall now use the following well known easily verifiable result.

If P is a prime and a and b are integers then

$\left(a+b\right)^p = a^p + b^p + a$ multiple of p

From the above result we find that

+ a multiple of 7

$$= \frac{\left(k+1\right)^7-k^7}{7} = \frac{1}{7} \text{ + an integer i.e. also by the same argument as above}$$

$$\frac{\left(k+1\right)^5-k^5}{5} = \frac{1}{5} \text{ + an integer}$$

$$\frac{\left(k+1\right)^3-k^3}{3} = \frac{1}{3} \text{ + an integer}$$

Using the above result in eq. (1), we have

$f\left(k+1\right) - f\left(k\right) = +$ an integer −

= an integer

Since $f\left(k\right)$ is true (an integer) it follows that $f\left(k+1\right)$ is an integer and consequently $T\left(n\right)$ is true. For $n = k + 1$, since $T\left(1\right)$ is true and true of $T\left(k\right)$ implies $T\left(k+1\right)$, therefore $T\left(n\right)$ is true for every positive integer.

Illustration 20 By using mathematical induction principle prove that $27^n + 35^n - 5$ is divisible by 24 for all $n \in N$.

Solution

Let $T\left(n\right)$ be the statement

$f\left(n\right) = 27^n + 35^n - 5 = 24$ which is divisible by 24.

Step 1

$f\left(1\right) = 27^1 + 35^1 - 5 = 24$ which is divisible by 24.

Step 2

Let us assume that $T\left(k\right)$ is true.

$f\left(k+1\right) = 27^{k+1} + 35^{k+1} - 5$

$f\left(k\right) = 27^k + 35^k - 5$

$f\left(k+1\right) - f\left(k\right)$

$= 2\left(7^{k+1} - 7^k\right) + 3\left(5^{k+1} - 5^k\right)$

$= 127^k + 125^k$

$= 12\left(7^k + 5^k\right)$

which is divisible by 24, since $7k + 5k$ is always even.

Now $f\left(k+1\right) - f\left(k\right)$ is a multiple of 24 and $f\left(k\right)$ is also multiple of 24. It follows that $f\left(k+1\right)$ is a multiple of 24 and consequently $T\left(k+1\right)$ is true. Statement $T\left(n\right)$ is true for $n \in N$

Illustration 21 By mathematical induction principle prove that $\frac{2n!}{2^{2n}(n!)^2} \leq \frac{1}{(3n+1)^{1/2}}$ for all positive integer *n*.

Solution

Let $T(n) = \frac{2n!}{2^{2n}(n!)^2} \leq \frac{1}{(3n+1)^{1/2}}$

Step 1

When $n = 1$

$$\frac{2n!}{2^{2n}(n!)^2} \leq \frac{2!}{2^2(1!)^2} = \frac{1}{2}$$

and $\frac{1}{(3n+1)^{1/2}} = \frac{1}{4^{1/2}} = \frac{1}{2}$

Step 2

Suppose $p(k)$ is true.

$$\frac{(2k+2)!}{2^{2(k+1)}\left[(k+1)!\right]^2} = \frac{(2k+2)(2k+1)(2k)!}{2^2(k+1)^2\,2^{2k}(k!)^2}$$

$$= \frac{(2k+1)}{(2k+2)}\frac{(2k)!}{2^{2k}(k!)^2}$$

$$= \frac{(2k+1)}{(2k+2)}\frac{1}{(3k+1)^{1/2}}$$

$$= \frac{(2k+1)}{\left[(2k+2)^2(3k+1)\right]^{1/2}}$$

$$= \frac{1}{(3k+4)^{1/2}}\left[\frac{(2k+1)^2(3k+4)}{(2k+2)^2(3k+1)}\right]^{1/2}$$

$$= \frac{1}{(3k+4)^{1/2}}\left(\frac{12k^3+28k^2+19k+4}{12k^3+28k^2+20k+4}\right)^{1/2}$$

$\frac{1}{(3k+4)^{1/2}}$ =; since *k* is a positive integer

$$= \frac{1}{\left[3(k+1)+1\right]^{1/2}}$$

showing that $T(k+1)$ is true.

Statement is true for $n \in N$.

Illustration 22 The number $\left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n$ is an integer for each natural number of *n*.

Solution

Let $T(n) = (2+\sqrt{3})^n + (2-\sqrt{3})^n$ is an integer

Step 1

$T(1)$ is true

Step 2

Let $T(m)$ be true for all $m < k$

Then $a^k + b^k = (a^{k-1} + b^{k-1})(a+b) - ab(a^{k-2} + b^{k-2})$

where $a = 2+\sqrt{3}$, $b = 2-\sqrt{3}$

Now $a + b$, ab are integers by one hypothesis

$a^{k-1} + b^{k-1}$ and $a^{k-2} + b^{k-2}$ are also integers. This implies that $a^k + b^k$ is an integer.

By principle of mathematical induction we can say that $(2+\sqrt{3})^n + (2-\sqrt{3})^n$ is an integer for all $n \in N$.

Illustration 23 If $a_n = a_{n-1} + a_{n-2}$ for $n > 2$ and $a_1 = 1$, $a_2 = 1$ show that $a_n < \left(\frac{7}{4}\right)^n$ is true for all $n \in N$

Solution

Suppose $T(n)$ be the statement that

$$a_n < \left(\frac{7}{4}\right)^n$$

Step 1

$$a_1 = 1 < \left(\frac{7}{4}\right)^1$$

$T(1)$ is true

Step 2

Let $T(n)$ be true for all $n < k$

$$\begin{aligned}
a_k &= a_{k-1} + a_{k-2} \\
&< \left(\frac{7}{4}\right)^{k-1} + \left(\frac{7}{4}\right)^{k-2} \\
&= \left(\frac{7}{4}\right)^{k-2} \left(\frac{7}{4} + 1\right) \\
&= \left(\frac{7}{4}\right)^{k-2} \left(\frac{11}{4}\right) \\
&< \left(\frac{7}{4}\right)^{k-2} \left(\frac{49}{16}\right) \\
&= \left(\frac{7}{4}\right)^{k}
\end{aligned}$$

Statement $a_n < \left(\frac{7}{4}\right)^k$ is true for all $n \in N$

Illustration 24 Let $R = \left(5\sqrt{5}+11\right)^{2n+1}$ and $f = R - \left[R\right]$ where $\left[\ \right]$ denotes the greatest integer function prove that $R_f = 4^{2n+1}$

Solution

Let $\left[R\right]$ stand for all integral part of R and f is fractional part of R. We shall show that the fraction part of R is $\left(5\sqrt{5}+11\right)^{2n+1}$. In order to do so we must prove that

(1) $\left(5\sqrt{5}+11\right)^{2n+1} - \left(5\sqrt{5}-11\right)^{2n+1}$ is an integer for all n.

(2) $\left(5\sqrt{5}-11\right)^{2n+1}$ lies between 0 and 1 for all n.

Let us denote both the above statements by $T\left(n\right)$.

Step 1

$\left(5\sqrt{5}+11\right) - \left(5\sqrt{5}-11\right) = 22$ which is a positive integer

and $0 < \left(5\sqrt{5}-11\right) < 1$

therefore $T\left(0\right)$ is true

Step 2

Assuming that $T\left(n\right)$ is true for all $n < k$ we shall show that $T\left(k\right)$ is true. Consider the identity

$$a^{2k+1} - b^{2k+1} = \left(a^{2k-1} - b^{2k-1}\right)\left(a^2+b^2\right) - a^2b^2\left(a^{2k-3} - b^{2k-3}\right)$$

If we take $a = 5\sqrt{5}+11$ and $b = 5\sqrt{5}-11$ we find that by the induction hypothesis $a^{2k-1} - b^{2k+1}$ and $a^{2k-3} - b^{2k-3}$ are integers. Also $a^2 + b^2 = 492$; $a^2b^2 = 16$ are integers. Consequently $a^{2k+1} - b^{2k+1}$ is an integer since $a^{2k+1} > 1$ and $b^{2k+1} > 1$ for all k. Therefore $a^{2k+1} - b^{2k+1}$ is a positive integer.

Hence $T\left(k\right)$ is true. Consequently f the fractional part of R is $\left(5\sqrt{5}-11\right)^{2n+1}$

$$\begin{aligned} Rf &= \left(5\sqrt{5}+11\right)^{2n+1}\left(5\sqrt{5}-11\right)^{2n+1} \\ &= \left(5\sqrt{5}+11\right)\left(5\sqrt{5}-11\right)^{2n+1} = 4^{2n+1} \end{aligned}$$

Remark: Note that in the above example induction starts with $n = 0$

ANALYTICAL EXERCISES

Prove the following by using the principle of mathematical induction

1. $3 + 6 + 9 + \ldots + 3n = \dfrac{3n\left(n+1\right)}{2}$

2. $3 + 8 + 13 + 18 + \ldots + \left(5n - 2\right) = \dfrac{n}{2}\left(5n+1\right)$

3. $7 + 10 + 13 + \ldots + \left(3n + 4\right) = \dfrac{3n\left(n+1\right)}{2}$

4. $2 + 4 + 6 + 8 + \ldots + 2n = n\left(n + 1\right)$

5. $2 + 6 + 10 + 14 + \ldots + \left(4n - 2\right) = 2n^2$

6. $1 + 5 + 9 + 13 + \ldots + (4n - 3) = n(2n - 1)$

7. $3 + 9 + 15 + 21 + \ldots + (6n - 3) = 3n^2$

8. $5 + 10 + 15 + 20 + \ldots + 5n = \frac{n}{2}(5n + 1)$

9. $1 + 11 + 21 + 31 + \ldots + (10n - 9) = n(5n - 4)$

10. $2.5 + 4.8 + 6.11 + \ldots + 2n(3n + 2) = n(n + 1)(2n + 3)$

11. $2.4 + 4.6 + 6.8 + \ldots + 2n(2n + 2) = \frac{4n(n + 1)(n + 2)}{3}$

12. $1.4 + 2.5 + 3.6 + \ldots + n(n + 3) = \frac{n(n + 1)(n + 5)}{3}$

13. $1.3 + 2.5 + 3.7 + \ldots + n(2n + 1) = \frac{n(n + 1)(4n + 5)}{6}$

14. $1.3 + 2.4 + 3.5 + \ldots + n(n + 2) = \frac{n(n+1)(2n+2)}{6}$

15. $1.2.3 + 2.3.4 + 3.4.5 + \ldots + n(n + 1)(n + 2) = \frac{n}{4}(n+1)(n+2)(n+3)$

16. $\frac{1}{2.6}+\frac{1}{6.10}+\frac{1}{10.14}+\ldots+\frac{1}{(4n-2)(4n+2)}=\frac{n}{4(2n+1)}$

17. $\frac{1}{5.6}+\frac{1}{6.7}+\frac{1}{7.8}+\ldots+\frac{1}{(n+4)(n+5)}=\frac{n}{5(n+5)}$

18. $\frac{1}{1.5}+\frac{1}{5.9}+\frac{1}{9.13}+\ldots+\frac{1}{(4n-3)(3n+1)}=\frac{n}{4n+1}$

19. Prove that $3^{2n} + 7$ is divisible by 8

20. Prove that $2^{4n} - 1$ is divisible by 15

21. Prove that $10^n + 3.4^{n+2} + 5$ is divisible by 9

22. Prove that $10^{2n-1} + 1$ is divisible by 11

23. Prove that $11^{n+1} + 12^{2n-1}$ is divisible by 133

24. Prove that $n(n + 1)(2n + 1)$ is divisible by 6

25. Prove that $1 + 2 + 2^2 + 2^3 + \ldots 2^{n-1} = 2^n - 1$

26. Prove that $a + (a + d) + (a + 2d) + \ldots + [a + (n - 1)d] = \frac{n}{2}[2a+(n-1)d]$

Find the sum of the following series

27. $8^2 + 9^2 + \ldots + 15^2$

28. $6^3 + 7^3 + \ldots + 12^3$

29. $21^2 + 22^2 + \ldots + 30^2$

30. $11^3 + 12^3 + \ldots + 20^3$

Find the sum of *n* terms of the following series (31 to 50)

31. $1.5 + 2.6 + 3.7 + \ldots$

32. $2.6 + 5.10 + 8.14 + \ldots$

33. $3.5 + 4.7 + 5.9 + \ldots$

34. $2.6 + 4.8 + 6.10 + \ldots$

35. $1.2.5 + 2.3.7 + 3.4.9 + \dots$

36. $3.5.7 + 6.7.9 + 9.9.11 + \dots$

37. $3^2 + 5^2 + 7^2 + 9^2 + \dots$

38. $2^2 + 4^2 + 6^2 + \dots$

39. $2^3 + 4^3 + 6^3 + \dots$

40. $1^3 + 3^3 + 5^3 + 7^3 + \dots$

41. $2.1^2 + 4.2^2 + 6.3^2 + \dots$

42. $1.4^2 + 2.5^2 + 3.6^2 + \dots$

43. If $10 < n < 20$ and $T_n = n^2 + 4$ find S_n

44. $\frac{1}{1} + \frac{1+2}{2} + \frac{1+2+3}{3} + \dots$

45. $2.5 + 5.8 + 8.11 + \dots$

46. Find the sum of n terms of a series whose nth term is $6n^2 - n - 1$

47. Find the sum of n terms of a series whose nth term is $3n^2 + 4n + 5$

48. $1 + (1 + 2) + (1 + 2 + 3) + \dots$

49. $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots$

50. $3 + 8 + 13 + \dots$

51. Obtain the value of the following

(1) $\sum_{i=7}^{17} (i^2 + 2i + 2)$ (2) $\sum_{i=1}^{10} i(i+1)(i+2)$

(3) $\sum_{i=1}^{10} (5i+1)^2$ (4) $\sum_{i=1}^{20} 4i$

52. Find the value of

(1) $\sum_{i=31}^{50} i^2$ (2) $\sum_{i=1}^{35} (4i-5)$ (3) $\sum_{i=10}^{20} i^3$

Prove that the following statements are true by using mathematical induction principle (53 to 73)

53. $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right) \dots \left(1+\frac{2n+1}{n^2}\right) = (n+1)^2$

54. $1 + 2.2 + 3.2^2 + \dots n2^n = 1 + (n - 1)2^n$

55. $3^{3n} < 2^{5n}$

56. $5^{2n+n} - 24n - 25$ is divisible by 576

57. $n(n^2 + 5)$ is divisible by 6

58. $2^{2n} - 3n - 1$ is divisible by 9

59. $6^{2n} + 3^{n+2} + 3^n$ is divisible by 11

60. $3^{2n+1} + 40n - 67$ is divisible by 64

61. $10^n + 3.4^{n+2} + 5$ is divisible by 9

62. $7^{2n} + 16n - 1$ is divisible by 64

63. $\sum r2^{r-1} = 1 + (n-1)2^n$

64. $2.1! + 5.2! + 10.3! + \dots + (n^2 + 1)n! = n(n + 1)!$

65. $\frac{1}{1.2}+\frac{1}{2.3}+\ldots\frac{1}{n(n+1)}<1$

66. If $x>-1$ then $(1+x)^n+nx$

67. $1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}\leq 2-\frac{1}{n}$

68. For $n\geq 3 \quad 2^n>2n+1$

69. For $n>1$ $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{n}}>\sqrt{n}$

70. For $n\geq 2$ $\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\ldots\left(1-\frac{1}{n}\right)=\frac{(n+1)}{2^n}$

71. Prove that for all natural numbers n, $11^{n+2}+12^{2n+1}$ is divisible by 133

72. If p be a natural number then prove that $p^{n+1}+(p+1)^{2n-1}$ is divisible by p^2+p+1 for every positive integer n.

73. Prove that for all natural numbers n, $7^{2n}+(2^{3n-3})3^{n-1}$ is divisible by 25.

Prove the following by applying the principle of finite induction (74 to 81)

74. $\frac{1^2}{1.3}+\frac{2^2}{3.5}+\frac{3^2}{5.7}+\ldots+\frac{n^2}{(2n-1)(2n+1)}=\frac{n(n+1)}{2(2n+1)}$

75. $\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$

76. $\sum_{r=1}^{n}\frac{r}{2^r}=2-\left(\frac{1}{2}\right)^n(n+2)$

77. $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$

78. $\frac{1}{1.5}+\frac{1}{5.9}+\frac{1}{9.13}+\ldots+\frac{1}{(4n-3)(4n+1)}=\frac{n}{4n+1}$

79. $\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}>\frac{13}{24}$; for all natural number $n>1$

80. $\frac{4^n}{n+1}<\frac{(2n)!}{(n!)2}$ for all $n>!$

81. $1.1!+2.2!+3.3!+\ldots+nn!=(n+1)!-1$

82. $3.5^{2n+1}+2^{3n+1}$ is divisible by 17.

83. $5^{2n+1}+2^{n+4}+2^{n+1}$ is divisible by 23.

84. $2^{2n+1}-9n^2+3^{n-2}$ is divisible by 54.

85. $3^{2n+5}+160n^2-56n-243$ is divisible by 512.

ANSWERS

(27) 1,100

(28) 5,859

(29) 6,585

(30) 41,075

(31) $\frac{n(n+1)(2n+13)}{6}$

(32) $n\left(4n^2+7n-1\right)$

(33) $\frac{n}{6}\left(4n^2+27n+59\right)$

(34) $\frac{2}{3}n(n+1)(2n+7)$

(35) $\frac{n(n+1)(n+2)(3n+7)}{6}$

(36) $\frac{n}{2}(n+1)\left(6n^2+38n+61\right)$

(37) $\frac{n}{3}\left(4n^2+12n+11\right)$

(38) $\frac{2}{3}n(n+1)(2n+1)$

(39) $2n^2(n+1)^2$

(40) $n^2\left(2n^2-1\right)$

(41) $\frac{n^2(n+1)^2}{2}$

(42) $\frac{n}{4}(n+1)\left(n^2+9n+22\right)$

(43) 2,525

(44) $\frac{n(n+3)}{4}$

(45) $n\left(3n^2+6n+1\right)$

(46) $\frac{n}{2}\left(4n^2+5n-1\right)$

(47) $\frac{n}{2}\left(2n^2+7n+15\right)$

(48) $\frac{n(n+1)(n+2)}{6}$

(49) $\frac{n(n+1)^2(n+2)}{12}$

(50) $\frac{n}{2}(5n+1)$

(51) (1) 1,980 (2) 4,290
(3) 9,085 (4) 840

(52) (1) 33,470 (2) 42,075 (3) 2,345

26

Sequence and Series

LEARNING OBJECTIVES

After studying this chapter, the student will be able to understand:

- The concept of sequence and series
- Progression, means and their types
- Definition, properties and formulae of progression

INTRODUCTION

Sequence and series is a mathematical concept that draws majorly from the basic number system and the simple concepts of arithmetic. This is the reason that makes it an important topic for this exam. On an average 2–3 questions have been asked from the topic almost every year. This topic is important for other exams also for example, CAT, IIFT, SNAP, XAT, MAT and JMET. The application of logic or some very simple concepts of algebraic calculations can be solved simply.

SEQUENCE AND SERIES

Let us consider the following progressions:

1, 3, 5, 7, 9 . . .
and 2, 6, 8, 12 . . .

It can be observed here that each of these two series shares some or the other common properties.

If the terms of a sequence are written under same specific conditions then the sequence is called a *progression*.

With respect to preparation for the BBA, we will confine ourselves only to the following standard series of progression:

1. Arithmetic progression
2. Harmonic progression
3. Geometric progression

ARITHMETIC PROGRESSION

Definition

If in any progression consecutive difference between any two terms is same, then that progression is said to be an *arithmetic progression* (A.P.), e.g.

$a, a+d, a+2d, a+3d \ldots a+(n-1)d$ i.e. $T_n = a+(n-1)d$. T_n is the last term.

Note that $d = T_2 - T_1 = T_3 - T_2 = T_4 - T_3 = \ldots$

Sum of an A.P.

If S_n is the sum of first n terms of the A.P.

$a, a+d, a+2d \ldots$

i.e. If $S_n = a + (a+d) + (a+2d) + \ldots$ up to n terms

then

$$S_n = \begin{cases} \frac{n}{2}\left[2a+(n-1)d\right] \\ \text{or} \\ \frac{n}{2}(a+l) \end{cases}$$

$$\text{where } \begin{cases} a = \text{first term of progression} \\ d = \text{consecutive difference between two terms} \\ n = \text{number of terms} \\ l = \text{last term} \end{cases}$$

Arithmetic Mean

If three numbers a, b, c are in A.P., then b is called the arithmetic mean between a and c.

(1) The arithmetic mean between two numbers a and b is $\frac{a+b}{2}$.

(2) $A_1, A2 \ldots A_n$ are said to be n arithmetic means between two numbers a and b. If $a, A1, A2 \ldots A_n, b$ are in A.P. and if d is the common difference of this A.P., then $b = a + (n+2-1)d \Rightarrow d = \frac{b-a}{n+1}$.

$$A_1 = a + d = a + \left(\frac{b-a}{n+1}\right)$$

$$A_2 = a + 2d = a + 2\left(\frac{b-a}{n+1}\right)$$

Hence __________________

____

$$A_n = a + nd = a + n\left(\frac{b-a}{n+1}\right)$$

HARMONIC PROGRESSION

Definition

Non-zero numbers $a_1, a_2, a_3 \ldots a_n$ are said to be in a *harmonic progression* (H.P.) if $\frac{1}{a_1}, \frac{1}{a_2} \ldots \frac{1}{a_n}$ are in A.P.

(1) Three non-zero numbers a, b, c are in H.P. if $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

(2) If $\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b} \Rightarrow \frac{2}{b} = \frac{1}{a} + \frac{1}{c}$

(3) If a, b, c are in H.P. then b is called the harmonic mean between a and c.

(4) The harmonic mean between two numbers a and b is $\frac{2ab}{a+b}$.

(5) If $a, H_1, H_2 \ldots H_n, b$ are in H.P. then $H_1, H_2 \ldots H_n$ are called harmonic means between a and b.

(6) nth term of the H.P.

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d} \ldots \frac{1}{a+(n-1)d}$$

DO YOU KNOW?

(1) Three numbers a, b, c are in A.P. if $b = \frac{a+c}{2}$.

(2) Three numbers a, b, c are in A.P. if $\frac{a-c}{b-c} = \frac{a}{a}$.

(3) Three non-zero numbers a, b, c are in H.P. if $b = \frac{2ac}{a+c}$.

(4) Three non-zero numbers a, b, c are in H.P. if $\frac{a-b}{b-c} = \frac{a}{c}$.

(5) If A and H denote respectively the A.M. and H.M. between two distinct positive numbers then $A > +1$.

(6) If the terms of an A.P. are increased, decreased, multiplied and divided by the same non-zero constant then they remain in A.P.

(7) In an A.P, sum of terms equidistant from the beginning and end is constant.

GEOMETRIC PROGRESSION

If in any sequence, consecutive ratio between any two terms is same it is said to be a *geometric progression* (G.P.).

e.g. $a, ar^2, ar^3 \ldots ar^{n-1}$

$\therefore\ T_n = ar^{n-1}$

where $\begin{cases} a = \text{first term of progression} \\ r = \text{common ratio} \\ n = \text{number of terms} \end{cases}$

Sum of a G.P.

If S_n is the sum of first n terms of G.P. a, ar^2, ar^3 . . . ar^{n-1}

i.e. $s_n = a + ar + ar^2 + \ . \ . \ . \ + ar^{n-1}$

$$s_n = \begin{cases} \dfrac{a\left(r^n - 1\right)}{r - 1}\ ; r > 1 \\ \dfrac{a\left(1 - r^n\right)}{\left(1 - r\right)}\ ; r < 1 \end{cases}$$

We can take the above S_n formula for finite progression and $s_n = a + ar + ar^2 + \ . \ . \ .$ up to infinity.

$$\therefore s_n = \frac{a}{1 - r}\ ; -1 < r < 1$$

Geometric Mean

If three non-zero numbers a, b, c are in G.P. then b is called the geometric mean between a and b

i.e. $\dfrac{a}{b} = \dfrac{b}{c}$

$\therefore b^2 = ac$

$\therefore b = \sqrt{ac}$

DO YOU KNOW?

(1) Three non-zero numbers in a, b, c are in G.P. if $b^2 = ac$.

(2) Three non-zero numbers in a, b, c are in G.P. if $\dfrac{a - b}{b - c} = \dfrac{a}{b}$.

(3) If A, G, H denote respectively the A.M., G.M., H.M. between two distinct positive numbers then

   (1) A, G, H are in G.P.

   (2) $A > G > H$.

(4) In a G.P. the product of terms equidistant from the beginning and end is constant.

(5) A sequence (or a series) is both an A.P. as well as a G.P. if it is a constant sequence i.e. if all the terms are equal.

Important Notes on A.P. and G.P.

It is convenient to take

(1) Three numbers in A.P. as $a - d, a, a + d$

(2) Four numbers in A.P. as $a - 3d, a - d, a + d, a + 3d$

(3) Three numbers in G.P. as $\dfrac{a}{r}, a, ar$

(4) Five numbers in G.P. as $\dfrac{a}{r^2}, \dfrac{a}{r}, a, ar, ar^2$

Some Important Power Series

(1) $1 + 2 + 3 + \ldots + n = \sum n = \frac{n(n+1)}{2}$

(2) $1^2 + 2^2 + 3^2 + \ldots + n^2 = \sum n^2 = \frac{n(n+1)(2n+1)}{6}$

(3) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \sum n^3 = \left(\sum n^2\right) = \frac{n^2(n+1)^2}{4}$

(4) $1 + 1 + 1 + 1 + \ldots + 1 n \text{ terms} = \sum 1 = n$

ILLUSTRATIONS

Illustration 1 If for a sequence $\{a_n\}$, $S_n = 2(3m - 1)$, find its first four terms.

Solution

Let $S_n = 2\left[3^n - 1\right]$

$\Rightarrow S_{n-1} = 2\left[3^{n-1} - 1\right]$

$\therefore A_n = S_n - S_{n-1}$

$= 2(3^n - 1 - 3^{n-1} + 1)$

$= 2 \times (3^n - 3^{n-1})$

$= 2 \times 3^{n-1}(3 - 1)$

$= 2 \times 3^{n-1} \times 2$

$\therefore a_n = 4 \times 3^{n-1}$

$\therefore a_1 = 4 \times 3^{1-1} = 4 \times 3^0 = 4(1) = 4$

$a_2 = 4 \times 3^{2-1} = 4 \times 3^1 = 4(3) = 12$

$a_3 = 4 \times 3^{3-1} = 4 \times 3^2 = 4(9) = 36$

$a_4 = 4 \times 3^{4-1} = 4 \times 3^3 = 4(27) = 108$

$\therefore$ The first four terms of the sequence are 4, 12, 36, 108

Illustration 2 If *a*, *b* are positive real numbers, show that the sequence $\log\left(\frac{a^2}{b}\right)$, $\log\left(\frac{a^3}{b^2}\right)$, $\log\left(\frac{a^4}{b^3}\right)$ . . . is an A.P. Also find its general term.

Solution

2. Let $\log\frac{a^2}{b} - \log a = \log\frac{a^2}{b}\ \frac{1}{a} = \log\frac{a}{b}$

$\log\frac{a^3}{b^2} - \log\frac{a^2}{b} = \log\frac{a^3}{b^2}\ \frac{b}{a^2} = \log\frac{a}{b}$

$\log\frac{a^4}{b^3} - \log\frac{a^3}{b^2} = \log\frac{a^4}{b^3}\ \frac{b^2}{a^3} = \log\frac{a}{b}$

Here difference between any two consecutive terms remains constant = d = $\log\frac{a}{b}$ and first term $a = \log a$

$= \log a + (n-1)\log\frac{a}{b}ac$

$$= \log a + \log\left(\frac{a}{b}\right)^{n-1}$$

$$= \log a \frac{a^{n-1}}{b^{n-1}} = \log\frac{a^n}{b^{n-1}}$$

$$= \therefore f(n) = \log\frac{a^n}{b^{n-1}}$$

Illustration 3 Which term of the sequence −3, −7, −11, −15, . . . is −403? Also find which term if any of the given sequence is −500.

Solution

Here given sequence is an A.P.

$\therefore\ a = -3$ and $f(n) = -403$

$d = (-7) - (-3) = -7 + 3 = -4$

$\therefore\ f(n) = a + (n-1)d$

$-403 = -3 + (n-1)(-4)$

$n - 1 = 100$

$n = 101$

Hence −403 is the 101st term of the sequence.

Let $f(n) = -500$

$\Rightarrow -500 = -3 + (n-1)(-4)$

$\Rightarrow \dfrac{-497}{-4} = (n-1)$

$\Rightarrow\ n = \dfrac{497}{4} + 1 = \dfrac{501}{4} = 125\dfrac{1}{4}$

which is not a natural number.

$\therefore$ There is no term of the given sequence which is −500.

Illustration 4 Which term of the sequence $25, 24\frac{1}{4}, 23\frac{1}{2}, 22\frac{3}{4}$ . . . is the first negative term?

Solution

$25, 24\frac{1}{4}, 23\frac{1}{2}, 22\frac{3}{4}$ . . . are in A.P.

$\therefore$ Here $a = 25$ and $d = \dfrac{97}{4} - 25 = \dfrac{-3}{4}$

Let nth term of the given A.P. be the first negative term, then

$f(n) < 0$

$\Rightarrow 25 + (n-1)\left(\dfrac{-3}{4}\right) < 0$

$\Rightarrow 25 - \dfrac{3n}{4} + \dfrac{3}{4} < 0 \Rightarrow \dfrac{103}{4} < \dfrac{3n}{4}$

$\Rightarrow \dfrac{103}{3} < n \Rightarrow n > \dfrac{103}{3} = 34\dfrac{1}{3}$

$\Rightarrow n = 35$

Hence 35th term of the given sequence is the first negative term.
Hence the result.

Illustration 5 Which term of the sequence $12 + 8i$, $10 + 7i$, $8 + 6i$, . . . is (i) real (ii) purely imaginary?

Solution

Let $12 + 8i$, $10 + 7i$, $8 + 6i$ . . . are in A.P.

$\therefore a = 12 + 8i$ and

$d = (10 + 7i) - (12 + 8i) = 10 + 7i - 12 - 8i$

$\therefore d = -2 - i$

$\therefore f(n) = a + (n - 1)d$

$= (12 + 8i) + (n - 1)(-2 - i)$

$= 12 + 8i + (n - 1)(-2) - i(n - 1)$

$= (14 - 2n) + i(8 - n + 1)$

(i) Now nth term of given sequence is real

$9 - n = 0 \quad \Rightarrow n = 9$

(ii) nth term of given sequence is purely imaginary

$14 - 2n = 0 \quad \Rightarrow n = 7$

Illustration 6 If P times the pth term of an A.P. is q times the qth term then show that its $(p + q)$th term is zero.

Solution

Let a be the first term and d be the common difference of the given A.P., then $pf(p) = qp(q)$

$\Rightarrow p[a + (p - 1)d] = q[a + (q - 1)d]$

$\Rightarrow ap + p(p - 1)d - aq - q(q - 1)d = 0$

$\Rightarrow a(p - q) + (p^2 - p - q^2 + q)d = 0$

$\Rightarrow a(p - q) + [(p^2 - q^2) - (p - q)]d = 0$

$\Rightarrow (p - q)[a + (p + q - 1)d] = 0$

$\Rightarrow a + (p + q - 1)d = 0$

$\Rightarrow f(p + q) = 0$

Hence the result.

Illustration 7 Find the 15th term from the end of the sequence 7, 10, 13 . . . 130.

Solution

Let 7, 10, 13 . . . 130 be in A.P. where $a = 7$ and $d = 10 - 7 = 3$ and $f(n) = 130$

$\therefore f(n) = a + (n - 1)d$

$130 = 7 + (n - 1)3$

$$\frac{123}{3} = n - 1 \Rightarrow n - 1 = 41 \Rightarrow n = 42$$

Now, 15th term from end
= 42 − 15 + 1
= 28
∴ 28th term from beginning
$7 + (28 - 1)3 = 7 + 81 = 88$
Second method: 15th term from the end, we consider the given sequence as an A.P. with first term = 130, $d = -3$.
∴ The 15th term from end of the given sequence = $130 + (15 - 1)(-3)$
= 130 − 42 = 88
Hence the result.

Illustration 8 Prove that a, b, c are in A.P. $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$.

Solution

$\frac{1}{bc} - \frac{1}{ca} - \frac{1}{ab}$ are in A.P.

If $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in A.P

i.e. $\frac{1}{ca}, \frac{1}{bc} = \frac{1}{ab}, \frac{1}{ca}$

$\frac{b-a}{abc} = \frac{c-b}{abc} \Rightarrow b - a = c - b$

$\Rightarrow 2b = a + c$

$\Rightarrow b = \frac{a+c}{2}$

∴ a, b, c are in A. P.
Hence the result.

Illustration 9 If a, b, c are in A.P. prove that the following are also in A.P.

(i) $(b + c)^2 - a^2, (c + a)^2 - b^2, (a + b)^2 - c^2$

(ii) $a\left(\frac{1}{b} + \frac{1}{c}\right), b\left(\frac{1}{c} + \frac{1}{a}\right), c\left(\frac{1}{a} + \frac{1}{b}\right)$

Solution

(i) $(b + c)^2 - a^2, (c + a)^2 - b^2, (a + b)^2 - c^2$ are in A.P.

$\Rightarrow (b + c + a)(b + c - a), (c + a + b)(c + a - b),$
$(a + b + c)(a + b - c)$ are in A.P.

$\Rightarrow b + c - a, c + a - b, a + b - c$ are in A.P.

(Dividing each of them by $a + b + c$)

$\Rightarrow (c + a - b) - (b + c - a) = (a + b - c) - (c + a - b)$

$\Rightarrow \cancel{c} + a - b - b - \cancel{c} + a = \cancel{a} + b - c - c - \cancel{a} + b$

$2a - 2b = 2b - 2c$

$a - b = b - c$

$\Rightarrow b = \dfrac{a+c}{2}$

$\therefore$ *a, b, c* are in A. P. which is given to be true.

(ii) Let *a, b, c* be in A.P.

$\Rightarrow \dfrac{a}{abc}, \dfrac{b}{abc}, \dfrac{c}{abc}$ are in A.P.

$\Rightarrow \dfrac{1}{bc}, \dfrac{1}{ac}, \dfrac{1}{ab}$ are in A.P.

$\Rightarrow \dfrac{ab+bc+ca}{bc}, \dfrac{ab+bc+ca}{ac}, \dfrac{ab+bc+ca}{ab}$ are in A.P.

$\Rightarrow \dfrac{ab+bc+ca}{bc} - 1, \dfrac{ab+bc+ca}{ac} - 1, \dfrac{ab+bc+ca}{ab} - 1$ are in A.P.

$\Rightarrow \dfrac{ab+ca}{bc}, \dfrac{ab+bc}{ca}, \dfrac{bc+ca}{ab}$ are in A.P.

$\Rightarrow a\left(\dfrac{b+c}{bc}\right), b\left(\dfrac{a+c}{ac}\right), c\left(\dfrac{b+c}{ab}\right)$ are in A.P.

$\Rightarrow a\left(\dfrac{1}{c}+\dfrac{1}{b}\right), b\left(\dfrac{1}{c}+\dfrac{1}{a}\right), c\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$ are A.P.

Hence the result.

Illustration 10 If a^2, b^2, c^2, are in A.P. show that $\dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}$ are in A.P.

Solution

Let $\dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}$

$\Rightarrow \dfrac{a}{b+c} + 1, \dfrac{b}{c+a} + 1, \dfrac{c}{a+b} + 1$ are in A.P.

$\Rightarrow \dfrac{a+b+c}{b+c}, \dfrac{b+c+a}{a+c}, \dfrac{c+a+b}{a+b}$ are in A.P.

$\Rightarrow \dfrac{1}{b+c}, \dfrac{1}{a+c}, \dfrac{1}{a+b}$ are in A.P.

$\Rightarrow \dfrac{(a+b)\cancel{(b+c)}(c+a)}{\cancel{(b+c)}}, \dfrac{(a+b)(b+c)\cancel{(c+a)}}{\cancel{(a+c)}}, \dfrac{\cancel{(a+b)}(b+c)(c+a)}{\cancel{(a+b)}}$ are in A.P.

Multiplying each term by $(a + b)(c + a)(c + a)$

$\Rightarrow (a + b)(c + a), (a + b)(b + c), (b + c)(c + a)$ are in A.P.

$\Rightarrow ac + a^2 + bc + ab, ab + ac + b^2 + bc, bc + ba + c^2 + ac$ are in A.P.

$\Rightarrow a^2, b^2, c^2$ are in A.P.

(Subtracting $ab + bc + ca$ from each term)

which is given to be true.

Illustration 11 If $a^2 + 2bc$, $b^2 + 2ca$, $c^2 + 2ab$ are in A.P. show that $\frac{1}{b-c}$, $\frac{1}{c-a}, \frac{1}{a-b}$ are in A.P.

Solution

Let $a^2 + 2bc$, $b^2 + 2ac$, $c^2 + 2ab$ are in A.P.

$\Rightarrow a^2 + 2bc - (ab + bc + ca)$, $b^2 + 2ac - (ab + bc + ca)$,
$c^2 + 2ab - (ab + bc + ca)$ are in A.P.

$\Rightarrow a^2 + bc - ab - ac$, $b^2 + ac - ab - bc$, $c^2 + ab - bc - ca$ are in A.P.

$\Rightarrow (a - b)(a - c)$, $(b - c)(b - a)$, $(c - a)(c - b)$ are in A.P.

$\Rightarrow \frac{-1}{b-c}, \frac{-1}{c-a}, \frac{-1}{a-b}$ are in A.P.

(Dividing each term by $(a - b)(b - c)(c - a)$

$\Rightarrow \frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are in A.P.

(Dividing each term by -1)

Illustration 12 If $a^2(b + c)$, $b^2(c + a)$, $c^2(a + b)$ are in A.P., prove that either a, b, c are in A.P. or $ab + bc + ca = 0$

Solution

Let $a^2(b + c)$, $b^2(c + a)$, $c^2(a + b)$ are in A.P.

$\Rightarrow b^2(c + a) - a^2(b + c) = c^2(a + b) - b^2(c + a)$

$\Rightarrow (ab^2 - a^2b) + (b^2c - a^2c) = (c^2b - b^2c) + (c^2a - ab^2)$

$\Rightarrow ab(b - a) + c(b - a)(b + a) = bc(c - b) + a(c - b)(c + b)$

$\Rightarrow (b - a)(ab + bc + ac) = (c - b)(bc + ac + ab)$

$\Rightarrow (b - a - c + b)(ab + bc + ca) = 0$

$\therefore (2b - a - c)(ab + bc + ca) = 0$

Either $2b - a - c =$ or $ab + bc + ca = 0$

$\Rightarrow 2b = a + c$ or $ab + bc + ca = 0$

$\Rightarrow b = \frac{a+c}{2}$

$\therefore a, b, c$ are in A.P. or $ab + bc + ca = 0$

Illustration 13 If $\log_{10}^{2}$, $\log_{10}(2^x - 1)$, $\log_{10}(2n + 3)$ are in A.P. find the value of x.

Solution

Let $\log_{10}^{2}$, $\log_{10}^{(2^x - 1)}$, $\log_{10}^{(2^x + 3)}$ are in A.P.

$$\Rightarrow \therefore \log_{10}^{(2^x - 1)} = \frac{\log_{10}^{2} + \log_{10}^{(2^x + 3)}}{2}$$

$$\Rightarrow 2\log_{10}^{(2^x - 1)} = \log_{10}^{2} + \log_{10}^{(2^x + 3)}$$

$$\Rightarrow \log_{10}^{(2^x - 1)^2} = 2(2^x + 3)$$

$\Rightarrow (2^x - 1)^2 = 2(2^x + 3)$

$\Rightarrow (y - 1)^2 = 2(y + 3) \qquad (\because y = 2^x)$

$\Rightarrow y^2 - 2y + 1 - 2y - 6 = 0$

$\Rightarrow y^2 - 4y - 5 = 0$

$\Rightarrow (y - 5)(y + 1) = 0$

$\Rightarrow y = 5$ or $y = -1$

$\therefore 2^x = 5 \qquad (\because 2^x = -1$ is not possible$)$

$\Rightarrow x = \log_2^5$

Hence the result.

Illustration 14 (i) Sum the series $2 + 4 + 6 + \ldots$ up to 40 terms.

(ii) Find the sum of first 19 terms of the A.P. whose nth term is $(2n + 1)$.

(iii) Find the sum of the series $1 + 3 + 5 + \ldots 99$.

Solution

(i) $2 + 4 + 6 + \ldots$ up to 40 terms

$a = 2, d = 4 - 2 = 2, n = 40.$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

$$S_{40} = \frac{40}{2}\left[2 + (39)2\right]$$

$= 40(4 + 39)$

$= 40(41)$

$= 1640$

(ii) Let $f(n) = 2n + 1$

$\Rightarrow f(1) = 2 + 1 = 3$

$f(19) = 38 + 1 = 39$

$$\therefore S_{19} = \frac{19}{2}(3 + 39) = \frac{19}{2}(42)$$

$S_{19} = (19)(21) = 399$

(iii) Let $1 + 3 + 5 + \ldots + 99.$

$\therefore a = 1, d = 2, f(n) = 99$

$f(n) = a + (n - 1)d$

$99 = 1 + (n - 1)2$

$98 = (n - 1)2 \Rightarrow n - 1 = 49$

$\Rightarrow n = 50$

$$\therefore S_n = \frac{n}{2}(a + l) \qquad \left[\because l = f(n)\right]$$

$$S_{50} = \frac{50}{2}(1 + 99)$$

$= (50)(50)$

$S_{50} = 2500$

Hence the result.

Illustration 15 (i) Solve the equation $2 + 5 + 8 + 11 + \ldots + x = 155$.

(ii) Sum up $0.7 + 0.71 + 0.72 + \ldots$ up to 100 terms.

(iii) How many terms of the A.P. $17 + 15 + 13 + \ldots$ must be taken so that sum is 72? Explain the double answer.

Solution

(i) $2 + 5 + 8 + 11 + \ldots + x = 155.$

$a = 2,\ d = 3,\ f(n) = x,\ S_n = 155$

$$\therefore S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

$$155 = \frac{n}{2}\left[2a + (n-1)3\right]$$

$$310 = n(3n + 1)$$

$$\Rightarrow 3n^2 + n - 310 = 0$$

$$\therefore n = \frac{-1 \pm \sqrt{1 - 4(3)(-310)}}{2(3)}$$

$$\therefore n = \frac{-1 \pm \sqrt{3721}}{6} = \frac{-1 \pm 61}{6}$$

$$\therefore n = \frac{-1 - 61}{6},\ n = \frac{-1 + 61}{6} = 10$$

$$\therefore n = \frac{-62}{6} \text{ is not possible}$$

$$\therefore n = 10$$

$$\therefore f(n) = a + (n - 1)d$$

$$\Rightarrow x = 2 + (10 - 1)3$$

$$\Rightarrow x = 2 + 27$$

$$\therefore x = 29$$

(ii) Let $0.7 + 0.71 + 0.72 + \ldots$ be up to 100 terms.

$a = 0.7,\ d = 0.01,\ n = 100$

$$S_{100} = \frac{100}{2}\left[2(0.7) + (99)(0.01)\right]$$

$$= 50(1.4 + 0.99)$$

$$= 50(2.39)$$

$$= 119.5$$

(iii) Let $a = 17,\ d = 17,\ d = -2,\ S_n = 72$

$$\therefore S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

$$72 = \frac{n}{2}\left[34 + (n-1)(-2)\right]$$

$$72 = n\left[17 - n + 1\right]$$

$$72 = n(18 - n)$$

$$72 = 18n - n^2$$

$$\Rightarrow n^2 - 18n + 72 = 0$$
$$\Rightarrow (n - 12)(n - 6) = 0$$
$$\therefore\ n = 12 \text{ or } n = 6$$

Both values of n being positive integers are valid. We get double answer because sum of 7th to 12th terms is zero as some terms are positive and some are negative.

Illustration 16 If the first term of an A.P. is 2 and the sum of the first five terms is equal to one-fourth of the sum of the next five terms, find the sum of first 30 terms.

Solution

Let $a_1, a_2, a_3, \ldots$ be in A.P.

with common difference d and $a_1 = 2$

$$a_1 + a_2 + a_3 + a_4 + a_5 = (a_6 + a_7 + a_8 + a_9 + a_{10})$$
$$\Rightarrow 4S_5 = S_{10} - S_5$$
$$\Rightarrow 5S_5 = S_{10}$$
$$\Rightarrow \cancel{5}\frac{\cancel{5}}{2}\left[2(2) + 4d\right] = \frac{\overset{2}{\cancel{10}}}{\cancel{2}}\left[2(2) + 9d\right]$$
$$\Rightarrow 5(4 + 4d) = 2(4 + 9d)$$
$$\Rightarrow 5 \times \cancel{2}(2 + 2d) = \cancel{2}(4 + 9d)$$
$$10 + 10d = 4 + 9d$$
$$-6 = d \Rightarrow d = -6$$

$\therefore$ The sum of first 30th term

$$= \frac{30}{2}\left[2(2) + (29)(-6)\right]$$
$$= 15(4 - 174) = 15(-170)$$
$$= -2550$$

Illustration 17 The sums of n terms of three A.Ps are S_1, S_2 and S_3. The first term of each is unity and the common differences are 1, 2 and 3 respectively, prove that $S_1 + S_2 = 2 \cdot S_2$.

Solution

Let

S_1 = Sum of n terms of an A.P. with first term 1 and $d = 1$

$$= \frac{n}{2}\left[2(1) + (n - 1)(1)\right]$$
$$= \frac{n}{2}(n + 1)$$

S_2 = Sum of n terms of an A.P. with first term 1 and $d = 2$

$$= \frac{n}{2}\left[2(1) + (n - 1)(2)\right]$$
$$= n\left[\cancel{1} + n - \cancel{1}\right] = n(n)$$
$$= n^2 \qquad \text{and}$$

S_3 = Sum of n terms of an A.P. with first term 1 and $d = 3$

$$= \frac{n}{2}\left[2(1) + (n-1)3\right]$$

$$= \frac{n}{2}(2 + 3n - 3) = \frac{n}{2}(3n-1)$$

$$= \frac{n}{2}(3n-1)$$

$$\therefore \text{ L.H.S. } S_1 + S_3 = \frac{n}{2}(n+1) + \frac{n}{2}(3n-1)$$

$$= \frac{n}{2}\left[n + \cancel{1} + 3n - \cancel{1}\right] = \frac{n}{\cancel{2}}\left(\overset{2}{\cancel{4}}n\right)$$

$$= 2n^2 = 2S_2 = \text{R.H.S.}$$

Illustration 18 (i) Find the sum of all the three digit numbers which leave the remainder 2 when divided by 5.

(ii) The sum of n terms of two arithmetic series is in the ratio $(7m + 1) : (4n + 27)$. Find the ratio of their 11th terms.

Solution

(i) The three digit numbers which leave 2 as remainder when divided by 5 are 102, 107, 112 . . . 997

$\therefore\ a = 102,\ d = 5,\ f(n) = 997$

$\therefore\ f(n) = a + (n-1)d$

$997 = 102 + (n-1)5$

$$\frac{895}{5} - n - 1 \Rightarrow 179 = n - 1$$

$\Rightarrow n = 180$

$$\therefore S_n = \frac{n}{2}(a + l)$$

$$S_{180} = \frac{180}{2}(102 + 997)$$

$$= 90(1{,}099)$$

$$= 98{,}910$$

Hence the required sum is 98,910.

(ii) Let

$$\frac{S_n}{S'_n} = \frac{7n+1}{4n+27}$$

$$\Rightarrow \frac{\cancel{(n/2)}\left[2a + (n-1)d\right]}{\cancel{(n/2)}\left[2\text{A} + (n-1)\text{D}\right]} = \frac{7n+1}{4n+27}$$

$$\Rightarrow \frac{a + \left[(n-1)/2\right]d}{\text{A} + \left[(n-1)/2\right]\text{D}} = \frac{7n+1}{4n+27}$$

Now, the ratio of the 11th term $\dfrac{a+10d}{\mathrm{A}+10\mathrm{D}}$

$$\therefore \frac{n-1}{2}=10 \Rightarrow n=20+1=21$$

$$\therefore \frac{a+10d}{\mathrm{A}+10\mathrm{D}}=\frac{147+1}{84+27}=\frac{148}{111}=\frac{4}{3}$$

$$\therefore \frac{f(11)}{f'(11)}=\frac{4}{3}$$

Hence the result.

Illustration 19 The interior angles of a polygon are in arithmetic progression. The smallest angle is 120° and the common difference is 5°. Find the number of sides in the polygon.

Solution

The sum of all exterior angles of a polygon = 360. If the interior angles are in A.P., the exterior angles are also in A.P.

$\therefore$ Largest exterior angle is $180° - 120° = 60°$ and common difference $= -5°$.

$$\therefore \text{Sum} = 360° = \frac{n}{2}\left[2a+(n-1)d\right]$$

$$\Rightarrow 720° = n\left[2.60° + (n-1)(-5°)\right]$$

$$= 120°\,n - 5n^2 + 5n$$

$$\therefore 5n^2 - 125n + 720° = 0$$

$$n^2 - 25n + 144 = 0$$

$$(n-16)(n-9) = 0$$

$$\therefore n = 9 \text{ or } n = 16$$

Here if $n = 16$, then internal angle $120° + (16-1)(5°) = 195°$, so one of the internal angles will be 180° which is not possible in a polygon.

$\therefore n = 9$

Hence the result.

Illustration 20 A sum of Rs. 6,240 is paid off in 30 instalments, such that each instalment is Rs. 10 more than the preceding instalment. Calculate the value of the first instalment.

Solution

Let $S_n = 6240$

$n = 30,\ d = 10,\ \Rightarrow a = ?$

$$\therefore S_n = \frac{n}{2}\left[2a+(n-1)d\right]$$

$$S_{30} = \frac{30}{2}\left[2a+(30-1)10\right]$$

$$S_{30} = \frac{30}{2}(2a+290)$$

$$6{,}240 = 15(2a + 290)$$

$$\frac{6{,}240}{15} = 2a+290$$

$$416 - 290 = 2a$$

$$\Rightarrow 2a = 126$$

$$\Rightarrow a = 63$$

Hence the first instalment is Rs. 63.

Illustration 21 If the sum of p term of an A.P. is a and the sum of q term is p, show that the sum of $(p + q)$ terms is $-(p + q)$. Also find $l + m$ the sum of first $p - q$ terms $(p > q)$.

Solution

Let $S_p = q$ and $S_q = p$

$$\therefore \frac{p}{2}\left[2a+(p-1)d\right] = q, \frac{q}{2}\left[2a+(q-1)d\right] = p$$

$$2a+(P-1)d = \frac{2q}{p}$$

$$2a+(q-1)d = \frac{2p}{q}$$

$$-\quad -\qquad -$$

$$(p-1-q+1)\,d = \frac{2q}{p} - \frac{2p}{q}$$

$$(p-q)d = -\frac{2(p^2-q^2)}{pq}$$

$$\Rightarrow d = \frac{-2(p+q)}{pq}$$

Now $2a = \frac{2q}{p} - (p-1)d$

$$\therefore S_{p+q} = \frac{p+q}{2}\left[2a+(p+q-1)d\right]$$

$$= \frac{p+q}{2}\left[\frac{2q}{p} - (p-1)d + (p+q-1)d\right]$$

$$= \frac{p+q}{2}\left[\frac{2q}{p} - d(\cancel{p} - \cancel{1} - \cancel{p} - q + \cancel{1})\right]$$

$$= \frac{p+q}{2}\left(\frac{2q}{p} + dq\right)$$

$$= \frac{p+q}{2}\left[\frac{2q}{p} - \frac{2(p+r)\cancel{q}}{p\cancel{q}}\right]$$

$$= (p+q)\frac{(\cancel{q} - p - \cancel{q})}{p}$$

$$= -(p+q)$$

$$\therefore S_{p+q} = -(p+q)$$

Similarly sum of first $(p-q)$ terms

$$= \frac{p-q}{2}[2a + (p-q-1)d]$$

$$= \frac{p-q}{2}\left[\frac{2q}{p} - (p-1)d + (p-q-1)d\right]$$

$$= \frac{p-q}{2}\left[\frac{2q}{p} - d(\cancel{p} - \cancel{1} - \cancel{p} + q + \cancel{1})\right]$$

$$= \frac{p-q}{2}\left(\frac{2q}{p} - dq\right)$$

$$= \frac{p-q}{2}\left[\frac{2q}{p} + \frac{\cancel{q}^2(p+q)}{p\cancel{q}}\right] = \left[\frac{(p-q)(p+2q)}{p}\right]$$

$$S_{p-q} = \frac{(p-q)(p+2q)}{p}$$

Hence the result.

Illustration 22 If a, b and c be the respective sums of q, q and r terms of an A.P. show that $\frac{a}{p}(q-r) + \frac{b}{q}(r-p) + \frac{c}{r}(p-q) = 0.$

Solution

Let A be the first term and D be the common difference of the given A.P.

$$\therefore S_p = a = \frac{p}{2}[2A + (p-1)D]$$

$$S_q = b = \frac{q}{2}[2A + (q-1)D]$$

$$S_r = C = \frac{r}{2}[2A + (r-1)D]$$

$$\Rightarrow \frac{a}{q} = A + \frac{p-1}{2}D,\ \frac{b}{q} = A + \frac{q-1}{2}D \text{ and } \frac{c}{r} = A + \frac{r-1}{2}D$$

$$\therefore \text{L.H.S. } \frac{a}{p}(q-r) + \frac{b}{q}(r-p) + \frac{c}{r}(p-q)$$

$$\left(A + \frac{p-1}{2}D\right)(q-r) + \left(A + \frac{q-1}{2}D\right)(r-p) + \left(A + \frac{r-1}{2}D\right)(p-q)$$

$= A(q - r + r - p + p - q) + \frac{D}{2}[(p - 1)(q - r) + (q - 1)(r - p) +$
$(r - 1)(p - q)]$

$= A(0) + \frac{D}{2}(\cancel{pq} - \cancel{pr} - \cancel{q} + \cancel{r} + \cancel{qr} - \cancel{qp} - \cancel{r} + \cancel{p} + \cancel{rp} - \cancel{rq} - \cancel{p} + \cancel{q})$

$= A(0) = (0) = 0 + 0 = 0 =$ R.H.S.

Hence the result.

Illustration 23 Let A.P. proves that the sum of terms equidistant from the beginning and end is always same and equal to the sum of the first and the last terms.

Solution

Let $T_1, T_2, T_3, \ldots T_n$ be in A.P. with common difference d. Then kth term from beginning is

$T_k = a + (k - 1)d$ and kth term from end $= (n - k + 1)$th term from beginning.

$\therefore k$th term from beginning + kth term from end

$= [a + (k - 1)d] + [a + (n - k + 1 - 1)d]$

$= 2a + (k - 1 + n - k)d$

$= 2a + (n - 1)d$

$= a + [a + (n - 1)d]$

$= T_1 + T_n$

$\therefore\ T_k + T_{n-k+1} = T_1 + T_n$ for all $k = 1, 2, 3 \ldots n$.

Hence the sum of term equidistant from the beginning and end is always same and equals to the sum of the first and the last terms.

Hence the result.

Illustration 24 If $a_1, a_2 \ldots a_n$ are in arithmetic progression where $a_i > 0$ for all i show that.

$$\frac{1}{\sqrt{a_1} + \sqrt{a_2}} + \frac{1}{\sqrt{a_2} + \sqrt{a_3}} + \ldots + \frac{1}{\sqrt{a_{n-1}} + \sqrt{a_n}} = \frac{n-1}{\sqrt{a_1} + \sqrt{a_n}}$$

Solution

Let $a1, a2, a3 \ldots a_n$ are

$\Rightarrow a_2 - a_1 = a_2 - a_3 = \ldots a_n - a_{n-1} = d$

$\therefore$ L.H.S.

$$\frac{1}{\sqrt{a_1} + \sqrt{a_2}} + \frac{1}{\sqrt{a_2} + \sqrt{a_3}} + \ldots + \frac{1}{\sqrt{a_{n-1}} + \sqrt{a_n}}$$

$$= \frac{\sqrt{a_2} - \sqrt{a_1}}{a_2 - a_1} + \frac{\sqrt{a_3} - \sqrt{a_2}}{a_3 - a_2} + \ldots + \frac{\sqrt{a_n} + \sqrt{a_{n-1}}}{a_n - a_{n-1}}$$

$$= \frac{\sqrt{a_2}+\sqrt{a_1}}{d} + \frac{\sqrt{a_3}+\sqrt{a_2}}{d} + \ldots + \frac{\sqrt{a_n}+\sqrt{a_{n-1}}}{d}$$

$$= \frac{\sqrt{a_2}-\sqrt{a_1}+\sqrt{a_3}-\sqrt{a_2}+\ldots+\sqrt{a_n}+\sqrt{a_{n-1}}}{d}$$

$$= \frac{\sqrt{a_n}-\sqrt{a_1}}{d} = \frac{a_n - a_1}{d\left(\sqrt{a_n}+\sqrt{a_1}\right)}$$

but $a_n = a_1 + (n - 1)d$

$$= \frac{\cancel{a_1}+(n-1)d-\cancel{a_1}}{d\left(\sqrt{a_n}+\sqrt{a_1}\right)} = \frac{\cancel{d}(n-1)}{\cancel{d}\left(\sqrt{a_n}+\sqrt{a_1}\right)}$$

$$= \frac{n-1}{\sqrt{a_1}+\sqrt{a_n}} = \text{R.H.S.}$$

Hence the result.

Illustration 25 (i) The sum of three numbers in A.P. is 18 and their product is 192. Find the numbers.

(ii) The sum of four numbers in A.P. is 24 and product of the squares of extremes is 729. Find the numbers.

Solution

(i) Let $a - d, a, a + d$ three numbers are in A.P.

$\therefore (a - d) + (a) + (a + d) = 18$

$\Rightarrow 3a = 18 \Rightarrow a = 6$

and $(a - d)(a)(a + b) = 192$

$\Rightarrow (a^2 - d^2)6 = 192$

$\Rightarrow a^2 - d^2 = \dfrac{192}{6} = 32$

$\Rightarrow 36 - d^2 = 32d^2 = 4$

$\therefore d = \pm 2$

$\therefore a = 6, d = 2$ and $a = 6, d = -2$

$a - d, a, a + d$	$a - d, a, a + d$
4, 6, 8	8, 6, 4

Hence required numbers are 4, 6, 8.

(ii) Let $a - 3d, a - d, a + d, a + 3d$ are four numbers in A.P.

$a - 3d + a - d + a + d + a + 3d = 24$

$4a = 24 \Rightarrow a = 6$

Now square of extremes is 729

$(a^2 - 9d^2)^2 = 729$

$(36 - 9d^2)^2 = 729$

$36 - 9d^2 = \pm 27$

$$36 - 9d^2 = 27$$
$$9d^2 = 9$$
$$d^2 = 1$$
$$d = \pm 1$$
$$a = 6 \quad d = \pm 1$$
$$a - 3d,\ a - d,\ a + d,\ a + 3d$$
$$3,\ 5,\ 7,\ 9$$

$$36 - 9d^2 = -27$$
$$63 = 9d^2$$
$$d^2 = 7$$
$$d = \pm\sqrt{7}$$
$$a = 6,\ d = \pm\sqrt{7}$$
$$a - 3d,\ a - d,\ a + d,\ a + 3d$$
$$6-3\sqrt{7},\ 6-\sqrt{7},\ 6+\sqrt{7},\ 6+3\sqrt{7}$$

Hence there are two sets of numbers satisfying the given conditions. These are $(3,\ 5,\ 7,\ 9)$ and $\left(6-3\sqrt{7},\ 6-\sqrt{7},\ 6+\sqrt{7},\ 6+3\sqrt{7}\right)$.

Hence the result.

Illustration 26 Insert six arithmetic means between 2 and 16. Also prove that their sum is 6 times the A.M. between 2 and 16.

Solution

Let A_1, A . . . A_6 be six AMs between 2 and 16.

$\therefore\ a = 2,\ b = 16,\ n = 6.$

$$\therefore d = \frac{b-a}{n+1} = \frac{16-2}{7} = \frac{14}{7} = 2$$

$$\therefore A_1 = a + d = 2 + 2 = 4$$
$$A_2 = a + 2d = 2 + 4 = 6$$
$$A_3 = a + 3d = 2 + 6 = 8$$
$$A_4 = a + 4d = 2 + 8 = 10$$
$$A_5 = a + 5d = 2 + 10 = 12$$
$$A_6 = a + 6d = 2 + 12 = 14$$

Now sum of these means

$= 4 + 6 + 8 + 10 + 12 + 14$

$$= \frac{6}{2}(4 + 14) = 3(18) = 54$$

= 6 times the A.M. between 2 and 16.

Hence the result.

Illustration 27 The ratio of second to seventh of n A.M. between −7 and 65 is 1 : 7. Find n.

Solution

Let A_1, A . . . A_n be n AMs between − 7 and 65, then as 65 is the $(n + 2)$th term

$$\therefore 65 = -7 + (n + 2 - 1)d$$
$$72 = (n + 1)d$$
$$\therefore d = \frac{72}{n+1}$$

Now $\dfrac{A_2}{A_7} = \dfrac{1}{7} \Rightarrow \dfrac{-1+2d}{-7+7d} = \dfrac{1}{7}$

$$\frac{-7+2d}{-1+d}=1 \Rightarrow -7+7d = -1 + d$$

$\therefore\ d = 6$

$$\therefore 6=\frac{72}{n+1} \Rightarrow n+1=\frac{72}{6}=12$$

$\Rightarrow n + 1 = 12 \Rightarrow n = 11$

Hence the result.

Illustration 28 The sum of two numbers is $\frac{13}{6}$. An even number of arithmetic means are inserted between them and their sum exceeds their number by 1. Find the number of means inserted.

Solution

Let a and b the two numbers so $a+b=\frac{13}{6}$

Let $2n$ AMs be inserted between a and b.

$\therefore$ The sum of these $2n$ AMs

$= 2n \times$ A.M. between a and b

$$= 2n\times\frac{a+b}{2} = n(a + b)$$

$$= \frac{13n}{6}$$

Also by given condition, sum of these $2n$ AMs = number of means + 1.

$$\frac{13n}{6}=2n+1$$

$\Rightarrow 13n = 12n + 6$

$n = 6$

Hence number of means inserted

$= 2n = 12$

Hence the result.

Illustration 29 (i) If A is the A.M. between a and b show that

$$\frac{A+2a}{A-b}+\frac{A+2b}{A-a}=4$$

(ii) Find n so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the A.M. between a and b.

Solution

(i) Let A be the A.M. between a and b

$$\therefore A=\frac{a+b}{2}$$

$$\therefore \text{L.H.S.} = \frac{A+2a}{A-b}+\frac{A+2b}{A-a}$$

$$= \frac{[(a+b)/2]+2a}{[(a+b)/2]-b} + \frac{[(a+b)/2]+2b}{[(a+b)/2]-a}$$

$$= \frac{5a+b}{a-b} + \frac{a+5b}{b-a}$$

$$= \frac{(5a+b)-(5b+a)}{a-b}$$

$$= \frac{4a-4b}{a-b} = \frac{4(a-b)}{(a-b)}$$

= 4 R.H.S.

(ii) Let A.M. between a and b is $\frac{a+b}{2}$

Thus we have

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{a+b}{2}$$

$\Rightarrow 2a^{n+1} + 2b^{n+1} = a^{n+1} + ba^n + ab^n + b^{n+1}$

$\Rightarrow a^{n+1} - ba^n - ab^n(a - b) = 0$

$(a - b)(a^n - b^n) = 0$

$\Rightarrow a^n - b^n = 0 \qquad (\because a \neq b \Rightarrow a - b \neq 0)$

$\Rightarrow a^n = b^n$

$$\Rightarrow \left(\frac{a}{b}\right)^n = 1 = \left(\frac{a}{b}\right)^0$$

$\therefore$ $n = 0$ is the required solution, hence the result.

Illustration 30 (i) Find the nth and the 15th term of the series

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ . \ . \ .$$

(ii) Which term of the sequence 18, −12, 8 . . . is $\frac{512}{729}$?

Solution

Let $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{4} - \frac{1}{8} + \ . \ . \ .$ are in G.P.

(i) $\therefore$ $a = 1$, $r = -\frac{1}{2}$

$\therefore f(n) = ar^{n-1}$

$$= 1\left(\frac{-1}{2}\right)^{n-1} = (-1)^{n-1} 2^{1-n}$$

$\therefore f(15) = (-1)^{14} \times 2^{-14} = 2^{-14}$

$\therefore$ nth term is $(-1)^{n-1} \times 2^{1-n}$ and 15th term is 2^{-14}

(ii) Let 18, −12, 8 . . . be in G.P.

$a = 18,\ r = \frac{-12}{18} = -\frac{2}{3}$ and

$f(n) = \frac{512}{729}$

$\therefore f(n) = ar^{n-1}$

$\frac{512}{729} = 18\left(-\frac{2}{3}\right)^{n-1}$

$\Rightarrow \left(-\frac{2}{3}\right)^{n-1} = \frac{2^8}{3^8} = \left(\frac{-2}{3}\right)^8$

$\therefore n - 1 = 8$

$\Rightarrow n = 9$

$\therefore$ 9th term of G.P. is $\frac{512}{729}$

Illustration 31 Find the 5th term from the end of the sequence 2, 6, 18 . . . 39, 366.

Solution

2, 6, 18 . . . 39,366 is a G.P. with common ratio = $\frac{6}{2}$ = 3 and last term =

l = 39,366

$\therefore$ The fifth term from the end

$= l\left(\frac{1}{r}\right)^{n-1}$

$= 39{,}366\left(\frac{1}{3}\right)^4 = \frac{39{,}366}{81} = 486$

Illustration 32 If the third, sixth and the last terms of a G.P. are 6, 48 and 3,072 respectively, find the first term in the G.P.

Solution

Let $f(3) = 6$, $f(6) = 48$ and
$f(n) = 3{,}072$ be in G.P.
$\therefore f(3) = ar^2$, $f(6) = 48$
$6 = ar^2 \qquad ar^5 = 48$

$\frac{ar^5}{ar^2} = \frac{48}{6} \Rightarrow r^3 = 8$

$\Rightarrow r = 2$

and $f(n) = 3{,}072 \qquad 6 = a4$

$\Rightarrow ar^{n-1} = 3{,}072 \qquad a = \frac{3}{2}$

$$\frac{\cancel{3}}{2}(2)^{n-1} = \overset{1,024}{\cancel{3,072}}$$

$\Rightarrow 2^{n-1} = 2,048 \Rightarrow 2^{n-1} = 2^{11}$

$n - 1 = 11$

$n = 12$

$\therefore$ First term $= a = \frac{3}{2}$ $a = \frac{3}{2}$ $a = \frac{3}{2}$ v and number of terms = 12.

Illustration 33 The first term of a G.P. with real terms is 2. If the sum of its third and fifth terms is 180, find the common ratio of the G.P.

Solution

Let first term of G.P. $= a = 2$

and $f(3) + f(5) = 180$

$\Rightarrow ar^2 + ar^4 = 180$

$\Rightarrow 2r^2 + 2r^4 = 180$

$\Rightarrow r^2 + r^4 = 90$

$\Rightarrow r^4 + r^2 - 90 = 0$

$(r^2 + 10)(r^2 - 9) = 0$

$r^2 = -10$ or $r^2 = 9$

is not possible $\because$ r is real

$\Rightarrow r = \pm 3$

Hence the result.

Illustration 34 If the sum of the first two terms of a G.P. is −8 and the fifth term is 9 spale times the third term, find the G.P.

Solution

Let $T_1 + T_2 = -8$ of a G.P.

$\Rightarrow a + ar = -8$ and $T_5 = 9T_3 \Rightarrow ar^4 = 9ar^2$

$\therefore r = 3$	$r = -3$	$r^2 = 9$
$a + 3a = -8$	$a - 3a = -8$	$r = \pm 3$
$4a = -8$	$-2a = -8$	
$a = -2$	$a = 4$	

$\therefore a = -2, r = 3$

Required G.P. is

$-2, -6, -18 \ldots$

and $a = 4, r = -3$

Required G.P. is

$4, -12, +36 \ldots$

Hence the result.

Illustration 35 Three numbers whose sum is 21 are in A.P. If 2, 2, 14 are added to them respectively, the resulting numbers are in G.P. Find the numbers.

Solution

Let $a - d, a, a + d$ be in A.P.

$\therefore (a - b) + (a) + (a + b) = 21$

$3a = 21 \Rightarrow a = 7$

$\therefore$ Numbers are $7 - d, 7, 7 + d$.

If 2, 2, 14 respectively are added to the term we get $9 - d, 9, 21 + d$.

$\therefore 81 = (9 - d)\,(21 + d)$

$81 = 189 - 12d - d^2$

$\therefore d^2 + 12d - 108 = 0$

$(d + 18)(d - 6) = 0$

$\therefore d = -18$ or $d = 6$

$\therefore a = 7, d = -18 \Rightarrow a - d, a, a + d$

$= 25, 7, -11$

and $a = 7, d = 6 \Rightarrow a - d, a, a + d$

$= 1, 7, 13$

Thus two sets of numbers are 1, 7, 13 or 25, 7, −11.

Hence the result.

Illustration 36 (i) The third term of a G.P. is 4. Find the product of its first five terms.

(ii) If the fourth, seventh and tenth terms of a G.P. are *p, q, r* respectively then show that $q^2 = pr$.

Solution

(i) Let *a* be the first term and *r* be the common ratio between any two consecutive terms.

$\therefore ar^2 = 4$

Now product of first five terms is

$(a)(ar)(ar^2)(ar^3)(ar^4)$

$a^5r^{10} = (ar^2)^5 = (4)^5$

$= 2^{10} = 1024$

Hence the result.

(ii) Let $f(4) - \text{p}, f(4) = \text{P}, f(7) = q$ and $f(10) = r$.

$\therefore ar^3 = p, ar^6 = q, ar^9 = r.$

$\therefore$ L.H.S. q^2

$= (ar^6)^2 = a^2r^{12}$

$= (ar^3)(ar^9)$

$= pr =$ R.H.S.

Illustration 37 If x, $2x + 2$, $3x + 3$ are first three terms of a geometric sequence, find its fourth term.

Solution

Let x, $2x + 2$, $3x + 3$ be in G.P.

$$\therefore\ (2x+2)^2 = x(3x+3)$$

$$4(x+1)^2 = x3(x+1)$$

$$4x + 4 = 3x$$

$$x = -4 \quad (x \neq -1 \qquad x + 1 \neq 0)$$

$\therefore$ First three terms are −4, −6, −9 which is a G.P.

with $r = \dfrac{3}{2}$

$$\therefore\ f(4) = ar^3 = (-4)\left(\frac{3}{2}\right)^3 = \cancel{-4}\frac{27}{\underset{2}{\cancel{8}}}$$

$$\therefore\ f(4) = -\frac{27}{2}$$

Hence the result.

Illustration 38 (i) If a, b, c, d are in G.P. prove that $a + b$, $b + c$, $c + d$ are also in G.P.

(ii) For all sequences which are simultaneously arithmetic and geometric progression.

Solution

a, b, c, d are in G.P.

(i) $\Rightarrow \dfrac{b}{a} = \dfrac{c}{b} = \dfrac{d}{c} = r$

$$\begin{array}{l|l} \therefore\ b = ar,\ c = br & d = cr \\ \qquad\qquad\ \ = arr & \quad = ar^2 r \\ \qquad\qquad\ \ = ar^2 & \quad = ar^3 \end{array}$$

$$\begin{array}{l|l} \text{Now } (b+c)^2 & (a+b)(c+d) \\ = (ar + ar^2)^2 & (a+ar)(ar^2+ar^3) \\ = a^2r^2(1+r)^2 & a(1+r)\cdot ar^2(1+r) \\ & a^2r^2(1+r)^2 \end{array}$$

$$\therefore\ (b+c)^2 = (a+b)(c+d)$$

$\Rightarrow a + b$, $b + c$, $c + d$ are in G.P.

(ii) Let $a_1, a_2, a_3 \ldots a_n \ldots$ be in A.P. as well as in G.P.

Since it is in A.P.

$$a_{n+1} = \frac{a_n + a_{n+2}}{2},\ n \geq 1$$

Since it is in G.P.

$$a_n = a_1 r^{n-1}$$

$$a_{n+1} = a_1$$

$$a_{n+1} = a_1$$

$$\therefore\ a_1 r^n = \frac{a_1 r^{n-1} + a_1 r^{n+1}}{2}$$

$$2\cancel{a_1 r^n} = \cancel{a_1 r^n}\left(\frac{1}{r} + r\right)$$

$\therefore$ Only *a* constant sequence $a, a, a \ldots \left(a \neq 0\right)$ is both an A.P. and G.P. Hence the result.

Illustration 39 (i) If *p*th, *q*th and *r*th terms of a G.P. are themselves in G.P. show that *p, q, r* are in A.P.

(ii) If the *p*th, *q*th and *r*th terms of a G.P. are *a*, *b* and *c* respectively show that $a^{r-r}, b^{r-p}, c^{p-q} = 1$.

(iii) If *x, y, z* are three positive numbers forming a geometric sequence, then show that $\log_a^x, \log_a^y, \log_a^z$ form an arithmetic sequence, *a* being a positive number equal to 1.

Solution

(i) Let A be the first term and R be the common ratio then

$T_p = AR^{p-1}, T_q = AR^{q-1}, Tr = AR^{r-1}$

Now T_p, T_q, T_r are in G.P.

$T_q^2 = T_p\, T_r$

$\cancel{A^2}\, R^{2q-2} = \cancel{A}R^{p-1}\cancel{A}R^{r-1}$

$R^{2q-2} = R^{p+r-2}$

$\therefore\ 2q - \cancel{2} = P + r - \cancel{2}$

$$q = \frac{p+r}{2}$$

$\Rightarrow p, q, r$ are in A.P.

Hence the result.

(ii) Let *p*th, *q*th and *r*th terms of G.P. be

$T_p = AR^{p-1}, T_q = AR^{q-1}, T_r = AR^{r-1}$

$\therefore\ a = AR^{p-1}, b = AR^{q-1}, c = AR^{r-1}$

$\therefore$ L.H.S. $a^{q-r}\, b^{r-p}\, c^{p-q}$

$A^{q-r}\, R^{(p-1)(q-r)} A^{r-p} R^{(q-1)(r-p)}$

$= A^{\cancel{q}-\cancel{r}+\cancel{r}-\cancel{p}+\cancel{p}-\cancel{q}} R^{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)}$

$\cancel{p}q - \cancel{p}r - \cancel{q} + \cancel{r} + \cancel{q}r - \cancel{q}p - \cancel{r} + \cancel{p} + \cancel{r}p - \cancel{r}q - \cancel{p} + \cancel{q}$

$= A^0\, R$

$= A^0\, R^0$

$= \left(1\right)\left(1\right)$

$= \left(1\right)$ = R.H.S.

Hence the result.

(iii) Let *x*, *y*, *z* be in G.P.

$y^2 = xz$

Taking log on both sides with base *a*

$$\log_a^{y^2} = \log_a^{xz}$$

$$\Rightarrow 2\log_a^y = \log_a^x + \log_z^z$$

$\therefore \log_a^x, \log_a^y, \log_a^z$ are in A.P.

Hence the result.

Illustration 40 If a, b, c are the pth, qth and rth terms respectively of an A.P. and also the pth, qth and rth terms of a G.P. prove that a^{b-c}, b^{c-a}, $c^{a-b} = 1$.

Solution

Let the A.P. be a, $A + D$, $A + 2D$ . . . and G.P. be x, xR, xR^2 . . . then

$$\left.\begin{aligned} a &= A+(p-1)D \\ b &= A+(q-1)D \\ c &= A+(r-1)D \end{aligned}\right\} \Rightarrow \begin{aligned} a-b &= (p-q)D \\ b-c &= (q-r)D \\ c-a &= (r-p)D \end{aligned}$$

$\therefore$ L.H.S. $a^{b-c}b^{c-aa-b}$

$$\Rightarrow \left[xR^{(p-1)}\right]^{(q-r)D}\left[xR^{(q-1)}\right]^{(r-p)D}\left[xR^{(r-1)}\right]^{(p-q)D}$$

$$\Rightarrow x^{D(\cancel{q}\cancel{r}+\cancel{r}-\cancel{p}+\cancel{p}-\cancel{q})}R^{D[(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)]}$$

$$\Rightarrow x^{D(\cancel{q}\cancel{r}+\cancel{r}-\cancel{p}+\cancel{p}-\cancel{q})}R^{D[(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)]}$$

$$= x^0R^0 = (1)(1) = 1 = \text{RHS}$$

Hence the result.

Illustration 41 (i) Find three numbers in G.P. whose sum is 19 and product is 216.

(ii) The sum of four numbers in G.P. is 60 and the arithmetic mean of the first and last is 18. Find the numbers.

Solution

(i) Let $\frac{a}{r}, a, ar$ be in G.P.

$$\Rightarrow \frac{a}{r} + a + ar = 19 \quad and \quad \frac{a}{\cancel{r}}a\,a\cancel{r} = 216$$

$$\frac{a}{r} + 6 + 6r = 19 \qquad a^3 = 6^3$$

$$a = 6$$

$$\frac{6}{r} + 6r = 13 \Rightarrow 6r^2 - 13r + 6 = 0 \text{ v}$$

$\therefore 6r^2 - 9r - 4r + 6 = 0$

$\Rightarrow 3r(2r - 3) - 2(2r - 3) = 0$

$(3r - 2)(2r - 3) = 0$

$\therefore r = \frac{2}{3}$	$r = \frac{3}{2}$
and $a = 6$	and $a = 6$

$\therefore \frac{a}{r}, a, ar$

$\frac{6}{2/3}, 6, 6$

9, 6, 4

$\frac{a}{r}, a, ar$

$\frac{\overset{2}{\cancel{6}}}{\cancel{3}/2}, 6, \overset{3}{\cancel{6}}\frac{3}{\cancel{2}}$

4, 6, 9

Hence three numbers are

9, 6, 4 or 4, 6, 9

Hence the result.

(ii) Let a, ar, ar^2, ar^3 be in G.P.

$a + ar + ar^2 + ar^3 = 60$ and $\frac{a + ar^3}{2} = 18$

$(ar + ar^2) + 36 = 60$

$ar(1 + r) = 24$

$a(1 + r^3) = 36$

$$\therefore \frac{\cancel{a}\cancel{(1+r)}(1-r+r^2)}{\cancel{a}r\cancel{(1+r)}} = \frac{36}{24} = \frac{3}{2}$$

$2 - 2r + 2r^2 = 34$

$2r^2 - 5r + 2 = 0$

$2r^2 - 4r - r + 2 = 0$

$2r(r - 2) - (r - 2) = 0$

$(r - 2)(2r - 1) = 0$

$\therefore r = 2$

$a(1 + 8) = 36$

$a = \frac{36}{9} = 4$

a, ar, ar^2, ar^3

4, 8, 16, 32

$r = \frac{1}{2}$ $a\left(1+\frac{1}{8}\right) = 36$

$a = \frac{\overset{4}{\cancel{36}}\times 8}{\cancel{9}} = 32$

$a = 32$

$a, ar, ar^2, ar^3,$

3, 2, 16, 8, 4

$\therefore$ Required numbers are 4, 8, 16, 32

Illustration 42 (i) Find the sum of the series $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+ \ldots$ up 8 terms.

(ii) Find the sum of the series. $\frac{2}{9}+\frac{1}{3}+\frac{1}{2} \ldots + \frac{81}{32}$

Solution

(i) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+ \ldots$ up to 8 terms

$a = 1, r = -\frac{1}{2}(r < 1)\ n = 8$

$$\therefore S_n = \frac{a\left(1-r^n\right)}{1-r} \Rightarrow S_8 = \frac{1\left[1-\left(-1/2\right)^8\right]}{1-\left(1/2\right)}$$

$$= \frac{2}{3}\left(1-\frac{1}{256}\right)$$

$$= \frac{85}{128}$$

Hence the result.

(ii) $\frac{2}{9}+\frac{1}{3}+\frac{1}{2}+\ .\ .\ .\ +\frac{81}{32}$

$$a = \frac{2}{9},\ r = \frac{3}{2},\ f\left(n\right) = \frac{81}{32},\ f\left(n\right) = \frac{81}{32}$$

$$\therefore\ f\left(n\right) = ar^{n-1}$$

$$\frac{81}{32} = \frac{2}{9}\left(\frac{3}{2}\right)^{n-1}$$

$$\Rightarrow \left(\frac{3}{2}\right)^6 = \left(\frac{3}{2}\right)^{n-1} \Rightarrow 6 = n-1$$

$$\Rightarrow n = 7$$

$$\therefore S_n = \frac{a\left(r^n-1\right)}{r-1}\left(r>1\right)$$

$$S_7 = \frac{\left(2/9\right)\left[\left(3/2\right)^7-1\right]}{\left(3/2\right)-1} = \frac{4}{9}\left(\frac{2187}{128}-1\right)$$

$$= \frac{\cancel{4}}{9}\left(\frac{2059}{\underset{32}{\cancel{128}}}\right) = \frac{2059}{288}$$

Illustration 43 (i) How many terms of the series $\sqrt{3}+3+3\sqrt{3}+\ .\ .\ .$ will make the sum $39+13\sqrt{3}$?

(ii) Evaluate $\sum_{n=1}^{50}\left(2^n-1\right)$.

Solution

(i) Let $a=\sqrt{3}$, $r=\sqrt{3}\left(r>1\right)$ and $S_n = 39+13\sqrt{3}$

$$\therefore S_n = \frac{a\left(r^n-1\right)}{r-1}\ \text{vv}$$

$$\Rightarrow 39+13\sqrt{3} = \frac{\sqrt{3}\left[\left(\sqrt{3}\right)^n-1\right]}{\sqrt{3}-1}$$

$$\Rightarrow \frac{39\sqrt{3} - \cancel{39} + \cancel{39} - 13\sqrt{3}}{\sqrt{3}} = \left(\sqrt{3}\right)^n - 1$$

$$\frac{26\cancel{\sqrt{3}}}{\cancel{\sqrt{3}}} + 1 = \left(\sqrt{3}\right)^n$$

$$27 = 3^{n/2}$$

$$\Rightarrow 3^{n/2} = 3^3 \Rightarrow \frac{n}{2} = 3$$

$\Rightarrow n = 6$

Hence the required number of terms = 6.

Hence the result.

(ii) $\sum_{n=1}^{50}\left(2^n - 1\right)$

$$= \sum_{n=1}^{50} 2n - \sum_{n=1}^{50} 1$$

$$= \left(2 + 2^2 + 2^3 + \ldots + 2^{50}\right) - 50$$

$$= \frac{2\left(2^{50} - 1\right)}{2 - 1} - 50 = 2^{51} - 2 - 50$$

$$= 2^{51} - 52$$

Hence the result.

Illustration 44 Find the number of terms of a geometric sequence $\{a_n\}$ if $a_1 = 3$, $a_n = 96$ and $s_n = 189$.

Solution

Let $a_1 = 3$ $\quad an = a_1 r^{n-1}$

$96 = 3r^{n-1}$

$r^{n-1} = 32 \Rightarrow r^n = 32r$

$$S_n = \frac{a_1\left(r^n - 1\right)}{r - 1}$$

$$189 = 3\frac{\left(r^n - 1\right)}{r - 1} \Rightarrow 63\left(r - 1\right) = r^n - 1$$

$63r - 62 = r^n$

$63r - 62 = 32r$

$31r = 62$

$\Rightarrow r = 2$

$\therefore r^{n-1} = 32$

$2^{n-1} = 2^5$

$n - 1 = 5$

$\Rightarrow n = 6$

Hence the number of terms = 6.

Hence the result.

Illustration 45 In an increasing G.P. the sum of the first and last term is 66, the product of the second and the last but one term is 128. If the sum of the series is 126, find the number of terms in the series.

Solution

Let $T_1 + T_n = 66a + a2^{n-1} = 66 \Rightarrow a2^{n-1} = 66 - a$

and $T_2T_{n-1} = 128 \Rightarrow ar^1ar^{n-2} = 128$

$\Rightarrow aar^{n-1} = 128$

$\Rightarrow a(66 - a) = 128$

$a^2 - 66a + 128 = 0$

$\Rightarrow a = 2$ or 64

$\therefore S_n = 126 \Rightarrow a\dfrac{r^n - 1}{r - 1} = 126$

Now $a = 2$ $\quad 2 + 2r^{n-1} = 66$

$2r^{n-1} = 64$

$r^{n-1} = 32 \Rightarrow r^{n-1} = 32$

$\therefore 2\dfrac{r^n - 1}{r - 1} = 126 \Rightarrow \dfrac{r^n - 1}{r - 1} = 63$

$\Rightarrow \dfrac{r\,r^{n-1} - 1}{r - 1} = 63$

$\Rightarrow r32 - 1 = 63r - 63$

$62 = 31r \Rightarrow r = 2$

$\therefore 2^{n-1} = 2^5 \Rightarrow n - 1 = 5 \Rightarrow n = 6$

Now $a = 64$, $64 + 64r^{n-1} = 66$

$64r^{n-1} = 2$

$r^{n-1} = \dfrac{1}{32}$

$\therefore 64\dfrac{r^n - 1}{r - 1} = 126$

$32\dfrac{r\,r^{n-1} - 1}{r - 1} = 62$

$32\dfrac{r(1/32) - 1}{r - 1} = 63$

$32\left(\dfrac{r}{32} - 1\right) = 63r - 63$

$r - 32 = 63r - 63$

$$31 = 62r \Rightarrow r = \frac{1}{2}$$

but $r > 1$. Hence $a = 64$ is not possible.

$\therefore$ $a = 2$. We get the number of terms in the given G.P. is 6.

Hence the result.

Illustration 46 (i) Find the sum of first n terms of the series $5 + 55 + 555 + \ldots$

(ii) $0.5 + 0.55 + 0.555 + \ldots$

(iii) $\left(x+\frac{1}{x}\right)^2, \left(x^2+\frac{1}{x^2}\right)^2, \left(x^3+\frac{1}{x^3}\right)^2$

Solution

(i) $5 + 55 + 555 + \ldots$ up to n terms

$$= 5(1+11+111+\ldots \text{ up to } n \text{ term})$$

$$= \frac{5}{9}(9+99+999+ \ldots \text{ up to } n \text{ term})$$

$$= \frac{5}{9}(9+99+999+ \ldots \text{ up to } n \text{ term})$$

$$= \frac{5}{9}\left[(10-1)+(10^2-1)+(10^3-1)+ \ldots \text{ up to } n \text{ term}\right]$$

$$= \frac{5}{9}\left[(10+10^2+ \ldots \text{ up to } n \text{ term}) - (1+1+1 \ldots \text{ up to } n \text{ term})\right]$$

$$= \frac{5}{9}\left[\frac{10(10^n-1)}{10-1} - n\right]$$

$$= \frac{5}{81}(10^{n+1}-10) - \frac{5n}{9}$$

$$= \frac{5}{81}(10^{n+1}-10-9n) \text{ v}$$

Hence the result.

(ii) $0.5 + 0.55 + \ldots$ up to n terms.

$$\frac{5}{9}(0.9+0.99+ \ldots \text{ up to } n \text{ term})$$

$$\frac{5}{9}\left[(1-1)+(1-01)+ \ldots \text{ up to } n \text{ term}\right]$$

$$\frac{5}{9}\left[(1+1+1 \ldots \text{ up to } n \text{ term}) - \left(\frac{1}{10}+\frac{1}{10^2}+ \ldots \text{ up to } n \text{ term}\right)\right]$$

$$\frac{5}{9}\left\{n - \frac{1}{10}\frac{\left[1-(1/10)^n\right]}{1-(1/10)}\right\}$$

$$\frac{5}{9}\left[n-\frac{1}{9}\left(1-\frac{1}{10^n}\right)\right] \mathrm{V}$$

$$\frac{5}{81}\left(9n-1+\frac{1}{10^n}\right)$$

(iii) $$S_n = \left(x+\frac{1}{x}\right)^2+\left(x^2+\frac{1}{x^2}\right)^2+\ \ldots\ +\left(x^n+\frac{1}{x^n}\right)^2$$

$$= \left(x^2+\frac{1}{x^2}+2\right)+\left(x^4+\frac{1}{x^4}+2\right)+\left(x^6+\frac{1}{x^6}+2\right)+\ \ldots\ +\left(x^{2n}+\frac{1}{x_{2n}}+2\right)$$

$$= \left(x^2+x^4+x^6+\ \ldots\ +x2^n\right)+\left(\frac{1}{x^2}+\frac{1}{x^4}+\frac{1}{x^6}+\ \ldots\ +\frac{1}{x2^6}\right)+$$

$$\left(2+2+2+\ \ldots\ +\text{up to } n \text{ term}\right)$$

$$= \frac{x^2\left(x^{2n}-1\right)}{x^2-1}+\frac{\left(1/x^2\right)\left[\left(1/x^2\right)^n-1\right]}{\left(1/x^2\right)-1}+2n$$

$$= \frac{x^2\left(x^{2n}-1\right)}{x^2-1}+\frac{1-x^{2n}}{x^{2n}\left(1-x^2\right)}+2n$$

$$= \frac{x^{2n}-1}{x^2-1}\left(x^2+\frac{1}{x^{2n}}\right)+2n$$

Hence the result.

Illustration 47 Sum the series $\left(x+y\right)+\left(x^2+xy+y^2\right)+\left(x^3+x^2y+xy^2+y^3\right)$ + . . . up to *n* terms.

Solution

$$S_n = \left(x+y\right)+\left(x^2+xy+y^2\right)+\left(x^3+x^2y+xy^2+y^3\right)+\ \ldots\ \text{up to } n \text{ terms}$$

$$= \frac{1}{(x-y)}\left[\left(x^2-y^2\right)+\left(x^3-y^3\right)+\left(x^4-y^4\right)+\ \ldots \text{up to } n \text{ term}\right]$$

$$= \frac{1}{x-y}\left[\left(x^2+x^3+x^4+\ \ldots \text{up to } n \text{ term}\right)\right.$$

$$\left.-\left(y^2+y^3+y^4+\ \ldots \text{up to } n \text{ term}\right)\right]$$

$$= \frac{1}{x-y}\left[\frac{x^2\left(1-x^n\right)}{1-x}-\frac{y^2\left(1-y^n\right)}{1-y}\right]$$

Hence the result.

Illustration 48 If S_n represents the sum of *n* terms of a G.P. with first term *a* and common ratio *r*, then find the value of $S_1 + S_2 + S_5 + \ldots + S_{2n-1}$

Solution

Let $S_n = a\dfrac{1-r^n}{1-r}$ ss

$$\therefore S_1 + S_3 + S_5 + \ldots S_{2n-1} = \sum_{k=1}^{n} S_{2k-1}$$

$$= \sum_{k=1}^{n} a\,\frac{1-r^{2k-1}}{1-r}$$

$$= \frac{a}{1-r}\sum_{k=1}^{n}\left(1-r^{2k-1}\right)$$

$$= \frac{a}{1-r}\left[n-\sum_{k=1}^{n}\left(1-r^{2k-1}\right)\right]$$

$$= \frac{a}{1-r}\left[n-r\,\frac{1-\left(r^2\right)^n}{1-r^2}\right]$$

$$= \frac{na}{1-r}-\frac{ar\left(1-r^{2n}\right)}{\left(1-r\right)^2\left(1+r\right)}$$

Hence the result.

Illustration 49 Insert 4 geometric means between 3 and 96. Also show that their product is the 4th power of the G.M. between them.

Solution

Let G_1, G_2, G_3, G_4 be 4 GMs between 3 and 96.
$a = 3$, $b = 96$. Total no. of terms = 6

$$r = \left(\frac{b}{a}\right)^{1/n+1} = \left(\frac{96}{3}\right)^{1/5} \qquad (\because n = 4)$$

$$= (32)^{1/5} = (25)^{1/5} = 2$$

$\therefore G_1 = ar = 6$
$G_2 = ar^2 = 12$
$G_3 = ar^3 = 24$
$G_4 = ar^4 = 48$

G.M. of 3 and 96 = G = $\sqrt{3 \times 96}$

$$= \sqrt{288} = 12\sqrt{2}$$

and $G_1G_2G_3G_4 = 6 \times 12 \times 24 \times 48 = 12^4 \times 2^2$

$$= \left(12\sqrt{2}\right)^4$$

$$= G_4$$

Hence the result.

Illustration 50 (i) The A.M. between two numbers is 34 and G.M. is 16. Find the numbers.

(ii) Show that A.M. between two distinct positive numbers is always greater than G.M.

(iii) If x, y, z are distinct positive numbers, prove that. $(x+y)\,(y+z)\,(z+x) > 8xyz$.

Further if $x+y+z=1$, show that $(1-x)\,(1-y)\,(1-z) > 8xyz$.

Solution

(i) Let a and b be any two numbers.

$$\therefore \text{ A.M.} = 34 = \frac{a+b}{2} \Rightarrow \quad a + b = 68 \qquad \text{(i)}$$

$$b = 68 - a$$

and G.M. $= 16 = \sqrt{ab} \Rightarrow ab = 256$

$\therefore\ a(68-a) = 256$

$\Rightarrow a^2 - 68a + 256 = 0$

$\Rightarrow (a-64)(a-4) = 0$

$\therefore\ a = 64$ or $a = 4$

$b = 68 - a \Rightarrow b = 4$ or 64.

Required numbers are 4 and 64.

Hence the result.

(ii) Let a and b be any two positive numbers.

$\therefore$ A.M. $= \dfrac{a+b}{2}$ and

G.M. $= \sqrt{ab}$

$$\therefore A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2}$$

$$= \frac{\left(\sqrt{a}\right)^2 + \left(\sqrt{b}\right)^2 - 2\sqrt{a}\,\sqrt{b}}{2}$$

$$= \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2} > 0$$

$\therefore A - G > 0 \Rightarrow A > G$

Hence the result.

(iii) Let x, y, z be distinct positive numbers.

$$\therefore \frac{x+y}{2} > \sqrt{xy} \Rightarrow x+y > 2\sqrt{xy} \qquad (\because A > G)$$

$\therefore\ y+z > 2\sqrt{yz}$ and $z+x > 2\sqrt{zx}$.

$\therefore (x+y)(y+z)(z+x) > 2\sqrt{xy}\ 2\sqrt{yz}\,2\sqrt{zx}$

Now $x+y+z=1 \Rightarrow x+y = 1-z,\ y+z = 1-x$ and $z+x = 1-y$

$$\therefore\ (1-z)(1-x)(1-y) > 8xyz$$

$$\therefore\ (1-z)(1-y)(1-z) > 8xyz$$

Hence the result.

Illustration 51 (i) If G is the geometric mean between two distinct positive numbers a and b, then show that $\dfrac{1}{G-a}+\dfrac{1}{G-b}=\dfrac{1}{G}$

(ii) The arithmetic mean between two distinct positive real numbers is twice the geometric mean between them. Find the ratio of the greater to the smaller.

Solution

(i) Let $G = \sqrt{ab}$

$$\text{LHS } \frac{1}{G-a}+\frac{1}{G-b}$$

$$= \frac{1}{\sqrt{ab}-a}+\frac{1}{\sqrt{ab}-b}$$

$$= \frac{1}{\sqrt{a}\left(\sqrt{b}-\sqrt{a}\right)}-\frac{1}{\sqrt{b}\left(\sqrt{b}-\sqrt{a}\right)}$$

$$= \frac{\cancel{\sqrt{b}-\sqrt{a}}}{\sqrt{a}\,\sqrt{b}\left[\cancel{\sqrt{b}-\sqrt{a}}\right]}=\frac{1}{\sqrt{ab}}=\frac{1}{G}$$

(ii) Let a and b be given any two distinct numbers where $a > b > 0$.

$\therefore$ A.M = 2 G.M.

$$\frac{a+b}{2}=2\sqrt{ab}$$

$$\frac{a+b}{2\sqrt{ab}}=\frac{2}{1}$$

Take componendo to dividendo.

$$\frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}}=\frac{2+1}{2-1}$$

$$\frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{\left(\sqrt{a}-\sqrt{b}\right)^2}=\frac{3}{1}$$

$$\Rightarrow \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\frac{\sqrt{3}}{1}$$

Again take componendo to dividendo.

$$\frac{\cancel{2}\sqrt{a}}{\cancel{2}\sqrt{b}}=\frac{\sqrt{3}+1}{\sqrt{3}-1}\Rightarrow\frac{a}{b}=\frac{\left(\sqrt{3}+1\right)^2}{\left(\sqrt{3}-1\right)^2}$$

$$\Rightarrow \frac{a}{b} = \frac{4+2\sqrt{3}}{4-2\sqrt{3}} = \frac{2+\sqrt{3}}{2-\sqrt{3}}$$

$\therefore a : b :: \left(2+\sqrt{3}\right) : \left(2-\sqrt{3}\right)$

Hence the result.

Illustration 52 If y is the geometric mean between x and z, and $a^x + b^y = c^z$, then prove that $\log_b^a \cdot \log_b^c = 1$.

Solution

Let $a^x = b^y = c^z = k$ (say)

$x = \log_a^k,\ y = \log_b^k,\ z = \log_c^k$

Now y is the G.M. between x and z.

$\Rightarrow y^2 = xz$

$$\left(\log_b^k\right)^2 = \left(\log_a^k\right)\left(\log_c^k\right)$$

$$\frac{\log_b^k}{\log_a^k} = \frac{\log_c^k}{\log_b^k}$$

$\Rightarrow \log_b^a = \log_c^b$

$$\Rightarrow \log_b^a = \frac{1}{\log_b^c}$$

$\Rightarrow \log_b^a \log_b^c = 1$

Hence the result.

Illustration 53 Let x be the arithmetic mean and y, z be two geometric means between any two positive numbers, prove that $y^3 + z^3 = 2xyz$.

Solution

Let a and b be any two positive numbers.

$\therefore x = \dfrac{a+b}{2}$, $2x = a + b$

Now y, z are two GMs between a and b

$\therefore$ a, y, z, b are in G.P.

$$\therefore \frac{y}{a} = \frac{z}{y} = \frac{b}{z} = r \ \text{(say)}$$

$\therefore y = ar,\ z = yr = ar^2,\ b = zr$

$\Rightarrow b = ar^3$

$$\Rightarrow \frac{b}{a} = r^3$$

$\therefore$ L.H.S. $y^3 + z^3$

$a^3r^3 + a^3r^6$

$a^3\left(r^3 + r^6\right)$

$$= a^3\left(\frac{b}{a}+\frac{b^2}{a^2}\right)$$

$$= a^2b + ab^2$$

$$= ab(a + b)$$

$$\text{R.H.S.} = 2xyz$$

$$= (2x)(y)(z)$$

$$= (a + b)arar^2$$

$$= (a + b)a^2r^3$$

$$= (a+b)\,\cancel{a}\,\frac{b}{\cancel{a}} \qquad \left(\because \frac{b}{a} = r^3\right)$$

$$= ab(a + b)$$

$$\therefore y^3 + z^3 = 2xyz$$

Hence the result.

Illustration 54 The 7th term of an H.P. is $\frac{1}{10}$ and 12th term is $\frac{1}{25}$ find the 20th term, and the *n*th term.

Solution

Let H.P. be $\frac{1}{a}, \frac{1}{a+b}, \frac{1}{a+2b}$

The 7th term $= \frac{1}{a+6d} = \frac{1}{10} \Rightarrow a+6d = 10$

and 12th term $= \frac{1}{a+11} = \frac{1}{25} \Rightarrow a+11d = 25$

$$\therefore a + 6d = 10$$

$$a + 11d = 25$$

$$- \quad - \qquad -$$

$$-5d = -15$$

$$d = 3$$

$$a + 18 = 10$$

$$a = -8$$

Hence 20th term $= \frac{1}{a+19d} = \frac{1}{-8+57}$

$$= \frac{1}{49}$$

and *n*th term $= \frac{1}{a+(n-1)d} = \frac{1}{-8+(n-1)3}$

$$= \frac{1}{3n-11}$$

Hence the result.

Illustration 55 The *m*th term of an H.P. is *n* and *n*th term is *m*. Find its
(i) $(m + n)$th term (ii) (mn)th term (iii) *r*th term

Solution

Let the given H.P. be $\frac{1}{a}, \frac{1}{a+b}, \frac{1}{a+2d}$

$$\therefore m\text{th term } = n = \frac{1}{a+(m-1)d}$$

$$\Rightarrow \frac{1}{n} = a+(m-1)-d$$

$$n\text{th term} = m = \frac{1}{a+(n-1)d}$$

$$\Rightarrow \frac{1}{m} = a+(n-1)d$$

$$\therefore \frac{1}{n} - \frac{1}{m} = d\left(m - \cancel{1} - n + \cancel{1}\right)$$

$$\frac{m-n}{mn} = d(m-n) \Rightarrow d = \frac{1}{mn}$$

$$\rightarrow \frac{1}{m} = a+(n-1)\frac{1}{mn}$$

$$\therefore \frac{1}{m} = a + \frac{1}{m} - \frac{1}{m}$$

$$\cancel{\frac{1}{m}} = a + \cancel{\frac{1}{m}} - \frac{1}{mn}$$

$$\Rightarrow a = \frac{1}{mn}$$

(i) $\therefore (m + n)$th term of H.P. is

$$= \frac{1}{a+(m+n-1)d} = \frac{1}{(1/mn)+[(m+n-1)/mn]}$$

$$= \frac{mn}{\cancel{1}+m+n-\cancel{1}} = \frac{mn}{m+n}$$

(ii) $\therefore (mn)$th term of H.P. is

$$= \frac{1}{a+(mn-1)d} = \frac{1}{(1/mn)+[(m-n-1)/mn]}$$

$$= \frac{mn}{\cancel{1}+mn+\cancel{1}} = \frac{mn}{mn} = 1$$

(iii) Let the H.P. be $\dfrac{1}{a+(r-1)d}$

$$= \frac{1}{(1/mn)+[(r-1)/mn]} = \frac{mn}{1+r-1}$$

$$= \frac{mn}{r}$$

Hence the result.

Illustration 56 If *p*th, *q*th and *r*th terms of an H.P. are *a*, *b*, *c* respectively, prove that $\dfrac{q-r}{a}+\dfrac{r-p}{b}+\dfrac{p-q}{c}=0$

Solution

Let the H.P. be $\dfrac{1}{\mathrm{A}}, \dfrac{1}{\mathrm{A}+\mathrm{D}}, \dfrac{1}{\mathrm{A}+2\mathrm{D}}$

$\therefore$ *p*th term = $a = \dfrac{1}{\mathrm{A}+(p-1)\mathrm{D}} \Rightarrow \dfrac{1}{a} = \mathrm{A}+(p-1)\mathrm{D}$

*q*th term = $b = \dfrac{1}{\mathrm{A}+(q-1)\mathrm{D}} \Rightarrow \dfrac{1}{b} = \mathrm{A}+(q-1)\mathrm{D}$

*r*th term $c = \dfrac{1}{\mathrm{A}+(r-1)\mathrm{D}} \Rightarrow \dfrac{1}{c} = \mathrm{A}+(r-1)\mathrm{D}$

$$\frac{q-r}{a}+\frac{r-p}{b}+\frac{p-q}{c}$$

$= \mathrm{A}(q-r) + (p-1)(q-r)\mathrm{D} + \mathrm{A}(r-p) + (q-1)(r-p)\mathrm{D} + \mathrm{A}(p-q) +$
$(r-1)(p-q)\mathrm{D}$

$= \mathrm{A}(\cancel{q}-\cancel{r}+\cancel{r}-\cancel{p}+\cancel{q}-\cancel{q})+D(\cancel{pq}-\cancel{pr}-\cancel{q}+\cancel{r}+\cancel{qr}-\cancel{qp}-\cancel{r}+\cancel{p}+\cancel{pr}-\cancel{p}+\cancel{qr}+\cancel{q})$

$= \mathrm{A}(0) + \mathrm{D}(0)$

$= 0 + 0$

$= 0 =$ R.H.S.

Hence the result.

Illustration 57 Prove that three quantities *a*, *b*, *c* are in A.P., G.P. or H.P. if. $\dfrac{a-b}{b-c} = \dfrac{a}{a}, \dfrac{a}{b}, \dfrac{a}{c}$.

Solution

a, *b*, *c* are in A.P. if

$b - a = c - b$ if $\dfrac{a-b}{b-c} = 1 = \dfrac{a}{a}$

Also *a*, *b*, *c* are in G.P. if

$$\frac{b}{a} = \frac{c}{b} \Rightarrow 1-\frac{b}{a} = 1-\frac{c}{b}$$

$$\Rightarrow \frac{a-b}{a} = \frac{b-c}{b} \Rightarrow \frac{a-b}{b-c} = \frac{a}{b}$$

Similarly a, b, c are in H.P. if

$$\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}$$

$$\Rightarrow \frac{a-b}{a\cancel{b}}=\frac{b-c}{\cancel{b}c} \Rightarrow \frac{a-b}{b-c}=\frac{a}{c}$$

$$\therefore \frac{a-b}{b-c}=\frac{a}{a},\frac{a}{b},\frac{a}{c}$$

Hence the result.

Illustration 58 (i) If a, b, c are in A.P., b, c, a are in G.P., then show that c, a, b are in H.P.

(ii) If a, b, c are in A.P., p, q, r are in H.P. and ap, bq, cr are in G.P., show that $\frac{p}{r}+\frac{r}{p}=\frac{c}{a}+\frac{a}{c}$.

(iii) If a^2, b^2, c^2 are in A.P. show that $\frac{b+c}{a},\frac{c+a}{b},\frac{a+b}{c}$ are in H.P.

Solution

(i) a, b, c are in A.P. if $b=\frac{a+c}{2}$

Since b, c, a are in G.P. $c^2 = ab$

$$\Rightarrow bc^2=\frac{a+c}{2}ab$$

$$\therefore 2bc^2 = (a + c)ab$$

$$= \left(\frac{c^2}{b}+c\right)ab \qquad \left(\because \frac{c^2}{a}=b\right)$$

$$\Rightarrow 2bc^{\cancel{2}} = \frac{(c+b)}{\cancel{b}}a\cancel{b}\cancel{c}$$

$$\Rightarrow 2bc = (c + b)a$$

$$\Rightarrow \frac{2bc}{c+b} = a \qquad \left(\because \frac{1}{a}=\frac{1}{c}+\frac{1}{b}2\frac{1}{a}=\frac{b+c}{2bc}\right)$$

$\therefore$ c, a, b are in H.P.

(ii) Let a, b, c be in A.P. $b=\frac{a+c}{2}$

p, q, r in H.P. $q=\frac{2pr}{p+r}$

and ap, bq, cr are in G.P. $(bq)^2 = apcr$

$\therefore$ $b^2q^2 = acpr$

$$\frac{(a+c)^2}{\cancel{4}}\frac{\cancel{4}p^2r^2}{(p+r)^2}=acpr$$

$$\frac{(a+c)^2}{ac} = \frac{\cancel{pr}(p+r)^2}{p^{\cancel{2}}r^{\cancel{2}}}$$

$$\Rightarrow \frac{a^2+c^2+2ac}{ac} = \frac{p^2+r^1+2pr}{pr}$$

$$\Rightarrow \frac{a}{c}+\frac{c}{a}+\cancel{2} = \frac{p}{r}+\frac{r}{p}+\cancel{2}$$

$$\therefore \frac{a}{c}+\frac{c}{a} = \frac{p}{r}+\frac{r}{p}$$

Hence the result.

(iii) Let $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ be in H.P.

$$\Rightarrow \frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b} \text{ are in A.P.}$$

$$\Rightarrow \frac{a}{b+c}+1, \frac{b}{c+a}+1, \frac{c}{a+b}+1 \text{ are in A.P.}$$

$$\frac{a+b+c}{b+c}, \frac{a+b+c}{c+a}, \frac{a+b+c}{a+b} \text{ are in A.P.}$$

$$\Rightarrow \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \text{ are in A.P.}$$

$$\Rightarrow \frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$$

$$\frac{b+\cancel{c}-\cancel{c}-a}{\cancel{(c+a)}(b+c)} = \frac{c+\cancel{a}-\cancel{a}-b}{(a+b)\cancel{(c+a)}}$$

$$(a+b)(b-a) = (b+c)(c-b)$$

$$b^2 - a^2 = c^2 - b^2$$

$$2b^2 = a^2 + c^2$$

$\therefore$ a^2, b^2, c^2 are in A.P.

Hence the result.

Illustration 59 Find two numbers between $\frac{1}{16}$ and $\frac{1}{6}$ such that the first three may be in G.P. and the last three in H.P.

Solution

Le a and b be the two numbers between $\frac{1}{16}$ and $\frac{1}{6}$

$\therefore$ $\frac{1}{16}$, a, b are in G.P. $\Rightarrow a^2 = \frac{b}{16}$

and a, b, $\frac{1}{6}$ are in A.P. $\Rightarrow b = \frac{2a(1/6)}{a+(1/6)}$

$$= \frac{2a}{6a+1}$$

$$a^{\cancel{2}} = \frac{\cancel{2}\,\cancel{a}}{(6a+1)\underset{8}{\cancel{16}}}$$

$$8a(6a + 1) = 1$$

$$48a^2 + 8a - 1 = 0$$

$$(12a - 1)(4a + 1) = 0$$

$\Rightarrow\ a = \frac{1}{12}$ or $\frac{-1}{4}$ is not possible $\qquad (\because a > 0)$

$$\therefore a = \frac{1}{12} \Rightarrow a^2 = \frac{1}{144}$$

$$\therefore \frac{1}{144} = \frac{b}{16} \Rightarrow b = \frac{1}{9}$$

$\therefore$ Required numbers are $\frac{1}{12}$ and $\frac{1}{9}$

Hence the result.

Illustration 60 Insert 3 harmonic means between -2 and $2/11$.

Solution

Let H_1, H_2, H_3 be three harmonic means between -2 and $\frac{2}{11}$.

$\Rightarrow -2, H_1, H_2, H_3, \frac{2}{11}$ are in H.P.

$\Rightarrow \frac{-1}{2}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{11}{2}$ are in A.P.

$\therefore a = -\frac{1}{2}$ and 5th term $= \frac{11}{2}$

$$\Rightarrow f(5) = a + 4d$$

$$\frac{11}{2} = -\frac{1}{2} + 4d \Rightarrow \frac{11}{2} + \frac{1}{2} = 4d$$

$$\therefore 4d = 6 \ \therefore\ d = \frac{3}{2}$$

$$\therefore \frac{1}{H_1} = a + d = -\frac{1}{2} + \frac{3}{2} = 1$$

$$\frac{1}{H_2} = a + 2d = -\frac{1}{2} + 3 = \frac{5}{2}$$

$$\frac{1}{H_3} = a + 3d = \frac{-1}{2} + \frac{9}{2} = 4$$

$\therefore H_1 = 1, H_2 = \frac{5}{2}, H_3 = \frac{1}{4}$ are three harmonic means.

Hence the result.

Illustration 61 Two A.Ms A_1, A_2, two G.Ms G_1, G_2 and two H.Ms are inserted between any two numbers. Prove that $\frac{A_1 + A_2}{H_1 + H_2} = \frac{G_1G_2}{H_1 \times H_2}$.

Solution

Let two numbers be a and b

$\therefore\ a_1, A_1, A_2, b$ are in A.P.

$\therefore\ b - A_2 = A_1 - a$

$A_1 + A_2 = a + b$

as a, G_1, G_2 b are in G.P.

$$\therefore\ \frac{G_1}{a} = \frac{b}{G_2} \Rightarrow G_1\,G_2 = ab$$

and a, H_1, H_2, b are in H.P.

$$\frac{1}{H_1} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H_2}$$

$$\frac{1}{H_1} + \frac{1}{H_2} = \frac{1}{b} + \frac{1}{a}$$

$$\Rightarrow \frac{H_1 + H_2}{H_1\,H_2} = \frac{a+b}{ab}$$

$$\Rightarrow \frac{H_1 + H_2}{H_1\,H_2} = \frac{A_1 + A_2}{G_1\,G_2}$$

$$\Rightarrow \frac{G_1\,G_2}{H_1\,H_2} = \frac{A_1 + A_2}{H_1 + H_2}$$

$$\therefore\ \frac{A_1 + A_2}{H_1 + H_2} = \frac{G_1\,G_2}{H_1H_2}$$

Hence the result.

Illustration 62 If a, b, c, d are in H.P. show that

(i) $ad > bc$, (ii) $a + b > b + c$ given that a, b, c, d are positive real numbers.

Solution

Let a, b, c, d be in H.P.

(i) As G > H

$\Rightarrow \sqrt{ac} > b$ and $\sqrt{bd} > c$

$\Rightarrow \sqrt{ac}\,\sqrt{bd} > bc$

$\Rightarrow a\ b\ c\ d > b^2\ c^2 \Rightarrow ad > bc$

(ii) As A.M. > H.M.

$\frac{a+c}{2} > b$ and $\frac{b+d}{2} > c$

$a + c + b + d > 2b + 2c$

$a + d > b + c$

Hence the result.

Illustration 63 (i) Sum to n terms the series $2 + 5x + 8x^2 + 11x^3 + \ldots, |x| < 1$. Deduce the sum to infinity.

(ii) Find the sum of infinite series $1 - 3x + 5x^2 - 7x^3 + \ldots$, when $|x| < 1$.

Solution

(i) Let $2 + 5x + 8x^2 + 11x^3 + \ldots |x| < 1$

Hence nth term of arithmetic geometric series is

$[2 + (n - 1)3]x^{n-1}$

$= (3n - 1)x^{n-1}$

$\therefore x_n = 2 + 5x + 8x^2 + 11x^3 + \ldots + (3n - 1)x^{n-1}$

$xs_n = 2x + 5x^2 + 8x^3 + \ldots + (3n - 1)x^{n-1} + (n - 1)x^n$

$(1 - x)S_n = 2 + (3x + 3x^2 + 3x^3 + \ldots + 3x^{n-1})$

$$\Rightarrow S_n = \frac{2}{1-x} + \frac{3x(1-x^{n-1})}{(1-x)^2} - \frac{(3n-1)x^n}{(1-x)}$$

Now as $n \to \infty \Rightarrow x^n, x^{n-1}, nx^n \to 0$

$$\therefore S_n = \frac{2}{1-x} + \frac{3x}{(1-x)^2} = \frac{2(1-x)+3x}{(1-x)^2}$$

$$\therefore S_n = \frac{2+x}{(1-x)^2}$$

(ii) Let $1 - 3x + 5x^2 - 7x^3 + \ldots |x| < 1$.

Here 1, 3, 5, 7. . are in G.P.

Let $S = 1 - 3x + 5x^2 - 7x^3 + \ldots \infty$

$xS = x - 3x^2 + 5x^3 \ldots \infty$

$\Rightarrow (1 + x)S = 1 - 2x + 2x^2 - 2x^3 + \ldots$

$= 1 - 2x(1 - x + x^2 - 2x^3 + \ldots \infty)$

$$= 1 - 2x\left(\frac{1}{1+x}\right) \qquad \left(\because \lim_{x\to\infty} S_n = \frac{a}{1-r}\right)$$

$$= \frac{1+x-2x}{1+x}$$

$$\therefore (1 + x)S = \frac{1-x}{1+x} \Rightarrow S = \frac{1-x}{(1+x)^2}$$

Illustration 64 (i) Find the sum to infinity of the series $1+\frac{2}{3}+\frac{3}{3^2}+\frac{4}{3^3}+\ldots$

(ii) Sum to n term the series $\frac{2}{3}+\frac{5}{3^2}+\frac{8}{3^3}+\ldots$

Solution

(i) Let $\frac{1}{1}+\frac{2}{3}+\frac{2}{3^2}+\frac{4}{3^3}+\ldots\infty$

Here 1, 2, 3, 4 . . . are in A.P. and 1, 3, 3^2, 3^3 . . . are in G.P.

$$\therefore S = 1 + \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \ . \ . \ . \ \infty$$

$$\therefore \ \frac{1}{3}S = \frac{1}{3} + \frac{1}{3^2} + \frac{3}{3^3} + \ . \ . \ . \ \infty$$

$$\left(1 - \frac{1}{3}\right) S = 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} \ . \ . \ . \ \infty$$

$$\left(1 + \frac{1}{3}\right) S = \frac{1}{1 - (1/3)} \Rightarrow \frac{2}{3} S = \frac{3}{2}$$

$$\Rightarrow S = \frac{9}{4}$$

Hence the result.

(ii) Let $\frac{2}{3} + \frac{5}{3^2} + \frac{8}{3^3} + \ldots$

*n*th term of *a* series is

$$\therefore \text{ The } \left[2 + (n-1)3\right]\left[\frac{1}{3}\left(\frac{1}{3}\right)^{n-1}\right]$$

$$= \frac{3n-1}{3^n}$$

$$\text{Let } S_n = \frac{2}{3} + \frac{5}{3^2} + \frac{8}{3^3} + \ . \ . \ . \ + \frac{3n-4}{3^{n-1}} + \frac{3n-1}{3^n}$$

$$\frac{1}{3}Sn = \frac{2}{3^2} + \frac{5}{3^3} + \ . \ . \ . \ \frac{3n-4}{3^n} + \frac{3n-1}{3^{n+1}}$$

$$\left(1 - \frac{1}{3}\right) S_n = \frac{2}{3} + \left(\frac{3}{3^2} + \frac{3}{3^3} + \ . \ . \ . \ + \frac{3}{3^n}\right) - \frac{3n-1}{3^{n+1}}$$

$$= \frac{2}{3} + \frac{3}{3^2}\left[\frac{1 - (1/3)^{n-1}}{1 - (1/3)}\right] - \frac{3n-1}{3^{n+1}}$$

$$\frac{2}{3}S_n = \frac{2}{3} + \frac{1}{2}\left[1 - \frac{1}{3^{n-1}}\right] - \frac{3n-1}{2 \times 3^n}$$

$$S_n = 1 + \frac{3}{4}\left(1 - \frac{1}{3^{n-1}}\right) - \frac{3n-1}{3^{n+1}}$$

Illustration 65 If the sum to infinity of the series $3 + (3+d)\frac{1}{4} + (3+2d)\frac{1}{4^2} + \ . \ . \ .$ is $3\frac{1}{3}$ find *d*.

Solution

$$\text{Let } S = 3 + (3+d)\frac{1}{4} + (3+2d)\frac{1}{4^2} + \ . \ . \ .$$

$$\frac{1}{4}S = \frac{3}{4} + (3+d)\,\frac{1}{4^2} + \ . \ . \ .$$

$$\left(1-\frac{1}{4}\right)S = 3+\frac{d}{4}+\frac{d}{4^2}+\ ...$$

$$\frac{3}{4}S = 3+\frac{(d/4)}{1-(d/4)} = 3+\frac{d}{3}$$

but $S = \frac{10}{3}$ given

$$\frac{\cancel{3}}{4}\frac{10}{\cancel{3}} = 3+\frac{d}{3}$$

$$\frac{5}{2}-3=\frac{d}{3} \Rightarrow \frac{-1}{2}=\frac{d}{3} \Rightarrow d=-\frac{3}{2}$$

Hence the result.

Illustration 66 Sum to infinity the series $1^2 + 2^2x + 3^2 \times x^2 + \ldots; |x| < 1$.

Solution

Here given series is not an arithmetic-geometric series.

$$S = 1 + 4x + 9x^2 + 16x^3 + \ldots$$

$$xS = x + 4x^2 + 9x^3 + \ldots$$

$$(1-x)S = x + 4x^2 + \ldots$$

which is arithmetic geometric series

$$x(1-x)S = x + 3x^2 + 5x^3 + \ldots$$

$$(1 - x - x + x^2)S = 1 + 2x + 2x^2 + 2x^3 + \ldots$$

$$(1-x)^2\ S = 1 + \frac{2x}{1-x}$$

$$= \frac{1-x+2x}{1-x}$$

$$\therefore (1-x)^2\ S = \frac{1+x}{1-x}$$

$$\Rightarrow\ S = \frac{(1+x)}{(1-x)}$$

Hence the result.

Illustration 67 Find the sum of the series $2 \times 5 + 5 \times 8 + 8 \times 11 + \ldots$ up to n terms.

Solution

Let

$S = 2 \times 5 + 5 \times 8 + 8 \times 11 + \ldots$ up to n term.

$\therefore$ 2, 5, 8 . . . are in A.P.

$2 + (n-1)3 = 3n - 1$

5, 8, 11 . . . are in A.P.

$5 + (n - 1)3 = 3n + 2$

$$\therefore S_n = \sum_{1}^{n} (3n-1)(3n+2)$$

$$= \sum (9n^2 + 3n - 2)$$

$$= 9\Sigma n^2 + 3\Sigma n - 2\Sigma 1$$

$$= \frac{\overset{3}{\cancel{9}}\, n(n+1)(2n+1)}{\underset{2}{\cancel{6}}} + \frac{3n(n+1)}{2} - 2n$$

$$= \frac{3n(n+1)}{2}(2n+1+1) - 2n$$

$$= \frac{3n(n+1)\cancel{2}(n+1)}{\cancel{2}} - 2n$$

$$= n[3(n^2 + 2n + 1) - 2]$$

$$S_n = n(3n^2 + 6n + 1)$$

Hence the result.

Illustration 68 Sum the series 1 × 2 × 3 + 2 × 3 × 4 + 3 × 4 × 5 + . . . up to *n* terms.

Solution

Let 1 × 2 × 3 + 2 × 3 × 4 + 3 × 4 × 5 + . . . up to *n* terms

$$\therefore S_n = \sum n(n+1)(n+2)$$

$$= \sum n(n^2 + 3n + 2)$$

$$= \Sigma n^3 + 3\Sigma n^2 + 2\Sigma n$$

$$= \frac{n^2(n+1)^2}{4} + \frac{\cancel{3}n(n+1)(2n+1)}{\underset{2}{\cancel{6}}} \frac{\cancel{2}n(n+1)}{\cancel{2}}$$

$$= n(n+1)\left[\frac{n(n+1)}{4} + \frac{2n+1}{2} + 1\right]$$

$$= \frac{n(n+1)}{4}\left[n^2 + n + 4n + 2 + 4\right]$$

$$= \frac{n(n+1)(n^2 + 5n + 6)}{4}$$

$$= \frac{n(n+1)(n+2)(n+3)}{4}$$

Hence the result.

Illustration 69 Find the sum of the following series:

(i) $1^2 + 3^2 + 5^2 + \ldots$ up to n terms.

(ii) $10^3 + 11^3 + \ldots + 20^3$.

Solution

(i) Let $1^2 + 3^25 + 5^2 + \ldots$ upto n terms

$\therefore$ nth term is $\left[1 + (n-1)2\right]^2$

$= (2n-1)^2$

$$\therefore S_n = \sum (2n-1)^2$$

$$= \sum \left(4n^2 - 4n + 1\right)$$

$$= 4\Sigma n^2 - 4\Sigma n + \Sigma 1$$

$$= \frac{\overset{2}{\cancel{4}}\, n(n+1)(2n+1)}{\underset{3}{\cancel{6}}} - \frac{\overset{2}{\cancel{4}}\, n(n+1)}{\cancel{2}} + n$$

$$= n\left[\frac{2}{3}\left(2n^2 + 3n + 1\right) - 2(n+1) + 1\right]$$

$$= \frac{n}{3}\left[4n^2 + 6\cancel{n} + 2 - 6\cancel{n} - 6 + 3\right]$$

$$= \frac{n}{3}\left(4n^2 - 1\right)$$

Hence the result.

(ii) Let $10^3 + 11^3 + \ldots + 20^3$

$$= \sum_1^{20} n^3 - \sum_1^{9} n^3$$

$$= \left[\frac{n^2(n+1)^2}{4}\right]_{n=20} - \left[\frac{n^2(n+1)^2}{4}\right]_{n=9}$$

$= (210)^2 - (45)^2$

$= (210 + 45)\,(210 - 45) = 255 \times 165$

$= 42{,}075$

Hence the result.

Illustration 70 Find the sum of first n terms of the series

(i) $1 + (1 + 2) + (1 + 2 + 3) + \ldots$

(ii) $1^2 + \left(1^2 + 2^2\right) + \left(1^2 + 2^2 + 3^2\right) + \ldots$

Solution

(i) Let

$S_1 = 1$

$S_2 = 1 + 2$

$S_3 = 1 + 2 + 3$

$$S_n = 1 + 2 + 3 + \ldots + \Sigma n = \frac{n(n+1)}{2}$$

$$\sum_1^n S_n = \sum_1^n \frac{n^2+n}{2}$$

$$= \frac{1}{2}\left(\Sigma n^2 + \Sigma n\right)$$

$$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$$

$$= \frac{n(n+1)}{4}\left(\frac{2n+1}{3} + 1\right)$$

$$= \frac{n(n+1)}{4}\left(\frac{2n+4}{3}\right)$$

$$= \frac{n(n+1)\cancel{2}(n+2)}{\underset{2}{\cancel{4}} \times 3}$$

$$= \frac{n(n+1)(n+2)}{6}$$

Hence the result.

(ii) Let

$S_1 = 1^2$

$S_2 = 1^2 + 2^2$

$S_3 = 1^2 + 2^2 + 3^2$

$$S_n = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\therefore \Sigma S_n = \frac{1}{6}\sum n(n+1)(2n+1)$$

$$= \frac{1}{6}\sum n\left(2n^2 + 3n + 1\right)$$

$$= \frac{1}{6}\left(2\Sigma n^3 + 3\Sigma n^2 + \Sigma n\right)$$

$$= \frac{1}{6}\left[\frac{\cancel{2}n^2(n+1)^2}{\underset{2}{\cancel{4}}} + \frac{\cancel{3}n(n+1)(2n+1)}{\underset{2}{\cancel{6}}} + \frac{n(n+1)}{2}\right]$$

$$= \frac{n(n+1)}{12}\left[n(n+1) + 2n + 1 + 1\right]$$

$$= \frac{n(n+1)}{12}\left(n^2 + 3n + 2\right)$$

Hence the result.

Illustration 71 Find the sum of n terms of the series whose nth term is $n^2 + 2n + 2 + 2n$.

Solution

Let $f(n) = n^2 + 2n + 2 + 2^n$

$$\Sigma f(n) = \Sigma n^2 + 2\Sigma n + 2\Sigma 1 + \Sigma 2^n$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2} + 2n + 2\left(\frac{2^n - 1}{2 - 1}\right) \quad (\because 2 + 2^2 + 2^3 + \ldots \text{ are in G.P.})$$

$$= \frac{n}{6}\left[(n+1)(2n+1) + 6(n+1) + 12\right] + 2^{n+1} - 2$$

$$= \frac{n}{6}\left(2n^2 + 3n + 1 + 6n + 6 + 12\right) + 2^{n+1} - 2$$

$$= \frac{n}{6}\left(2n^2 + 9n + 19\right) + 2^{n+1} - 2$$

Hence the result.

Illustration 72 (i) Find the nth term of the series $1 + 3 + 7 + 13 + \ldots$ and hence find the sum of first n term.

(ii) Sum the series $1 + 3 + 7 + 15 + \ldots$ up to n terms.

Solution

(i) Let $S_n = 1 + 3 + 7 + 13 + \ldots + T_{n-1} + T_n$

$\Rightarrow S_n = 1 + 3 + 7 + \ldots + T_{n-1} + T_n$

$0 = 1 + 2 + 4 + 6 + \ldots$ upto n terms $- T_n$

$\therefore$ Tn = 1 + $\left[2 + 4 + 6 + \ldots \text{ upto } (n - 1) \text{ terms}\right]$

$$= 1 + \frac{n-1}{2}\left[2(2) + (n - 1 - 1)2\right]$$

$$= 1 + \frac{(n+1)}{\cancel{2}}\cancel{2}\left[\cancel{2} + n - \cancel{2}\right]$$

Tn = $n^2 - n + 1$

$$\therefore \ \Sigma T_n = \Sigma n^2 - \Sigma n + \Sigma 1$$

$$= \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + n$$

$$= \frac{n}{6}\left[(2n^2 + 3n + 1) - 3(n+1) + 6\right]$$

$$= \frac{n}{6}\left(2n^2 + \cancel{3n} + 1 - \cancel{3n} - 3 + 6\right)$$

$$= \frac{n}{6}\left(2n^2 + 4\right)$$

$$= \frac{n}{3}\left(n^2 + 2\right)$$

Hence the result.

(ii) Let 1 + 3 + 7 + 15 + . . . the successive differences are 2, 4, 8 . . . which are in G.P.

$\therefore$ $Sn = 1 + 3 + 7 + 15 + \ldots + T_{n-1} + T_n$

$Sn = 1 + 3 + 7 + \ldots + T_{n-1} + T_n$

$0 = 1 + 2 + 4 + 8 + \ldots T_n$

$T_n = 1 + 2 + 4 + 8 + \ldots T_n$

$= 2^0 + 2^1 + 2^2 + 2^3 + \ldots$

$$= 2^0 \frac{(2^n - 1)}{2 - 1}$$

$\therefore$ $T_n = 2^n - 1$

$$\Sigma T_n = \sum(2^n - 1) \quad V$$

$$= \Sigma 2^n - \Sigma 1$$

$$= \frac{2(2n - 1)}{2 - 1} - n$$

$$= 2^{n+1} - 2 - n$$

Hence the result.

Illustration 73 Sum up .3 + 33 + 333 + . . . up to n terms.

Solution

Let 3 + 33 + 333 + . . . upto n terms

$$= 3(1 + 11 + 111 + \ldots)$$

$$= \frac{3}{9}(9 + 99 + 999 + \ldots)$$

$$= \frac{3}{9}\left[(10 - 1) + (10^2 - 1) + (10^3 - 1) + \ldots\right]$$

$$= \frac{3}{9}\left[\frac{10(10^n - 1)}{10 - 1} - n\right]$$

$$= \frac{1}{3}\left[\frac{10}{9}(10^n - 1) - n\right]$$

$$= \frac{10}{27}(10^n - 1) - \frac{n}{3}$$

Hence the result.

Illustration 74 (i) Find the nth term, the sum of n terms and sum to infinity of the series $\frac{1}{2 \times 5} + \frac{1}{5 \times 8} + \frac{1}{8 \times 11} + \ldots$

(ii) $\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots$

Solution

(i) Let

$$+\ \frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}\ \ . . .$$

$$\therefore\ \mathrm{T}_n = \frac{1}{(3n-1)(3n+2)}$$

$$= \frac{1}{3}\left(\frac{1}{3n-1}-\frac{1}{3n-2}\right)$$

Putting $n = 1, 2, 3\ . . .$

$$\mathrm{T}_1 = \frac{1}{3}\left(\frac{1}{2}-\frac{\cancel{1}}{\cancel{5}}\right)$$

$$\mathrm{T}_2 = \frac{1}{3}\left(\frac{\cancel{1}}{\cancel{5}}-\frac{1}{8}\right)$$

$$\mathrm{T}_n = \frac{1}{3}\left(\frac{1}{3n-1}-\frac{1}{3n+2}\right)$$

$$\Sigma\mathrm{T}_n = \mathrm{S}_n = \frac{1}{3}\left(\frac{1}{2}-\frac{1}{3n+2}\right)$$

$$= \frac{1}{\cancel{3}}\left[\frac{\cancel{3}n+\cancel{2}-\cancel{2}}{2(3n+2)}\right]$$

$$\mathrm{S}_n = \frac{n}{2(3n+2)}$$

$$\therefore\ \underset{x\to\infty}{\text{limit}}\ \mathrm{S}_n = \underset{x\to\infty}{\text{limit}}\ \frac{n}{2(3n+2)}$$

$$= \frac{1}{2}\underset{x\to\infty}{\text{limit}}\ \frac{1}{3+(2/n)}$$

$$= \frac{1}{2}\,\frac{1}{3+0} \qquad \left(\because n\to\infty \Rightarrow \frac{1}{n}\to 0\right)$$

$$= \frac{1}{6}$$

(ii) $\dfrac{1}{1}+\dfrac{1}{1+2}+\dfrac{1}{1+2+3}+\ . . .$

$$\therefore\ \mathrm{T}_n = \frac{1}{1+2+\ . . .\ +n} = \frac{1}{\Sigma n} = \frac{1}{[n(n+1)]/2}$$

$$\therefore\ \mathrm{T}_n = \frac{2}{n(n+1)}$$

$$T_n = 2\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

Putting $n = 1, 2, 3 \ldots$

$$T_1 = 2\left(1-\frac{1}{2}\right)$$

$$T_2 = 2\left(\frac{1}{2}-\frac{1}{3}\right)$$

$$T_3 = 2\left(\frac{1}{3}-\frac{1}{4}\right)$$

$$T_n = 2\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$\therefore\ T_1 + T_2 + \ldots T_n\ \Sigma S_n = 2\left(1-\frac{1}{n+1}\right)$$

$$= 2\left(\frac{n+\cancel{1}-\cancel{1}}{n+2}\right)$$

$$\therefore\ \underset{x\to\infty}{\text{limit}}\ S_n = \underset{x\to\infty}{\text{limit}}\ \frac{2n}{n+1}$$

$$= 2\,\underset{x\to\infty}{\text{limit}}\ \frac{1}{1+(1/n)}$$

$$= 2\ \frac{1}{1+0} \qquad \left(\because n \to \infty \Rightarrow \frac{1}{n} \Rightarrow 0\right)$$

Hence the result $= 2$.

ANALYTICAL EXERCISES

1. The sides of a right angled triangle are in arithmetic progression. Find the ratio of the sides
2. Find the sum of all odd numbers between 100 and 200
3. How many terms of the A.P. 3, 6, 9, 12 . . . must be taken to make the sum 108?
4. Prove that if the mth term of an A.P. is n and its nth term is m its pth term is $m + n - p$.
5. The sum of all 2-digit numbers which are odd is 2,475. Prove that.
6. Prove that if the sum of n terms of a progression be a quadratic expression in n then it is A.P.
7. If a, b, c are in A.P. then prove that bc, ca and ab are in H.P.
8. Find the 8th term of the G.P. 2, 6, 18, 54 . . . .

9. If $\frac{1}{3}$, x_1, x_2, 9 are in G.P. then find the value of $\frac{x}{2}$

10. If $a = \left(x + \frac{x}{r} + \frac{x}{r^2} + \ldots\infty\right)$; $b = \left(y + \frac{y}{r} + \frac{y}{r^2} + \ldots\infty\right)$

$c = \left(z + \frac{z}{r} + \frac{z}{r^2} + \ldots\infty\right)$ then find the value of $\frac{ab}{c}$

11. If $a = 1 + x + x^2 + \ldots\infty$; $b = 1 + y + y^2 + \ldots\infty$ where $|x| < 1$ and $|y| < 1$ then find $\left(1 + xy + x^2y^2 + \ldots\infty\right)$.

12. If $x = y + y^2 + y^3 + \ldots\infty$ then find the value of y.

13. Find the values 5 + 55 + 555 + . . . to 10th term.

14. Prove that if p, q, r are in A.P. and p, q, s are in G.P. then p, $(p - q)$, $(s - r)$ are in G.P.

15. If nth term of the series $3, \sqrt{3}, 1, \ldots$ is $\frac{1}{243}$ then find n.

16. Find the harmonic mean of $\frac{x}{1 - xy}$ and $\frac{x}{1 + xy}$

17. If H is the harmonic mean between a and b then find the value of $\frac{H}{a} + \frac{H}{b}$.

18. The sum of n terms of an A.P. is $\left(3m^2 + 5n\right)$ which term of this is A.P.?

19. If xth, yth and zth terms of a G.P. be a, b, c respectively then find $a^{(y - z)}b^{(z - x)}c^{(x - y)}$

20. If a, b, c are in G.P. and $x^a = y^b = z^c$, then prove that $\log_y^x = \log_z^y$.

21. If xth, yth and zth terms of a G.P. be a, b, c respectively, then find $(y - z)\log a + (z - x)\log b + (x - y)\log c$.

22. Prove that if p, q, r, s are in G.P. then $p + q$, $q + r$, $r + s$ are in G.P.

23. Find four numbers in G.P. in which the third term is greater than the first by q and the second term is greater than the fourth by 18.

24. Find three numbers in G.P. whose sum is $\left(\frac{57}{2}\right)$ and product 729.

25. Find the sum of all numbers between 500 and 1,000 which are divisible by 13.

26. How many numbers of two digits are divisible by 7?

27. Which term of the series 7 + 11 + 15 + . . . is 403?

28. How many terms are there in an A.P. 5, 8, 11, 14 . . . 149?

29. The sum of digits of a three digit number is 12. The digits are in arithmetic progression. If the digits are reversed, then the number is diminished by 396. Find the numbers.

30. If the fifth term of G.P. is 81 and the second term is 24 then find the series.

31. Find the sum of all natural numbers between 100 and 1,000 which are multiples of 5.

32. Find the sum of all integers between 84 and 719 which are divisible by 5.

33. If $\sum n = 55$ then find $\sum n^2$.

34. If Σn, $\frac{\sqrt{10}}{3}$, Σn^2, Σn^3 are in G.P. then find the value of n.

35. Find the value of $2^{1/4}, 4^{1/8}, 8^{1/16}, 16^{1/32} \ldots \infty$

36. If x, y, z, w are in G.P. then $\frac{1}{x^3 + y^3}$; $\frac{1}{y^3 + z^3}$; $\frac{1}{z^3 + w^3}$ are in G.P. Prove that.

37. Find the sum of the series $0.5 + 0.55 + 0.555 + \ldots n$ terms.

38. The ratio of the sums of x and y terms of an A.P. is $x^2 = y^2$, then find the ratio of xth and yth terms.

39. If the sum of n terms of an A.P. is $(3n^2 - n)$ and its common difference is 6 then find first term.

40. Let a be the A.M. and b, c be two G.Ms between two positive numbers. Find the value of $\frac{b^3 + c^3}{abc}$.

41. If the sum of n terms of two A.Ps is in the ratio $(3m + 1) = (m + 4)$, the ratio of the fourth term is?

42. Three numbers are in G.P. With their sum 130 and their product 27,000, find the numbers.

43. If a, b, c are in A.P. and $a, b - a, c - a$ are in G.P. then find $a:b:c$.

44. If a and 1 of an A.P. are −4 and 146, the sum of the term is 7,171. Find the numbers of terms.

45. Find the numbers between 74 and 25,556 divisible by 5.

46. Prove that if p, q, r, s are in G.P., then $(p - q)^2$, $(q - r)^2$, $(r - s)^2$ are in G.P.

47. Find the sum of the series $2^3 + 4^3 + 6^3 + 8^3 \ldots$ to n terms.

48. If p, q, r are in A.P. and $(q - p)$, $(r - q)$, p are in G.P. then find the value of $p : q : r$

49. Find the nth term of the sequence 2, 5, 8, ...

50. Find the fourth term of the sequence $3, \frac{3}{2}, 1 \ldots.$

51. Find the value of progression $\log a^2, \log\frac{a^2}{b}, \log\frac{a^3}{b^2}$.

52. Find the fourth term of the sequence $\frac{1}{1+\sqrt{x}}; \frac{1}{1-x}; \frac{1}{1-\sqrt{x}}$

53. If $\log_2^{(5\cdot 2^x + 1)}; \log_4^{(2^{1-x} + 1)}; 1$ are in A.P. then find the value of x.

54. Find the sum of the series $a + (a + d) + (a + 2d) - (a + 3d) + \ldots$ up to $(2n + 1)$ terms.

55. Find the numbers of terms in the A.P. $a + b + \ldots + c$.

56. If the sides of a right triangle are in A.P. then find the sum of the sides of the two acute angles.

57. If a, b, c are three distinct numbers in G.P, $a + x, b + x, c + x$ are in H.P, then find the value of x.

58. If $\log_{\sqrt{3}}^{x} + \log_{3^{1/4}}^{x} + \log_{3^{1/6}}^{x} + \ldots + \log_{3^{1/16}}^{x} = 36$ then find x.

59. Find the sum to n terms of the series $\frac{1}{\sqrt{2}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{8}} + \frac{1}{\sqrt{8}+\sqrt{11}} + \ldots$

60. If in an A.P., whose first term is 2, the sum of first six terms is equal to one third of the sum of next six terms, then find the common difference of the A.P.

61. If $M = 1 + r^{a} + r^{2a} + \ldots$ then find r.

62. If $|x| < 1$ then find $1 + 3x + 5x^2 + 7x^2 + \ldots.$

63. If a, b, c are in A.P. then ath, bth and cth terms of a G.P. are in G.P. Prove that.

64. If $\log_{x}^{a}; a^{x/z}; \log_{b}^{x}$ are in G.P. then find x.

65. If a, 4, b are in A.P.; a, 2, b are in G.P., then a, 1, b are in H.P. Prove that.

66. If the sum of n terms of a series is $An^2 + bn$, then find the nth term.

67. The sum of 40 A.Ms between two numbers is 120. Then find the sum of 50 A.Ms between them.

68. Find the nth term of the sequence $\frac{1}{2}, \frac{1}{6}, \frac{1}{12} \ldots$

69. Find the sum of the nth term of the series $\sqrt{2} + \sqrt{8} + \sqrt{16} \ldots.$

70. Prove that if a, b, c are in A.P. as well as in G.P, then prove that $a = b = c$.

71. Find the sum of the series $\frac{1}{3 \times 7} + \frac{1}{7 \times 11} + \frac{1}{11 \times 15} + \ldots \infty$

72. Prove that if a, b, c are in A.P. then $\frac{1}{bc}; \frac{1}{c}; \frac{1}{b}$ are in A.P.

73. Prove that if the sum of first n terms of an A.P. is $Pn + Qn^2$ where P and Q are constants, then common difference of A.P. will be $2a$.

74. Prove that if p, q, r, s are in H.P., then $pq + qr + rs = 3ps$

75. Suppose the sequence $a_1, a_2, a_3 \ldots a \ldots$ forms an A.P, then find the value of $a_1^2 - a_2^2 + a_3^2 - a_4^2 + \ldots + a_{2n-1}^2 - a_{2n}^2$

76. If the ratio of 2nd to the 7th of n A.Ms between -7 and 65 is 1:7 then find the value of n

77. If A.P. consists of $n(\text{odd})$ terms and its middle term in m. then the sum of A.P. is mn Prove that.

78. If a, b, c are in A.P., then $(a + 2b - c)(a + c - b)(c + 2b - a) = 4abc$. Prove that.

79. Prove that if $x > 0, y > 0, z > 0$ and x, y, z are distinct then $(x + y)(y + z)(z + x) > 8xyz$.

80. Find the sum of the series $\frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \ldots \infty$ v

81. If A_1 and A_2 be the two A.Ms between two numbers a and b then $A_1 - A_2 = \frac{b-a}{3}$. Prove that.

82. If $\frac{1}{a}; \frac{a^n + b^n}{a^{n+1} + b^{n+1}}; \frac{1}{b}$ are in A.P., then find the value of n.

83. If p, q, r, s, t are in G.P., then prove that $\frac{t}{r} = \frac{s}{q}$.

84. Prove that the sum of the first 10 terms of the series $\sqrt{2} + \sqrt{6} + \sqrt{18} + \ldots = 121\left(\sqrt{6} + \sqrt{2}\right)$

85. Prove that the value of $\sum_{n=1}^{50} (-1)^n = 0$.

86. Prove that the sum of first n terms of the series $1^2 + 3^2 + 5^2 + \ldots = \frac{n(2n+1)(2n-1)}{3}$.

87. Find the value of 20th term of the series $\frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \ldots =$

88. Prove that the nth term of the sequence $1, \sqrt{2}, \sqrt[3]{3}, \sqrt{2} \ldots = n^{1/n}$.

89. Find the sum of n terms of the series $\left(1^2 - 2^2\right) + \left(3^2 - 4^2\right) + \left(5^2 - 6^2\right) + \ldots =$

90. Prove that if $x > 1, y > 1, z > 1$ are in G.P. then $\frac{1}{1 + \log x}; \frac{1}{1 + \log y}; \frac{1}{1 + \log z}$ are in H.P.

91. Prove that let A_1, A_2 be two A.Ms and G_1, G_2 be two GMs between x and y then $\frac{A_1 + A_2}{G_1 \cdot G_2} = \frac{x + y}{xy}$.

92. Prove that if H is the harmonic mean between x and y then $\frac{H + x}{H - x} + \frac{H + y}{H - y} = 2$.

ANSWERS

(1) 3 : 4 : 5
(2) 7,500
(3) 9
(8) 4,374
(9) 3
(10) $\frac{xy}{z}$
(11) $\frac{ab}{a + b - 1}$
(12) $\frac{x}{1 + x}$
(13) $\frac{5}{81}\left(10^{11} - 100\right)$
(15) 13
(16) x
(17) 2
(18) 27th
(19) 1
(21) 0
(23) 3, −6, 12, −24
(24) 6, 9, $\frac{27}{2}$
(25) 28,405
(26) 13
(27) 100
(28) 49
(29) 6, 4, 2
(30) 16, 24, 36
(31) 98,450
(32) 50,800
(33) 385
(34) 4
(35) 2

(37) $\frac{5n}{9}-\frac{5}{81}\left[1-(0.1)^n\right]$

(38) $(2x-1)=(2y-1)$

(39) 2

(40) 2

(41) 2

(42) 90, 30, 10

(43) 1 : 3 : 5

(44) 101

(45) 5,097

(47) $2n^2(n+1)^2$

(48) 1 : 2 : 3

(49) $3n-1$

(50) $\frac{4}{3}$

(51) A.P.

(52) $\frac{1+2\sqrt{x}}{1-x}$

(53) $1-\log_2^5$

(54) $a+nd$

(55) $\frac{b+c-2a}{b-a}$

(56) $\frac{7}{5}$

(57) b

(58) $\sqrt{3}$

(59) $\frac{1}{3}\left(\sqrt{3n+2}-\sqrt{2}\right)$

(60) 4

(61) $\left(\frac{A-1}{A}\right)^{1/a}$

(62) $\frac{1+x}{(1-x)^2}$

(64) $\frac{\log(\log a)-\log(\log b)}{\log a}$

(66) $\mathrm{A}(2n-1)+\mathrm{B}$

(67) 150

(68) $\frac{1}{n^2+n}$

(69) $\sum n$

(70) $a=b=c$

(71) $\frac{1}{12}$

(75) $\frac{n}{2n+1}\left(a_1^2-a_{2n}^2\right)$

(76) 11

(80) $\frac{1}{2}$

(82) -1

(87) $\frac{441}{4}$

(89) $n(2n+1)$